On the Implementation of an Interior-Point SQP Filter Line Search Algorithm

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Abstract: - We present a filter line search sequential quadratic programming (SQP) method based on an interior-point framework for nonlinear programming. Here we provide a comprehensive description of the algorithm, including the feasibility restoration phase for the filter method. The proposed method has been implemented in Fortran 90, and preliminary numerical testing seems to indicate that the method is effective.

Key-Words: - Nonlinear programming, SQP, Interior-point method, Filter method, Line search

1 Introduction

The proposed algorithm is a filter line search algorithm for solving nonlinear optimization problems of the form

$$\min_{x \in R^n} F(x) \qquad (1)$$
$$\text{s.t. } b \le h(x) \le b + r,\ l \le x \le u,$$

where $h_k : R^n \to R$ for $k = 1,\ldots,m$ and $F : R^n \to R$ are nonlinear and twice continuously differentiable functions. r is the vector of ranges on the constraints $h(x)$, u and l are the vectors of upper and lower bounds on the variables x and b is assumed to be a finite real vector. Elements of the vector r, l and u are real numbers subject to the following limitations: $0 \le r_k \le \infty$, $-\infty \le l_i, u_i \le \infty$ for $k = 1,\ldots,m$, $i = 1,\ldots,n$. Constraints of the form $b \le h(x) \le b + r$ are denoted by range constraints. Note that equality constraints can still be treated as range constraints with $r = 0$.

Let $\nabla F(x)$ denote the gradient of $F(x)$ and $\nabla h(x)$ denote de Jacobian matrix of the constraint vector

$$h(x)^T = [h(x) - b, b + r - h(x), x - l, u - x].$$

A solution of (1) will be denoted by x^*, and we assume that there is a finite number of solutions. We also assume that the first order Kuhn-Tucker (KT) conditions hold (with strict complementarity) at x^*. Thus, the constraints are verified and there exists a Lagrange multiplier vector $\lambda^* \ge 0$ such that

$$\nabla F(x^*) = \nabla h(x^*)^T \lambda^*,\ h(x^*)^T \lambda^* = 0. \qquad (2)$$

Given a starting point x_0, the proposed line search algorithm generates a sequence of improved estimates x_k of the solution for the problem (1) using a sequential quadratic programming (SQP) method. At each iteration k the search direction Δ_k is the solution of a quadratic programming subproblem whose objective function approximates the Lagrangian function $L(x,\lambda) = F(x) - \lambda^T h(x)$ and whose constraints are linear approximations to the constraints in (1). The usual definition of the QP subproblem is the following:

$$\min_{\Delta \in R^n} \Delta^T H_k \Delta + \nabla F_k^T \Delta \qquad (3)$$
$$\text{s.t. } b \le \nabla h_k \Delta + h_k \le b + r,\ l \le \Delta + x_k \le u$$

where $\nabla h(x)$ denotes the Jacobian matrix of the constraint vector $h(x)$, and ∇F_k, h_k and ∇h_k denote the relevant quantities evaluated at x_k. The matrix H_k is a symmetric positive definite approximation to the Hessian of the Lagrangian function. This problem has a solution Δ_k and a Lagrange multiplier $\pi_k \ge 0$ that satisfy

$$H_k \Delta_k + \nabla F_k = \nabla h_k^T \pi_k,\ \pi_k^T(\nabla h_k \Delta_k + h_k) = 0.$$

Clearly the most common approach for solving (3) considers active set methods (see, for example, [6]). Solving QP subproblems with equality constraints is straightforward. However, problems that have inequality constraints are significantly more difficult to solve than problems in which all constraints are equations since it is not known in advance which inequality constraints are active at the solution.

In this paper, we describe a new SQP method that is based on the interior-point paradigm for solving the

QP subproblems (3). To promote the global convergence, the filter technique of Fletcher and Leyffer [2] is used to globalize the SQP algorithm, avoiding the use of a merit function and the updating of the penalty parameter. The underlying concept is that trial iterates are accepted if they improve the objective function or improve the constraints violation, instead of a combination of those two measures defined by a merit function.

The paper is organized as follows. Section 2 describes the interior-point method used to solve the QP subproblems. The filter mechanism is described in Section 3 and Section 4 contains the numerical results, some conclusions and future developments.

