# Optimum LCC of the k-out-of-n Parallel Redundancy System

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*Abstract:* In process industries, large power plants and high risk systems, the use of a number of parallel and identical units to obtain a given system reliability is a standard design practice. In many instances, failure of some of the parallel units, though leading to a reduction of output capacity of the system, it does not lead to a complete failure. Thus if (n) parallel units are used in a system, at least (k) units should be operative at any time for proper functioning. Economic reliability of the k-out-of-n parallel redundancy system for which the lifecycle cost is a minimum, is the subject matter of this work, and involves estimating how production and maintenance costs will interact for a given reliability.

In the k-out-of-n redundancy system, the same overall (system) reliability can be obtained by using different items in different combinations, with different levels of redundancy (different system configurations).

An economic analysis of the k-out-of-n redundancy system configuration is made. The overall system initial cost is the aggregate of the similar items cost. The minimum cost system configuration, or alternatively, total expected profit will be maximum if the system reliability is chosen according to a formula based on computed lifetime of equipment, actual service life of equipment till it fails to function properly, revenue from running the equipment and salvage value of equipment. This results in selecting a minimum cost configuration satisfying a specified redundancy level. Hence, the reliability allocation, as given by a proposed expression gives an indication of the economic lifecycle cost of the system when used in a specific situation. This can be interpreted that according to the type of application, the economic use of the system may be justified by a shorter or longer lifecycle than that implied by the desired system reliability

*Keywords:* reliability, k-out-of-n redundancy, hazard, economics, process industries, lifecycle cost

## Nomenclature

R = reliability
F = failure probability
K = number of items in a system
N = number of operating items
p(t) = probability that an item will fail at time (t)
t = computed lifetime of equipment
b = actual service life of equipment till it fails to function properly
f(t) = revenue from running the equipment a period (t)
s(t - b) = salvage value of equipment, a function of the time a piece of equipment runs after its computed service life
r(b) = running cost of machine or equipment, till it fails to run satisfactorily, expressed in terms of the service life
m(t - b) = penalty cost, or cost incurred by repair, corrective maintenance,..etc, to reclaim the equipment into a good working condition

## 1 Introduction

Determination of an optimal or a near optimum system design is very important to economically produce systems which meet customers' expectations for both reliability and performance. Engineering specifications prescribe minimum acceptable levels levels of reliability. The redundancy allocation problem involves the simultaneous evaluation and selection of components and a system-level design configuration, which can meet all design constraints, and at the same time, optimize some objective function of system cost and reliability.

The redundancy allocation problem has been previously analyzed using dynamic programming (DP), integer programming (IP), mixed integer and nonlinear programming and genetic algorithms (GA) as new approach, ref. [1] through [8].

In this paper, an economic analysis of the k-out-of-n parallel redundancy system of similar active components is made, which results in selecting a minimum cost configuration satisfying a specified redundancy level.

## 2 K-out-of-n Parallel Redundancy

A generalization of the expression for the reliability of a system of n parallel and similar components occurs when a requirement exists for at least k out of n identical and independent components to function for the system to function. Obviously k ≤ n. If k = 1, complete redundancy occurs, and if k = n, the n components are, in effect, in series. A special case is the consecutive-k-out-of-n:F System. It is a series system consisting of n components, in which the failure of one or more components results in system failure. The system is not considered failed until at least k components have failed; those k components must be consecutively ordered within the system. Such systems are known as consecutive-k-our-of-n:F systems.

The reliability of the k-out-of-n parallel redundancy at different system states is obtained from the binomial probability distribution

$$R(k;n) = \sum_{r=k}^{n} \binom{n}{r} R^{r} . F^{n-r} \qquad (1)$$

where, $\binom{n}{r} = \dfrac{n!}{r!.(n-r)!}$

along with, $R + F = 1$

Figure 1 presents a typical parallel system with k-out-of-n reliabilities. For the system, a minimum of k components must be chosen, from among n available choices. Additionally, there is the option of adding more components to improve the system reliability as an alternative to using a more reliable, and more costly, component.

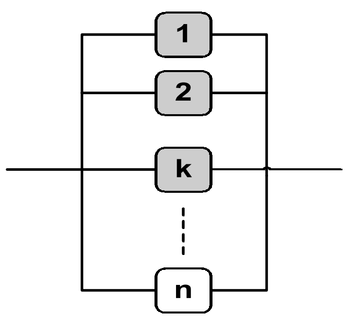


Fig.1 General k-out-of-n active redundancy

Considering the cases of the 3 and 4 components system, reliability at some system configurations are assessed in what follows.

