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# A Nonlinear Wave Propagation Model

ASU İNAN and LALE BALAS
Civil EngineeringDepartment
Gazi University
06570,Maltepe, Ankara
TURKEY

*Abstract:* The extended mild slope equation has been solved numerically to simulate wave propagation. Refraction, diffraction, shoaling, reflection, bottom friction, breaking energy dissipation and resonance with nonlinear wave celerity and group velocity have been considered. Mac Cormack Method and Point Gauss Seidel Method are applied together on an irregular mesh. In the predictor step, forward finite difference approximations are applied to first order derivatives and central finite difference approximations are applied to second order derivatives. In the corrector step, backward finite diffrence approximations are used for first order derivatives and central finite difference approximations are applied to second order derivatives. The developed numerical model has been applied to the Fethiye Bay located in the Mediterranen coast of Turkey.

*Key-Words:* -Mild slope equation, wave model, refraction, diffraction, shoaling,finite difference

## 1 Introduction

In every coastal or ocean engineering planning study, the information of wave distributions and conditions of the region has an important role. Therefore the wave propagation from deep water to shallow water has to be well simulated.

Numerical wave models are basically two types. First is phase- resolving models based on vertically integrated, time dependent mass and momentum balance equations; and the second is phase-averaged models based on the a spectral energy balance equation. Recently, the researchers have concentrated on the improvement of unified phase resolving models describing transient fully nonlinear wave propagation from deep water to shallow water over a large area (Liu & Losada, 2002). As the depth integrated models, these methos can be stated: Ray approximation, mild slope equation, parabolic approximation, Stokes waves approximation, Boussinesq approximation, highly nonlinear and dispersive models. The detailed explanation of these models can be found in the study of Liu and Losada (2002). In this study, on mild slope equations is focused. Linear wave transformation from deep water to shallow water like refraction, diffraction, shoaling, reflection can be solved together with the mild slope equation proposed by Berkhoff in 1972. This basic mild slope equation is an elliptical model. Bottom topography is assumed to be small, ($|\nabla h|/kh<<1$) where h is water depth, k is wave number, $\nabla$ is horizontal gradient operator. The equation below is called mild slope equation in the literature.

$$\frac{\partial}{\partial x}(CCg\frac{\partial\phi}{\partial x})+\frac{\partial}{\partial y}(CCg\frac{\partial\phi}{\partial y})+\sigma^2\frac{C_g}{C}\phi=0 \qquad (1)$$

where, $\sigma$: Angular frequency; C : Wave celerity; $C_g$: Group velocity; $\phi$: Two dimensional complex potential function.

Four approaches can be applied for solving of the elliptical equation: 1) Parabolic approach, 2) Hyperbolic approach, 3) Iteration methods, 4) Direct matrix equation solver.

Parabolic approach was proposed by Radder in 1979. Weak diffraction and negligible reflection are assumed in this approach. Parabolic equation is obtained after reducing the mild slope equation to Helmholtz equation. The advantage of this model is the applicability to short waves over large coastal areas with irregular bottom topography.

Hyperbolic approach is an unsteady approach. Mild slope equation is transformed to transient mild slope equation. This method includes reflection and it is the superioty to the parabolic approach. Hyperbolic method is a useful tool to solve wave transformation problems especially in harbours where reflection has a great importance.

The third approach is advanced iteration methods such as multi grid technique used by Li & Anastasiou (1992) and Generalized Minimum Residual Method applied by Walker (1982). These methods do not require large computer memory and have great convergence rate.

The fourth approach is solution of large band matrices with the Gauss Elimination Method using partial pivot algorithm. But large computer memory is necessary for this method.

If bottom topography has undulations, bottom curvature $\left((\nabla h)^2\right)$ and square of bottom slope $\left(\nabla^2 h\right)$ have to be accounted. Some researhers have studied on the modification of the mild slope equation for the rapidly varying bottom topographies.

Kirby (1986) developed mild slope equation for the rapidly varying bottom topographies and applied this modified mild slope equation to observe the reflection effect of propagating waves over sinusodial bathymetries. Massel (1993), studied on the transformations of propagating waves. He developed a modified mild slope equation including higher order bottom effect terms as wells as evanescent modes. Chamberlain and Porter (1995) solved modified mild slope equation for the not evanescent modes. It includes the general mild slope eqation and extended mild slope equation of Kirby. Suh et al. (1997) derived two equivalent time dependent wave equation for the propagation of water waves on rapidly varying bottom topography using Green's formula method and the Lagrangian formulation.