2 The Interior-Point Framework

This section describes an infeasible primal-dual interior-point method for solving the quadratic subproblem (3). We refer to [8] for details. Adding nonnegative slack variables w, p, g, t, (3) becomes

$$\min \Delta^T H_k \Delta + \nabla F_k^T \Delta \quad (4)$$

$$\text{s.t. } \nabla h_k \Delta - w = b - h_k,\ \nabla h_k \Delta + p = b + r - h_k,$$
$$\Delta - g = l - x_k,\ \Delta + t = u - x_k,\ w, p, g, t \geq 0.$$

The nonnegativity constraints are then eliminated by incorporating them in logarithmic barrier terms in the objective function transforming (4) into

$$\min \Delta^T H_k \Delta + \nabla F_k^T \Delta - \mu \sum_{j=1}^{m} \ln(w_j) - \mu \sum_{j=1}^{m} \ln(w_j) -$$
$$-\mu \sum_{i=1}^{n} \ln(g_i) - \mu \sum_{i=1}^{n} \ln(t_i)$$

subject to the same set of equality constraints, where μ is a positive barrier parameter. Optimality conditions for this subproblem produce the standard primal-dual system

$$\begin{aligned} &H_k \Delta + \nabla F_k - \nabla h_k^T y - z + s = 0,\ \ y + q - v = 0,\\ &WVe_1 = \mu e_1, PQe_1 = \mu e_1, GZe_2 = \mu e_2,\ TSe_2 = \mu e_2,\\ &\nabla h_k \Delta + h_k - b - w = 0, r - w - p = 0,\\ &\Delta + x_k - l - g = 0, u - \Delta - x_k - t = 0, \end{aligned} \quad (5)$$

where $V = diag(v_j)$, $Q = diag(q_j)$, $Z = diag(z_i)$, $S = diag(s_i)$, $W = diag(w_j)$, $P = diag(p_j)$, $G = diag(g_i)$ and $T = diag(t_i)$ are diagonal matrices, $y = v - q$, $e_1 = (1,...,1)^T$ and $e_2 = (1,...,1)^T$ are m and n vectors respectively. This is a nonlinear system of $5n+5m$ equations in $5n+5m$ unknowns. It has a unique solution in the strict interior of an appropriate orthant in primal-dual space $\{(\Delta,w,g,t,p,y,z,v,s,q): w,g,t,p,z,v,s,q \geq 0\}$.

The central path is an arc of strictly feasible points. It is parameterized by the scalar μ, and each point on the central path solves the primal-dual system (5). As μ tends to zero, the central path converges to an optimal solution to both primal and dual problems. For a value of μ let $(\Delta,w,g,...,q)$ denote the current point in the orthant. Our aim is to find the direction vectors $(\Delta\Delta,\Delta w, ..., \Delta q)$ such that the new point $(\Delta+\Delta\Delta,w+\Delta w, ..., q+\Delta q)$ lies approximately on the primal-dual central path at the point $(\Delta_\mu, w_\mu, ..., q_\mu)$. We see that the new point $(\Delta+\Delta\Delta, w+\Delta w, ..., q+\Delta q)$, if it were to lie exactly on the central path at μ, would be defined by

$$\begin{aligned} &-H_k \Delta\Delta + \nabla h_k^T \Delta y + \Delta z - \Delta s = \sigma\\ &-\Delta y - \Delta q + \Delta v = y + q - v \equiv \beta\\ &V^{-1} W \Delta v + \Delta w = \mu V^{-1} e_1 - w - V^{-1} \Delta V \Delta w \equiv \gamma_w\\ &P^{-1} Q \Delta p + \Delta q = \mu P^{-1} e_1 - q - P^{-1} \Delta P \Delta q \equiv \gamma_q\\ &G^{-1} Z \Delta g + \Delta z = \mu G^{-1} e_2 - z - G^{-1} \Delta G \Delta z \equiv \gamma_z\\ &T^{-1} S \Delta t + \Delta s = \mu T^{-1} e_2 - s - T^{-1} \Delta T \Delta s \equiv \gamma_s\\ &\nabla h_k \Delta\Delta - \Delta w = w + b - \nabla h_k \Delta - h_k \equiv \rho\\ &\Delta w + \Delta p = r - w - p \equiv \alpha\\ &\Delta\Delta - \Delta g = l - \Delta - x_k + g \equiv \upsilon\\ &\Delta\Delta + \Delta t = u - \Delta - x_k - t \equiv \tau \end{aligned} \quad (6)$$