The 3-elements system reliabilities at the mutually exclusive and exhaustive states are given by the expansion;

$(R+F)^{3} = R^{3} + 3\,R^{2}\,F + 3\,R\,F^{2} + F^{3}$

From which, the 2-out-of-3 system reliability is given by;

$R_{syatem} = R^{3} + 3\,R^{2}\,F$

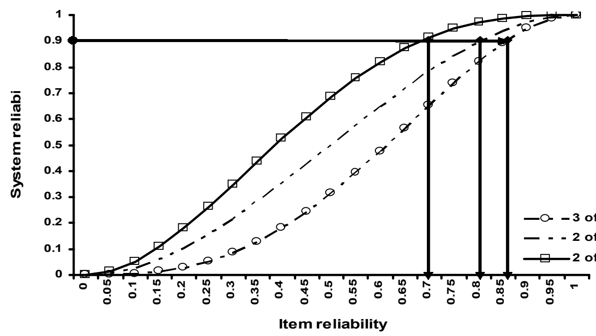


Fig.2 Relationship between item reliability and total system reliability

The 4-elements system reliabilities at the mutually exclusive and exhaustive states are given by the expansion;

$(R+F)^{4} = R^{4} + 4\,R^{3}\,F + 6\,R^{2}\,F^{2} + 4\,R\,F^{3} + F^{4}$

Hence, 2-out-of-4 system reliability is given by:

$R_{system} = R^{4} + 4\,R^{3}\,F + 6\,R^{2}\,F^{2}$

And the 3-out-of-4 system reliability is given by:

$R_{system} = R^{4} + 4\,R^{3}\,F$

The relationships given by the above expressions are shown in figure 2. It is clear that the same overall (system) reliability can be obtained by using different items in different combinations, with different levels of redundancy. The overall system initial cost is the aggregate of the similar items cost.

## 3 System Economics

One of the important aspects of any system is its cost. Orders are normally placed with the supplier who provides equipment or machines to the required specification at minimum initial cost. This approach takes into consideration only the initial cost and neglects the expense of keeping the equipment working satisfactorily once it has been purchased. How much maintenance and repair will cost depends on the reliability of the equipment Hence life cycle cost is the focal point of view.

Three separate cost factors are involved, viz the cost of design, the cost of production, and the cost of repair and maintenance. As the reliability of equipment increases, so will the cost of design and production increase, whereas the cost of repair and maintenance will go down. Design becomes more expensive because more precise assessments of the exact working conditions must be made, followed by more detailed development.

On the production side higher reliability means better quality and therefore more expensive parts. It may be necessary to use costlier materials, to work to finer limits, and to provide additional and more elaborate test and inspection facilities. Usually more skilled and, therefore, more highly paid assemblers are employed.

To make equipment more reliable is bound to increase its initial cost. This increase can be offset by economies in maintenance and repair costs. When equipment fails there is a loss of production or of service, which involves some form of financial loss. If the results of a failure are likely to be serious, then it may be necessary to provide spare equipments as replacements. Clearly the lower the reliability the greater will be the number of equipments or machines which are out of action at any given time, and therefore the higher the number of replacements which must be provided. The need to maintain a specified service level necessitate the provision of standby equipment or systems.

Besides the financial loss caused by an equipment failure, there is the cost of repair, which is more than the cost of the work and material involved, as it takes into account the expense of training the necessary skilled men, the cost of test and repair apparatus and installations, and the cost of spares. The less reliable an equipment is, the more repair work is needed, and the greater the number of men and the quantity of test apparatus and of spares which must be available. Quite apart from the cost of producing spares, they perform no useful function until they are used.

Maintenance costs may not only mean that it costs more to keep an item in working order,they may also add to the initial cost as provisions should be included in the selling price to cover the average amount of repair work which is estimated as necessary while the item is under warranty. The less reliable the equipment, the larger this amount will be. The manufacturer may also be affected in another way. The rate at which equipment can be produced may be lessened by the need to divert parts, which could be incorporated in complete equipments, for use as spares.

