Randomized Algorithm for Arrival and Departure of the Ships in a Simple Port

ALI DELAVARKHALAFI
Mathematics department
Shahrkord University, P.O. Box 115
Shahrkord, IRAN

Abstract: - In this paper we consider a simple port as multi server queuing system and determine optimum-group servers by a stochastic algorithm on the different types of servers, which minimize the demurrage cost in addition to service cost. Finally, it has been used for the ship arrival and departure from a port together with suitable computer programming.

Key-Words: - Queuing Theory, Quasi Birth and Death Process, Rate Matrix, Demurrage Reduction

1 Introduction

The multi server queuing system in which customers require a random number of identical servers has been studied originally by L. Green [4]. Gillent and Latouche presented an explicit solution for finding the rate matrix R associated with the model, and hence, simplified the computation of the steady state probability vector considerably [3]. Leeuwaarden and Winands determine the equilibrium distribution for a class of quasi birth and death (QBD) processes using the matrix geometric method, which requires the determination of the rate matrix R. In contrast to most QBD processes, the class under consideration allows for an explicit description of R, by exploiting its probabilistic interpretation [6]. Altiok propose bounds and an approximation for a modified port time which is a significant measure of performance in bulk-material port operations [1].

In this paper, our main focus is on the different types of servers, which minimize the demurrage cost in addition to service cost, a subject that was not considered in the aforementioned references.

2 The Model

We consider a simple port in which arrival of the ships are Poisson with rate λ. The system consists of s independent and identical servers. The service time of individual servers are exponentially distributed with mean μ. A ship requires simultaneous service from i servers with probability c_i, $1 \le i \le s$ depending on its tonnage, then service time of every ship is not exponentially distributed but it is distributed as

$$B_s(t) = \sum_{i=1}^{s} (1 - e^{-\mu t})^i c_i .$$

Supposing that arrival of the ships are Poisson with rate λ, the following lemma shows that i-type ships are also Poisson.

Lemma 1. *We consider a port system in which arrival of ships are Poisson with rate λ. If i-type ships with tonnage T_i arrive to port with probability p_i, then i-type ships are Poisson with rate $p_i\lambda$.*

Prove*: We define $N_i(t), i=1,\ldots,m$ as the number of i-type ships which arrive to port, with probability p_i and $N(t)= \sum_{i=1}^{m} N_i(t)$. Using total probability and because of $N(t)$ is Poisson with rate λ and $p(N_i(t)=n|N(t)=m)$ is a binomial distribution, so*

$P(N_i(t)=n)=$

$$\sum_{m=n}^{\infty} \frac{p_i^n (1-p_i)^{m-n} e^{-\lambda t} (\lambda t)^m}{n!(m-n)!} = \frac{(\lambda p_i t)^n e^{-\lambda p_i t}}{n!}$$ □

2.1 QBD processes and matrix-geometric solution

We present the model formulated as a QBD process. We express the state of the system at an epoch by a double *(i, j),* where *i* denotes the number of ships in queue and *j* denotes the number of busy *l*-type servers, where $1 \le l \le k$.

We order the states lexicographically, i.e. {(0, 0),…, (0,s), (1, 0),…, (1,s),…, (n, 0),…, (n,s),…} and assume that the infinitesimal generator Q has the following block tridiagonal structure:

$$Q = \begin{bmatrix} B_1 & B_0 & . & ... \\ B_2 & A_1 & A_0 & ... \\ . & A_2 & A_1 & ... \\ : & : & : & : \end{bmatrix}$$

where A_0, A_1 and A_2 are square matrices of order *s*. The matrices A_0, A_2, B_0 and B_2 are nonnegative and the matrices B_1 and A_1 have nonnegative off_-diagonal elements and strictly negative diagonals. We denote the diagonal elements of B_1 and A_1 by Δ, which are such that the row sums of Q equal zero [3].

The QBD process driven by Q is ergodic if and only if it satisfies the mean drift condition

$\omega A_0 e < \omega A_2 e$ (1)

where $\omega = (\omega_{0,\dots,}\omega_s)$ is the equilibrium distribution of the generator $A_0 + A_1 + A_2$ and e the unity vector. When (1) is satisfied, the stationary distribution of the QBD process exists. Denoting by $\pi(i, j)$ the stationary probability of the process being in state (i, j), and using the vector notation$\pi n = (\pi(n; 0),\dots, \pi(n;s))$, the balance equations of the QBD process are given by

$\pi_{n-1}A_0 + \pi_n A_1 + \pi_{n+1}A_2 = 0, \quad n \geq 2$ (2)

and

$\pi_0 B_1 + \pi_1 B_2 = 0,$ (3)

$\pi_0 B_0 + \pi_1 A_1 + \pi_2 A_2 = 0,$ (4)