where we have introduced notations $\sigma \equiv H_k \Delta + \nabla F_k - \nabla h_k^T y - z + s$ and $\beta, \rho, \alpha, \tau, \upsilon$, $\gamma_w, \gamma_q, \gamma_z, \gamma_s$ as short-hands for the right-hand side expressions. This is almost a linear system for the direction vectors $(\Delta\Delta,\Delta w, ..., \Delta q)$. The only nonlinearities appear on the right-side hand of the complementarity equations (*i.e.*, in $\gamma_w, \gamma_q, \gamma_z, \gamma_s$, the γ-vectors).

The algorithm implements a *predictor-corrector* [5] approach to find a good approximation solution to the equations (6). First, a predictor direction $(\Delta\Delta^p, \Delta w^p, ..., \Delta q^p)$ is computed from (6) ignoring the μ and Δ-terms of the γ-vectors. Then an estimate of an appropriate target value for μ is made using $\mu = \bar{\delta}\left(\bar{z}^T \bar{g} + \bar{s}^T \bar{t} + \bar{v}^T \bar{w} + \bar{p}^T \bar{q}\right) / (2m + 2n)$ with $\bar{z} = z + \bar{\alpha}^p \Delta z^p$, $\bar{g} = g + \bar{\alpha}^p \Delta g^p, ..., \bar{q} = q + \bar{\alpha}^p \Delta q^p$ and $\bar{\delta} = \left(\left(\bar{\alpha}^p - 1\right) / \left(\bar{\alpha}^p + 10\right)\right)^2$ where $\bar{\alpha}^p$ is the longest step length that can be taken along this direction before violating the nonnegative conditions $w, g, t, p, z, v, s, q \geq 0$ with an upper bound of 1. The corrector step $(\Delta\Delta,\Delta w, ..., \Delta q)$ is then obtained by reinstalling the μ and the Δ-terms on the γ-vectors in (6). This step is used to move to a new point in primal-dual space. Again we calculate the maximum step $\bar{\alpha}$ that can be taken along this direction before violating the nonnegativity conditions, yielding the new point $\Delta = \Delta + \bar{\alpha}\Delta\Delta, w = w + \bar{\alpha}\Delta w, ..., q = q + \bar{\alpha}\Delta q$.

Implementation details to provide initial values for all the variables in this interior-point paradigm as well as to solve system (6) are described in [1].

A solution of the quadratic subproblem is declared primal/dual feasible if the relative measures of primal and dual infeasibilities are less than 10^{-4}. Thus, the QP subproblem has a solution (Δ_k, π_k) with $\Delta_k = \Delta$ and $\pi_k^T = (v, q, z, s)$.

3 A Line Search Filter Method in SQP

After a search direction Δ_k has been computed, we consider a backtracking line search procedure, where a decreasing sequence of step sizes $\alpha_{k,l} \in (0,1]\,(l = 0,1,...)$, with $\lim_l \alpha_{k,l} = 0$, is tried until an acceptance criterion is satisfied. The procedure that decides which trial step is accepted is a "filter method". Traditionally, a trial step size $\alpha_{k,l}$ is accepted if the corresponding trial point $x_k(\alpha_{k,l}) = x_k + \alpha_{k,l}\Delta_k$, $\lambda_k(\alpha_{k,l}) = \lambda_k + \alpha_{k,l}\xi_k$, $ss_k(\alpha_{k,l}) = ss_k + \alpha_{k,l}\zeta_k$, provides sufficient reduction of a merit function, such as the augmented Lagrangian function [3], which has the form

$$L(x, \lambda, ss; \eta) = F(x) - \lambda^T \left(\bar{h}(x) - ss\right) + \frac{\eta}{2}\bar{\theta}(x, ss)^2 \qquad (7)$$