More often a user is interested only in the initial cost of equipment, instead of its total cost, namely how much it will cost him to buy the equipment and to keep it functioning normally throughout its working life. Paying more initially, to obtain more reliable equipment, maintenance costs can be lowered and the total cost reduced, however production and design costs rise as reliability increases, while maintenance and repair costs fall. These three added together represent the total cost equipment.

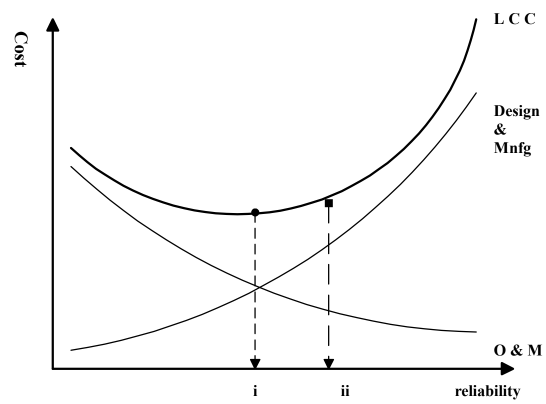


Fig.3 Functional form of the system LCC
i minimum system cost
ii constrained minimum system cost

Referring to figure 3, the system LCC is the sum of the elements outlined before. System reliability at the minimum LCC (point I on the curve) may be lower than the specified one (point ii on the curve) in which case desired reliability level is adhered to on the sake of cost. On the other hand, specified reliability may be lower than system reliability at the minimum LCC in which case the latter should be adhered to, and realized by procuring the dearer equipment (initial capital cost) while reducing running costs.

Optimum economic reliability, which is the system reliability for which the total cost is a minimum, is the subject matter of this work, and involves estimating how production and maintenance costs will vary for a given reliability.

## 4 Optimal System Configuration

For a system which comes into use in a specific application for the first time, or even if this system has been in use for more or less a long time, there arises the problem of lifetime at which a system is expected to operate satisfactorily and which is to be included in all future computations and applications. Even if the system reliability function is known beforehand, it is upon the economic considerations that the lifetime can be decided. Therefore the same system is introduced with differing life times in differing applications, depending upon the economy governing the problem in question.

The same system reliability can be obtained with different system configurations. Figure 4 shows the functional relationship between item reliability, system reliability and initial system cost, based on a monotone increasing convex cost function.

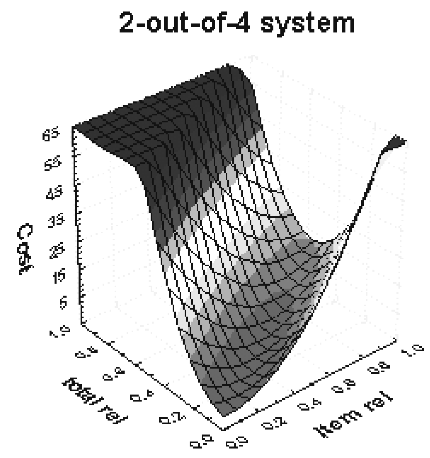


Fig.4 Functional form of the item reliability, total reliabilities and system cost relationship

Further, reliability of the system is directly related to the time the system is in use. It deteriorates by the passage of time and hence hazard increases. Therefore, overall desired system reliability, and hence hazard, is a techno-economic subject which requires a further investigation. In this case it is economically justifiable to allow the system to operate up to a certain level of hazard (with the corresponding reliability) so that the overall system cost is optimum.

Based upon an economic analysis of system , a formula is worked out to give the economic lifetime of a given system.

## 4. 1. Cost and revenue

The outcome of, or revenue from running the system for a certain time is given by f(t). The outcome resulting from running the system depends upon the time it is running until it is salvaged. The functional for of the relation can be expressed explicitly in terms of the lifetime C(t) as a monotone increasing convex, concave, or general function. Without losing generality, a straight line outcome-life relationship is rather a special case in which the effect of a declining ability to yield production is tolerated.

As a system fails to operate satisfactorily, it can be either used for less intricate tasks, or disposed of totally (salvaged), in which case, it yields a value depending upon its actual service life, and given as s(t - b). As the actual service life of the system exceeds the computed one, the salvage or scrap value can be introduced as a decreasing function of the running time after the computed service life.

The actual running costs of a system are expected to increase as the running time increases, but to be more than proportionate to the life time, due to overhauling and repair, mis-runs, run- outs... etc., and given by r(b). It is a function of the computed (theoretical) lifetime upon which depreciation of item is based and computed.