Introducing the rate matrix R as the minimal nonnegative solution of the nonlinear matrix equation

$A_0 + RA_1 + R^2 A_2 = 0,$ (5)

it can be proved that the equilibrium probabilities satisfy

$\pi_{n+1} = \pi_n R \quad n \geq 1,$ (6)

The vectors π_0 and π_1 follow from the boundary conditions (3-4) and the normalization condition

$$\sum_{i=0}^{\infty}\sum_{j=0}^{s}\pi(i,j) = \pi_0 e + \pi_1 (I-R)^{-1} e = 1 \qquad (7)$$

Where *I* represents the identity matrix. In order to obtain the stationary distribution, one should thus determine the rate matrix R. Several iterative procedures exist for solving (5). For example, we can use the following scheme

$R^{(k+1)} = -(A_0 + R^{(k)2}A_2)\, A_1^{-1}$,k = 0, 1,… (9)

starting with $R^{(0)}$ a matrix of zero-entries only. Now, the following result holds [6]:

Theorem 1. Assume (1) is satisfied and A2 $= \nu . e_s$, where ν is a column vector and e_s a row vector normalized by $e_s . e = 1$. Then

$R = -\lambda (A_1 + \lambda e.e_s)^{-1}$ (10)

We denote the stationary waiting time distribution at an arrival epoch by *W(t).* Consider an absorbing Continuous Time Markov Chain {*Y(t)*} with infinitesimal generator $\overline{Q}$:

$$\overline{Q} = \begin{bmatrix} b & B & . & ... \\ . & A_2 & D & ... \\ . & . & A_2 & ... \\ : & : & : & : \end{bmatrix}$$

which is obtained by setting λ equal to zero in *Q* and the set of transient states **i** ={(i, 1),…,(i, s)} for i≥0, an absorbing state 0=(0, 0). *Y(t)* gives the state of the system *(i, j)* at time t and time 0 is defined as an arbitrary arrival epoch, *i* is the number of ships still in queue who arrived before time 0, and *j* is the number of busy servers. Let $\mathbf{y}_i(t)=(y_{i1}(t),\dots,y_{is}(t))$, i≥0, t≥0 and $\mathbf{y}(t)=(y_0(t), \mathbf{y}_0(t),\dots)$, where **y**(t) be the solution of differential equations system: $\mathbf{y}'(t) = \mathbf{y}(t)\overline{Q}$, $\mathbf{y}(0)=\boldsymbol{\pi}$. Therefore **y**(t) gives the probabilities that the process is in the various states *(i, j)* and $y_0(t)$ is the probability that the process has reached state 0 at time *t.* Suppose $W_i(t)$ is the waiting time distribution for an arriving ship who requires *i* servers and $\mathbf{W}(t)=\{W_1(t),\dots,W_s(t)\}$ or equally:

$\mathbf{W}(t) = y_0(t)e + \mathbf{y}_0(t)C$ (11)

Such that *C* is an s×s matrix that the first *s-i* elements of column *i* are ones for each *i*, otherwise zeros.

Using total probability, we obtain $W(t)=\mathbf{cW}(t)$, where $\mathbf{c}=\{c_1,\dots,c_s\}$. Using (11) and Laplace Stieltjes transform (LST), we obtain:

$\hat{\mathbf{W}}(\xi) = y_0(t)\mathbf{e} + \hat{\mathbf{y}}(\xi)[\mathbf{be} + \xi C]$ (12)

where **b** is s-vector and $\hat{\mathbf{W}}(\xi)$ is LST of **W(t)**.

Using recursively method for solving the differential equation $\mathbf{y}'(t) = \mathbf{y}(t)\overline{Q}$,, we have

$$\hat{\mathbf{y}}_0(\xi) = \mathbf{y}_0(0)[\xi I - B]^{-1} + \qquad (13)$$

$$\sum_{k=1}^{\infty} y_k(0)[\xi I - D]^{-1}[A_2(\xi I - D)^{-1}]^{k-1} A_2 [\xi I - B]^{-1}$$

where $\hat{\mathbf{y}}_0(\xi)$ is LST of $\mathbf{y}_0(t)$. Let W_i denote the mean conditional waiting time given that an arriving ship requires i servers and $\mathbf{W} = (W_1, \ldots, W_s)$. From (12) and (13), we have $\mathbf{W} = \lim_{\xi \to \infty} -\frac{d}{d\xi} \hat{\mathbf{W}}(\xi)$. We define $\mathbf{r}_1 = \mathbf{e}_s(-B)^{-1}$, $\mathbf{r}_2 = \mathbf{e}_s(-B)^{-2}$ and $\mathbf{r}_3 = \mathbf{e}_s(-D)^{-1}$, we have [5]:

$$\mathbf{W} = \mathbf{y}_0(0) B^{-1}[C - E] + (\pi_1 (I-R)^{-1} e)[(r_2 b - r_3)e' - r_1 C] + r_3[\pi_1 (I-R)^{-2})e]e' + (\pi_1 [D(I-R)]^{-1} e)e' \quad (14)$$

where E is an $s \times s$ matrix of ones. Using (14), we can compute the mean waiting time as follows:

$$E[W] = \mathbf{cW}. \quad (15)$$

2. 2 Model formulation

Ships enter service in their order of arrival (FIFO) and leave the system only after all servers have finished service. In this paper we consider K-group servers with different types such that service rate of each group are different. Therefore we must choose optimum-group servers which minimize the total cost in the port i. e.

$$\min \mathrm{TC}(x, y, W, s) = \sum_{x,y} [W C_W(y) + s C_s(x)]$$
$$y \in \{T_1, \ldots, T_m\}, \quad x \in \{\mu_1, \ldots, \mu_k\} \quad (16)$$

Where $C_s(x)$, $C_W(y)$, W and s are service cost, demurrage cost, waiting time random variable and number of servers respectively. Since W is a random variable, the problem (16) is a stochastic programming. In the following Algorithm, we compute the probability distribution vector dependent on ship tonnages and their probabilities [2].

ALGORITHM 1.

Input: t[1],…, t[m]; u[1],…, u[m]; s.

Output: $\mathbf{c} = (c_1, \ldots, c_s)$.

Set **f=n=r=0**, h:=0, v:=0

1.For i=1 to m do e[i]:=s∗t[i]/t[m]
2. For j=1 to m-1 do
   n[j]:=trunc(e[j])
   e[j]:=e[j]-n[j]
3. f[s]:=u[m]
4. For i:=1 to m-1 do
5. If n[i]=0 then r[1]:=r[1]+u[i]∗ e[i]
   Else
6. If n[i]=v then
7. r[v+1]:=r[v+1]+u[i]∗e[i]
8. f[v]:=f[v]+u[i]∗(1-e[i])
9. Else
10. v:=v+1
11. Repeat step 6 while v≤s
12.For j:=1 to s do
   f[j]:=f[j]+r[j]
   h:=h+f[j]
13.For j:=1 to s do c[j]:=f[j]/h
14. Exit

One can select another scheme for computing the probability distribution vector, but our algorithm has the following property.

Lemma 2.

In algorithm 1, we suppose that s=m and T_i =iT, where u_i is that arriving probability to port for i-type ships with tonnage T_i, i=1,…,m, then we have $\boldsymbol{c} = (u_1, \ldots, u_m)$.

Prove: *step 1 gives **e**=(1, 2,…, m). Update **n** and **e** vectors in step 2 as follows: **n**=(1, 2,…, m-1, 0) , **e**=(0, 0,…., 0, m). The next step gives $f_s = u_m$. In For-loop for steps 4-11, our algorithm compute two vectors **r** and **f** as follows: **r**=(0,…, 0), **f**=$(u_1, \ldots, u_{m-1}, u_m)$. Using the results of previous steps and this fact that $\sum_{i=1}^{m} u_i = 1$, in step 12 we have h=1 and **f**=$(u_1, \ldots, u_{m-1}, u_m)$. Finally in step 13 we obtain the desired result. In addition, using lemma 1, we have probability distribution function as* $c_i(t) = \frac{(\lambda u_i t)^n e^{-\lambda u_i t}}{n!}$.

Lemma 2 is a good property for computing the probability distribution vecor.

The problem (16) is a simple stochastic programming. We can transform the problem (16) to deterministic problem by expectation value as follows:

$$\min \mathrm{E}(TC(x, y, W, s)) = \sum_{x,y} [E(W) C_W(y) + s C_s(x)]$$
$$y \in \{T_1, \ldots, T_m\}, \quad x \in \{\mu_1, \ldots, \mu_k\} \quad (17)$$

where $E(W) = \mathbf{cW}$, and $\mathbf{c} = (c_1, \ldots, c_s)$.

Now we can solve the problem (17) by the following algorithm, using algorithm 1, (10), (14) and (15).

ALGORITHM 2

[1.] Construct C vector using ship tonnage.

[2.] For $l = 1, \ldots, k$ Do

[2.1] Compute Rate Matrix for l-type server.

[2.2] Compute W_l vector and $E(W_l)$.

[3.] Determine optimal type server and its total cost.

In the next section we express an example and solve it, using our algorithm.