where the infeasibility measure $\bar{\theta}(x, ss)$ is given by

$$\bar{\theta}(x, ss) = \left\|\left(\bar{h}(x) - ss\right)\right\|_2$$

and η is a positive penalty parameter. Here, ss is a vector of nonnegative slack variables that are used only in the line search procedure and at the beginning of iteration k is taken as $ss_k = \max\left(0, \bar{h}(x_k)\right)$. We treat the elements of λ as additional variables so that π is used to define a "search direction", ξ, for the multiplier estimate λ, and the line search is performed with respect to x, λ and ss. At iteration k, a vector triple $d_k^T = (\Delta_k, \xi_k, \zeta_k)$ is computed to serve as direction of search for the variables (x_k, λ_k, ss_k). The vectors Δ_k and π_k are found from the QP subproblem (3). The ξ_k is defined as $\xi_k = \lambda_k - \pi_k$ and the vector ζ_k satisfies $\nabla\bar{h}_k\Delta_k + \bar{h}_k = \zeta_k + ss_k$, from which we can see that $\zeta_k + ss_k$ is simply the residual of the inequality constraints from problem (3). In order to avoid the determination of an appropriate value of the penalty parameter η, Fletcher and Leyffer [2] proposed the concept of a "filter method" in the context of a trust region SQP algorithm. The basic idea behind this approach is to interpret the optimization problem (1) as a biobjective optimization problem with two goals: minimizing the constraints violation $\hat{\theta}(x) = \left\|\min\left(0, \bar{h}(x)\right)\right\|_2$ and minimizing the objective function $F(x)$. A certain emphasis is placed on the first measure, since a point has to be feasible in order to be an optimal solution of (1).

Following this paradigm, we propose an approach based on the two components of the augmented Lagrangian function (7):

$$\bar{L}(x, \lambda, ss) = F(x) - \lambda^T \left(\bar{h}(x) - ss\right) \qquad (8)$$

and $\bar{\theta}(x, ss)$ (or, equivalently, $\bar{\theta}(x, ss)^2$) rather than on $\hat{\theta}(x)$ and $F(x)$. (Recently, in [7] a related approach using the Lagrangian function in a filter trust region based method is proposed.) The trial point $(x_k(\alpha_{k,l}), \lambda_k(\alpha_{k,l}), ss_k(\alpha_{k,l}))$ is accepted by the filter if it improves feasibility, i.e., if $\bar{\theta}(x_k(\alpha_{k,l}), ss_k(\alpha_{k,l})) < \bar{\theta}(x_k, ss_k)$, or if it improves the Lagrangian function (8), i.e., if $\bar{L}(x_k(\alpha_{k,l}), \lambda_k(\alpha_{k,l}), ss_k(\alpha_{k,l})) < \bar{L}(x_k, \lambda_k, ss_k)$. Note that this criterion is less demanding than the enforcement of decrease in the penalty function (7) and might in general allow larger steps.

3.1 Sufficient reduction

Line search methods that use a merit function ensure sufficient progress toward the solution by enforcing an Armijo condition for the augmented Lagrangian function (7). Following this idea, we might consider the trial point $(x_k(\alpha_{k,l}), \lambda_k(\alpha_{k,l}), ss_k(\alpha_{k,l}))$ during the backtracking line search to be acceptable, if the next iterate provides at least as much progress in one of the measures $\bar{\theta}$ or $\bar{L}$ that corresponds to a small fraction of the current constraints violation, $\bar{\theta}(x_k, ss_k)$, i.e, if

$$\bar{\theta}(x_k(\alpha_{k,l}), ss_k(\alpha_{k,l})) \leq (1 - \gamma_{\bar{\theta}})\bar{\theta}(x_k, ss_k), \quad \text{or} \qquad (9a)$$

$$\bar{L}(x_k(\alpha_{k,l}), \lambda_k(\alpha_{k,l}), ss_k(\alpha_{k,l})) \leq \bar{L}(x_k, \lambda_k, ss_k) - \gamma_{\bar{L}}\bar{\theta}(x_k, ss_k) \qquad (9b)$$

holds for fixed constants $\gamma_{\bar{\theta}}, \gamma_{\bar{L}} \in (0,1)$. However, we change to a different sufficient reduction criterion whenever, for the current iterate, we have