Penalty cost is the cost incurred by repair or overhauling a system to restore its normal working capacity when it fails after a service life less than the computed one and given by m(t - b). In many instances, this may not be a physical cost, but rather the cost of losing goodwill or even the loss of a customer.

## 4. 2. The mathematical model

If the reliability function is given, and the probability that an item will fail at time (t) – hazard - is p (t), and the feasible life – given by the mean time between failures MTBF - is ($t_o$), then for a life (b), where;

$$t_o \geq b \geq 0$$

is given, the following relations hold;

$$\int_0^{t_o} p(t)\,dt = 1 \qquad \text{and,}$$

$$\int_0^{b} p(t)\,dt + \int_b^{t_o} p(t)\,dt = 1 \qquad (2)$$

Using the same reasoning and terminology used before, the expected profit when cost and revenue parameters are computed for a service life (t) while an item fails after a life (b);

$$\int_b^{t_o} p(t)dt = 1 - \int_o^{b} p(t)dt \qquad b < t$$

$$\int_{b}^{t_o} [f(b).b - r(b).b)]\, p(t)\, dt \qquad t_o \geq b > t$$

but; $\int_{b}^{t_o} p(t)\, dt = 1 - \int_{o}^{b} p(t)\, dt$

and, p (0) = 0, from which it follows that;

$$\int_{b}^{t_o} [f(b).b - r(b).b]\; p(t)\, dt =$$

$$[f)b).b - r(b).b].[1 - \int_{0}^{b} p(t)\, dt \qquad (3)$$

hence, total expected profit is given by;

$$\int_{0}^{b} [f(t).t + s(t - b) - r(t).t - m(t - b)]\, p(t)dt +$$

$$[f(b).b - r(b).b].[1 - \int_{0}^{b} p(t)dt)]$$

To find the maximum profit, when the computed life of the system is (t), while the actual service life is (b), and the maximum possible service life is ($t_o$) - the expression for the total expected profit is differentiated with respect to (t) yielding;

$$\left[f(b).b.p(b)\right]_0^b + \left[s(o).p(b) \quad - s(b).p(o)\right] -$$

$$\left[\frac{d}{db} r(b).b \int_0^b p(t)\, dt + [r(b).b.p(\quad b)]_0^b\right] -$$

$$\left[m(0).p(b) \quad - m(b).p(0)\right] -$$

$$\left[f(b).b.p(b \quad) - r(b).b.p(b \quad)\right]$$

$$+ \left[\frac{d}{db} f(b).b \quad - \frac{d}{db} r(b).b\right] .$$

$$\left[1 - \int_0^b p(t)\, dt\right] = \text{zero}$$

by definition; s (0) = 0 , m(0)=0 , p(0) = 0.

The above expression is reduced to:

$$\frac{d}{db}[f(b).b - r(b).b] - \frac{d}{db} f(b).b . \int_0^b p(t)\, dt = \text{zero}$$

It follows that:

$$\int_0^b p(t)\, dt = 1 - \{[\frac{d}{db}[r(b).b]]/[\frac{d}{db}[f(b).b]]\} \qquad (4)$$

In this way, total expected profit will be maximum if the hazard (b) is chosen such that the integral will assume the value as given by the right hand side of the equality. Hence, the minimum-cost hazard, as given by the above expression gives an indication of the economic life of the system when used in a specific situation. This can be interpreted that the economic use of the system may be justified by a shorter or longer life than that implied by the desired system reliability.

## 5 Conclusion

In process industries, large power plants and high risk systems , a number of parallel units are used to obtain a given system reliability. The same overall (system) reliability of k-out-of-n redundancy system, can be obtained by different system configurations.

The overall system cost of the k-out-of-n redundancy system is the aggregate of the identical items cost. The minimum cost system configuration will be maximum if the system reliability is chosen according to a formula based on computed lifetime of equipment, actual service life of equipment till it fails to function properly, revenue from running the equipment and salvage value of equipment. This results in selecting a minimum cost configuration satisfying a specified redundancy level. This can be interpreted that the economic use of the system may be justified by a shorter or longer life than that implied by the desired system reliability.

Solution steps of the above equality depends upon the functional form of the different elements and hence can vary widely. Special cases can be covered by elaborative computational techniquea as in [9] and [10].

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