3 A numerical example

We consider a port in which ships arrive Poisson with rate $\lambda = 1.4$ and the number of servers equals to 5. The service cost and demurrage cost are given in following tables:

Table 1. table of service cost(s.c.)

i-server	server rate(μi)	s.c.
1	0.9	33
2	1.2	40
3	1.5	48
4	1.5	51

Table 2. table of demurrage cost(d.c.)

i-ship	tonnage	probability (pi)	d.c.
1	100	0.24	12
2	200	0.32	15
3	300	0.26	18
4	600	0.18	22

Now we use our algorithm to find optimum group-servers in which minimize the total cost in (17). The following computations are obtained by our algorithm.

THE PROBABILITY DISTRIBUTION VECTOR IS GIVEN BY

c_1	c_2	c_3	c_4	c_5
0.3194	0.3576	0.1354	0.0000	0.1875

THE TYPE OF SERVER=1 AND RATE OF SERVER=0.9

$$R.M. = \begin{pmatrix} .5532 & .0284 & .0507 & .1098 & .1649 \\ .3739 & .3946 & .0670 & .1451 & .2179 \\ .1722 & .1818 & .3248 & .1178 & .1770 \\ .0751 & .0792 & .1415 & .3065 & .1338 \\ .0511 & .0540 & .0963 & .2086 & .3133 \end{pmatrix}$$

$$X_0 = (.0678, .0753, .0675, .0521, .0375, .0270)$$
$$X_1 = (.01847, .01193, .01324, .01883, .01995)$$

MEAN CONDITIONAL WAITING TIMES IS GIVEN BY

$$\text{W} = (4.9412, 5.1530, 5.4603, 5.9632, 6.9690)$$

mean waiting time is given by $E(W) = 5.4674$.

THE TYPE OF SERVER=2 AND RATE OF SERVER=1.2

$$R.M. = \begin{pmatrix} .4782 & .0217 & .0352 & .0711 & .1010 \\ .3528 & .3234 & .0476 & .0962 & .1367 \\ .1716 & .1573 & .2559 & .0777 & .1104 \\ .0764 & .0700 & .1138 & .2299 & .0817 \\ .0520 & .0476 & .0775 & .1565 & .2223 \end{pmatrix}$$

$$X_0 = (.1947, .1623, .1228, .0823, .0531, .0359)$$
$$X_1 = (.0319, .0165, .0159, .0204, .0201)$$

MEAN CONDITIONAL WAITING TIMES IS GIVEN BY

$$\text{W} = (0.8768, 0.9777, 1.1463, 1.4669, 6.2257)$$

mean waiting time is given by $E(W) = 1.9523$.

THE TYPE OF SERVER=3 AND RATE OF SERVER=1.5

$$R.M. = \begin{pmatrix} .4202 & .0169 & .0258 & .0499 & .0864 \\ .3278 & .2732 & .0355 & .0685 & .0940 \\ .1649 & .1347 & .2104 & .0552 & .0757 \\ .0743 & .0619 & .0948 & .1831 & .0552 \\ .0506 & .0421 & .0645 & .1246 & .1709 \end{pmatrix}$$

$$X_0 = (.2963,.1976,.1331,.0807,.0484,.0319)$$
$$X_1 = (.0321,.0144,.0125,.0150,.0141)$$

MEAN CONDITIONAL WAITING TIMES IS GIVEN BY

$$\mathrm{W} = (0.3311,0.3879,0.4933,0.7173,1.7497)$$

mean waiting time is given by $E(W) = 0.6394$.

THE TYPE OF SERVER=4 AND RATE OF SERVER=1.8

$$R.M. = \begin{pmatrix} .3744 & .0134 & .0197 & .0369 & .0494 \\ .3035 & .2361 & .0274 & .0513 & .0687 \\ .1562 & .1215 & .1783 & .0413 & .0553 \\ .0710 & .0552 & .0810 & .1517 & .0399 \\ .0483 & .0376 & .0551 & .1033 & .1383 \end{pmatrix}$$

$$X_0 = (.3773,.2096,.1295,.0730,.0416,.0270)$$
$$X_1 = (.0291,.0117,.0094,.0108,.0098)$$

MEAN CONDITIONAL WAITING TIMES IS GIVEN BY

$$\mathrm{W} = (.1628,.1984,.2699,.4354,.9503)$$

mean waiting time is given by $E(W) = 0.3377$

THE TOTAL COST VECTOR FOR i-TYPE SERVERS IS GIVEN BY

$$\mathrm{E(TC(.))} = (1026.32,930.80,1002.84,1042.62)$$

MIN TOTAL COST=930.804 AND OPTIMAL TYPE SERVER=2

4 Conclusion

In this paper, we tried to formulate a simple port as multi server queuing system and we determined the optimum group servers using stochastic algorithm 2. It is hoped that we can construct a similar algorithm for reduction of traffic in call center and use it in the real world.

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