$\bar{\theta}(x_k, ss_k) \le \bar{\theta}^{\min}$, for some $\bar{\theta}^{\min} \in (0,\infty]$, and the following "switching conditions"

$$\nabla\bar{L}(x_k,\lambda_k,ss_k)^T d_k < 0 \quad \text{and}$$

$$\alpha_{k,l}\left[-\nabla\bar{L}(x_k,\lambda_k,ss_k)^T d_k\right]^{s_{\bar{L}}} > \delta\left[\bar{\theta}(x_k,ss_k)\right]^{s_{\bar{\theta}}}, \quad (10)$$

hold with fixed constants $\delta > 0,\ s_{\bar{\theta}} > 1,\ s_{\bar{L}} > 2s_{\bar{\theta}}$. If $\bar{\theta}(x_k, ss_k) \le \bar{\theta}^{\min}$ and (10) is true for the current iterate, the trial point $(x_k(\alpha_{k,l}),\lambda_k(\alpha_{k,l}),ss_k(\alpha_{k,l}))$, has to satisfy the Armijo condition

$$\begin{aligned}\bar{L}(x_k(\alpha_{k,l}),\lambda_k(\alpha_{k,l}),ss_k(\alpha_{k,l})) &\le \bar{L}(x_k,\lambda_k,ss_k) + \\ &+\eta_{\bar{L}}\alpha_{k,l}\nabla\bar{L}(x_k,\lambda_k,ss_k)^T d_k,\end{aligned} \quad (11)$$

instead of (9), in order to be acceptable. Here, $\eta_{\bar{L}} \in (0, 0.5)$ is a constant. In accordance with the previous publications on filter methods we may call a trial step size $\alpha_{k,l}$ for which (10) holds, a "$\bar{L}$-step size". Similarly, if an $\bar{L}$-step size is accepted as the final step size α_k in iteration *k*, we refer to *k* as an "$\bar{L}$-type iteration". At each iteration *k*, the algorithm also maintains a "filter", here denoted by $F_k \subseteq \left\{(\bar{\theta},\bar{L}) \in R^2 : \bar{\theta} \ge 0\right\}$. Following the ideas in [9, 10, 11], the filter here is not defined by a list but as a set F_k that contains those combinations of constraints violation values $\bar{\theta}$ and Lagrangian function values $\bar{L}$, that are "prohibited" for a successful trial point in iteration *k*. So, during the line search, a trial point $(x_k(\alpha_{k,l}),\lambda_k(\alpha_{k,l}),ss_k(\alpha_{k,l}))$ is rejected, if $\left(\bar{\theta}(x_k(\alpha_{k,l}),ss_k(\alpha_{k,l})),\bar{L}(x_k(\alpha_{k,l})\lambda_k(\alpha_{k,l}),ss_k(\alpha_{k,l}))\right) \in F_k$. The authors in [9, 10, 11] apply this simplified notation to active set SQP and barrier interior-point line search based algorithms. At the beginning of the optimization, the filter is initialized to

$$F_0 = \left\{(\bar{\theta},\bar{L}) \in R^2 : \bar{\theta} \ge \bar{\theta}^{\max}\right\} \quad (12)$$

for some $\bar{\theta}^{\max}$, so that the algorithm will never allow trial points to be accepted that have a constraint violation larger than $\bar{\theta}^{\max}$. Later, the filter is augmented, using the update formula

$$F_{k+1} = F_k \cup \left\{(\bar{\theta},\bar{L}) \in R^2 : \bar{\theta} \ge (1-\gamma_{\bar{\theta}})\bar{\theta}_k \text{ and } \bar{L} \ge \bar{L}_k - \gamma_{\bar{L}}\bar{\theta}_k\right\} \quad (13)$$

after every iteration in which the accepted trial step size does not satisfy the switching conditions (10). This ensures that the iterates cannot return to the neighborhood of x_k. On the other hand, if both (10) and (11) hold for the accepted step size, the filter remains unchanged.

Overall, this procedure ensures that the algorithm cannot cycle, for example between two points that alternatively decrease the constraints violation and the Lagrangian function $\bar{L}$. Finally, in some cases it is not possible to find a trial step size $\alpha_{k,l}$ that satisfies the above criteria. We define a minimum desired step size using linear models of the involved functions

$$\alpha_k^{\min} := \gamma_\alpha \begin{cases} \min\left\{\gamma_{\bar{\theta}}, \dfrac{\gamma_{\bar{L}}\bar{\theta}_k}{-\nabla\bar{L}_k^T d_k}, \dfrac{\delta\left[\bar{\theta}_k\right]^{s_{\bar{\theta}}}}{\left[-\nabla\bar{L}_k^T d_k\right]^{s_{\bar{L}}}}\right\} \\ \qquad \text{if } \nabla\bar{L}_k^T d_k < 0 \text{ and } \bar{\theta}_k \le \bar{\theta}^{\min} \\ \min\left\{\gamma_{\bar{\theta}}, \dfrac{\gamma_{\bar{L}}\bar{\theta}_k}{-\nabla\bar{L}_k^T d_k}\right\} \\ \qquad \text{if } \nabla\bar{L}_k^T d_k < 0 \text{ and } \bar{\theta}_k > \bar{\theta}^{\min} \\ \gamma_{\bar{\theta}}, \quad \text{otherwise} \end{cases} \quad (14)$$

with a safety factor $\gamma_\alpha \in (0,1]$. If the backtracking line search finds a trial step size $\alpha_{k,l} < \alpha_k^{\min}$, the algorithm reverts to a "feasibility restoration phase". Here, the algorithm tries to find a new iterate $(x_{k+1}, \lambda_{k+1}, ss_{k+1})$ which is acceptable to the current filter and for which (9) holds, by reducing the constraints violation within an iterative process.

Our interior-point SQP filter line search algorithm for solving inequality constrained optimization problems is as follows:

Algorithm 1

Given: Starting point (x_0,λ_0,ss_0) with $ss_0 = \max\left(0,\bar{h}(x_0)\right)$; constants $\bar{\theta}^{\max} \in \left(\bar{\theta}(x_0,ss_0),\infty\right]$, $\bar{\theta}^{\min} > 0$; $\gamma_{\bar{\theta}},\gamma_{\bar{L}} \in (0,1)$; $\delta > 0$; $\gamma_\alpha \in (0,1]$; $s_{\bar{\theta}} > 1$; $s_{\bar{L}} > 2s_{\bar{\theta}}$; $\eta_{\bar{L}},\eta_{\bar{\theta}_2} \in (0,1)$

1. *Initialize*. Initialize the filter (using (12)) and the iteration counter $k \leftarrow 0$.
2. *Check convergence.* Stop if x_k is a stationary point of the problem (1), i.e., if it satisfies the KT conditions (2) for some $\lambda \in R^m$.
3. *Compute search direction.* Compute the search direction Δ_k and the Lagrange multiplier π_k from the linear system (6) (using the interior-point strategy presented in Section 2).
4. *Backtracking line search.*

4.1 *Initialize line search.* Set $ss_k = \max\left(0,\bar{h}(x_k)\right)$,

$\xi_k = \lambda_k - \pi_k$, $\zeta_k = \nabla \bar{h}_k \Delta_k + \bar{h}_k - ss_k$, $\alpha_{k,l} = 1$, $l \leftarrow 0$.

4.2 *Compute new trial point.* If the trial step size becomes too small, i.e., $\alpha_{k,l} < \alpha_k^{\min}$ with $\alpha_k^{\min}$ defined by (14), go to feasibility restoration phase in step 8. Otherwise, compute the trial points $x_k(\alpha_{k,l}) = x_k + \alpha_{k,l}\Delta_k$, $\lambda_k(\alpha_{k,l}) = \lambda_k + \alpha_{k,l}\xi_k$, $ss_k(\alpha_{k,l}) = ss_k + \alpha_{k,l}\zeta_k$.

4.3 *Check acceptability to the filter.* If $\left(\bar{\theta}(x_k(\alpha_{k,l}), ss_k(\alpha_{k,l})), \bar{L}(x_k(\alpha_{k,l})\lambda_k(\alpha_{k,l}), ss_k(\alpha_{k,l}))\right) \in F_k$, reject the trial step size and go to step 4.5.

4.4 *Check sufficient decrease with respect to current iterate.*

Case I: $\alpha_{k,l}$ *is an* $\bar{L}$*-step size (i.e., (10) holds):* If the Armijo condition (11) for the $\bar{L}$ function holds, accept the trial step and go to step 5. Otherwise, go to step 4.5.

Case II: $\alpha_{k,l}$ *is not an* $\bar{L}$*-step size:* If (9) holds, accept the trial step and go to step 5. Otherwise, go to step 4.5.

4.5. *Choose new trial step size.* Set $\alpha_{k,l+1} \leftarrow \alpha_{k,l}/2$, $l \leftarrow l+1$, and go back to step 4.2.

5. *Accept trial point.* Set $\alpha_k \leftarrow \alpha_{k,l}$, $x_{k+1} \leftarrow x_k(\alpha_k)$, and $\lambda_{k+1} \leftarrow \lambda_k(\alpha_k)$.

6. *Augment filter if necessary* . If *k* is not an $\bar{L}$-type iteration, augment the filter using (13). Otherwise, leave the filter unchanged.

7. *Continue with next iteration*. Increase the iteration counter $k \leftarrow k+1$ and go back to step 2.

8. *Feasibility restoration phase*. Use a restoration algorithm to produce a point $(x_{k+1}, \lambda_{k+1}, ss_{k+1})$ that is acceptable to the filter, i.e., $(\bar{\theta}(x_{k+1}, ss_{k+1}), \bar{L}(x_{k+1}, \lambda_{k+1}, ss_{k+1})) \notin F_k$. Augment the filter using (13) and continue with the regular iteration in step 7.

3.2 Feasibility restoration phase

In this section we present a restoration algorithm. The task of the restoration phase is to compute a new iterate acceptable to the filter by decreasing the infeasibility, whenever the regular backtracking line search procedure cannot make sufficient progress and the step size becomes too small. To compute a trial point that sufficiently decreases infeasibility, we introduce the function $\bar{\theta}_2(x, ss) = \frac{1}{2}\left\|\left(\bar{h}(x) - ss\right)\right\|_2^2$. The restoration algorithm herein presented works with the step framework $\bar{d}^T = (\Delta, \zeta)$ that should be a descent direction for $\bar{\theta}_2(x, ss)$. In fact,

$$\bar{d}^T \nabla \bar{\theta}_2 = (\bar{h} - ss)^T \nabla \bar{h}\Delta - (\bar{h} - ss)^T \zeta = (\bar{h} - ss)^T (\nabla \bar{h}\Delta - \zeta)$$
$$= -(\bar{h} - ss)^T (\bar{h} - ss) = -2\bar{\theta}_2 < 0.$$

Additionally, we also ensure that the new iterate x_{k+1} does not deviate too much from the current iterate x_k (see step 5 in Algorithm 2).

Several other restoration algorithms are plausible but we chose the following one because it is consistent with the step calculation of our interior-point SQP filter line search method:

Algorithm 2 (restoration algorithm)

0. Set $x_{k,0} = x_k$, $\lambda_{k,0} = \lambda_k$, $ss_{k,0} = ss_k$, $l = 0$ and start with step 4.

1. If $(x_{k,l}, \lambda_{k,l}, ss_{k,l})$ is acceptable to the filter (conditions (9)) then set $x_{k+1} = x_{k,l}$, $\lambda_{k+1} = \lambda_{k,l}$, and stop restoration.

2. Compute $\Delta_{k,l}$ and $\pi_{k,l}$ by solving the QP subproblem (3), with $(x_k, \lambda_k) = (x_{k,l}, \lambda_{k,l})$.

3. Compute $ss_{k,l}$, $\xi_{k,l}$, $\zeta_{k,l}$ and define the vector $(\bar{d}_{k,l})^T = (\Delta_{k,l}, \zeta_{k,l})$ which is used as direction of search for the variables $(x_{k,l}, ss_{k,l})$.

4. Set $\alpha_k = 1$.

5. If
$\bar{\theta}_2(x_{k,l}(\alpha_k), ss_{k,l}(\alpha_k)) \le \bar{\theta}_2(x_{k,l}, ss_{k,l}) + \alpha_k \eta_{\bar{\theta}_2} \nabla \bar{\theta}_2^T \bar{d}_{k,l}$,
and $\|x_{k,l}(\alpha_k) - x_k\| \le \varepsilon_{\bar{\theta}_2}(1 + \|x_{k,l}(\alpha_k)\|)$
then set $x_{k,l+1} = x_{k,l}(\alpha_k)$, $ss_{k,l+1} = ss_{k,l}(\alpha_k)$, $\lambda_{k,l+1} = \lambda_{k,l}(\alpha_k)$, $l = l+1$, and return to step 1. Otherwise $\alpha_k \leftarrow \alpha_k/2$, and repeat step 5.

4 Results and Conclusions

To test this SQP framework based on the interior-point strategy with a filter line search method we selected 32 small inequality constrained problems from the Hock and Schittkowski (HS) collection [4]. The tests were done in double precision arithmetic with a Pentium 4 and Fortran 90. For the successful termination of the algorithm, the iterative sequence of *x*-values must converge and the final point must satisfy the first-order Kuhn-Tucker conditions (see (2)) with a 10^{-4} tolerance. The chosen values for the constants are similar to the ones used in [11]: $\bar{\theta}^{\max} = 10^4 \max\{1, \bar{\theta}(x_0, ss_0)\}$, $\bar{\theta}^{\min} = 10^{-4} \max\{1, \bar{\theta}(x_0, ss_0)\}$, $\gamma_{\bar{\theta}} = 10^{-5}$, $\gamma_{\bar{L}} = 10^{-5}$, $\delta = 1$, $\gamma_\alpha = 0.05$, $s_{\bar{\theta}} = 1.1$, $s_{\bar{L}} = 2.3$, $\eta_{\bar{L}} = 10^{-4}$, $\eta_{\bar{\theta}_2} = 10^{-4}$ and $\varepsilon_{\bar{\theta}_2} = 0.1$, where x_0 is the starting point and $ss_0 = \max(0, \bar{h}(x_0))$.

Our numerical results are reported in the first part of the Table 1. Each set of two columns contains the number of QP subproblems solved (N_{QP}) and the number of function evaluations (N_{fe}). For comparative purposes, we include in the second part of the table the results obtained when a line search method based on the merit function (7) is used as in [1] instead of the herein proposed filter method. In most problems the results are similar while in 9 of the 32 problems the filter method requires less function evaluations than the merit function based line search. Slightly better results were obtained with the merit function in 3 problems.

Table 1: Comparative results

Problem	*Filter Method*		*Merit Function*		*SNOPT*
	N_{QP}	N_{fe}	N_{QP}	N_{fe}	N_{QP}
HS1	44	59	44	59	33
HS2	13	16	13	16	18
HS3	1	2	1	2	2
HS4	2	3	2	10	2
HS5	9	31	9	31	10
HS15	8	9	10	61	8
HS16	5	6	5	6	1
HS17	13	15	12	15	12
HS18	9	43	6	67	15
HS19	7	8	7	8	2
HS20	8	9	7	9	1
HS21	2	13	2	13	1
HS23	6	7	6	7	2
HS24	6	7	6	7	6
HS30	11	12	11	12	5
HS31	7	8	7	68	15
HS32	5	6	5	6	4
HS33	5	6	5	6	1
HS34	7	17	8	31	5
HS35	2	3	2	3	16
HS36	6	7	6	29	3
HS37	9	10	9	10	12
HS38	13	14	13	14	160
HS41	12	13	12	13	11
HS44	6	7	6	16	2
HS45	5	6	5	6	8
HS53	3	4	3	4	8
HS55	1	2	1	2	3
HS60	8	9	12	52	12
HS63	12	15	9	12	30
HS64	19	173	19	173	33
HS65	9	10	10	54	22

We also include the results obtained by the solver SNOPT, a specific SQP implementation of an active-set method based on a smooth augmented Lagrangian merit function, which is available in the NEOS Server (http://www-neos.mcs.anl.gov/neos/). The comparison with SNOPT is not meant to be a rigorous assessment of the performance of our algorithm since the termination criteria are not comparable. However, the numerical results show that our interior-point SQP filter line search method is effective on small dimensional problems. Large scaling problems testing will follow in the near future. Despite some similarities our method has basically two differences from the filter line search SQP method proposed and analyzed in [9, 10]: definition (14) and the feasibility restoration phase. Future developments will focus on the global convergence analysis of the proposed method.

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