Maa et al. (2002) used the extended mild slope equation for solving water wave transformations (refraction, diffraction, shoaling, reflection, bottom friction, breaking energy dissipation and resonance).

In this study, the extended mild slope equation has been used to simulate refraction, diffraction, shoaling, reflection, bottom friction, breaking energy dissipation and resonance with nonlinear wave celerity and group velocity.

## 2 Theory

The extended mild slope equation used by Maa et al. (2002) has been solved in this study.

$$\nabla.\left(CC_g\nabla\phi\right)+k^2CC_g\left(1+if_{bd}\right)\phi+\left[f_1 g\nabla^2 h+f_2(\nabla h)^2 gk\right]\phi=0 \quad (1)$$

where

$$f_1=\frac{-4kh\cos(kh)+\sinh(3kh)+\sinh(kh)+8(kh)^2\sinh(kh)}{8\cosh^3(kh)\left[2kh+\sinh(2kh)\right]}-\frac{kh\tanh(kh)}{2\cosh^2(kh)} \quad (2)$$

$$f_2=\frac{\operatorname{sech}^2(kh)}{6\left[2kh+\sinh(2kh)\right]^3}f_3 \quad (3)$$

$$f_3=\left\{\begin{matrix}8(kh)^4+16(kh)^3\sinh(2kh)-9\sinh^2(2kh)\cos(2kh)+\\ 12(kh)\left[1+2\sinh^4(kh)\right]\left[kh+\sinh(2kh)\right]\end{matrix}\right\} \quad (4)$$

$$n=\frac{1}{2}\left(1+\frac{2kh}{\sinh 2kh}\right) \quad (5)$$

where, g:gravitational acceleration, $f_1$:bottom curvature coefficient, $f_2$:coefficient of square of the bottom slope, $\nabla$:horizontal gradient operator, $\phi$: velocity potential function, k:wave number, L:wave length, C:wave celerity, $C_g$:group velocity, $\nabla$h:bottom slope, $\nabla^2$h:bottom curvature, $f_{bd}=f_b+f_d$ : combined energy dissipation factor, $f_b$:bottom friction factor, $f_d$:energy dissipation factor after wave breaking. $f_b$ and $f_d$, were given by Hsu and Wen (2001) as follows:

$$f_b=\frac{4f_w}{3\pi}\frac{a\sigma^2}{ng\sinh^3 kh} \quad (6)$$

$$f_d=\frac{\Gamma}{kh}\left(1-\frac{K^2}{4\gamma^2}\right) \quad (7)$$

where, $f_w$ is the wave friction factor, $\gamma$=a/h is the ratio of the wave amplitude to the water depth, $\Gamma$ and K are emprical coefficients taken as 0.4 and 0.15, respectively.

Wave friction factor is determined from the following equation.( Jonsson and Carlsen (1975)).

$$\frac{1}{4\sqrt{f_w}}+\log_{10}\frac{1}{4\sqrt{f_w}}=m_f+\log_{10}\frac{a_{1m}}{k_N} \quad (8)$$

where, $a_{1m}$: Semi length of distance of moving fluid particle on bottom, $k_N$: Nikuradse roughness, $m_f$:an experimental constant ($m_f$=-0.08). When $a_{1m}/k_N < 2$ is, wave friction factor is taken as $f_w$=0.24 in the calculations. Otherwise the calculated value of $f_w$ by the equation (8) is used.

Breaking index is used to define breaker line. In the formulation below, m is bottom slope, and $L_0$ is the deep water wave length, R=$\sqrt{h/L_0}$ (Isobe, 1987).

$$\gamma_b=0.53-0.3\exp(-3R)+5m^{3/2}\exp\left[-45(R-0.1)^2\right] \quad (9)$$

$\gamma$ and $\gamma_b$ are calculated in each step and compared in the solution. If $\gamma<\gamma_b$ is $f_d$ is equalized to zero. Otherwise, $f_d$ value is used as calculated in the formula.

The use of nonlinear wave celerity and group velocity provides more accurate results in wave propagation solution. The importance increases especially in strong refraction condition in shallow water. Kirby and Dalrymple (1986) proposed a method valid either in deep water or in shallow water. Dispersion relationship is used to obtain nonlinear wave celerity and group velocity.

$$\sigma^2=gk\left(1+f_1{}'(kh)\varepsilon_*{}^2D\right)\tanh\left(kh+\varepsilon_* f_2{}'(kh)\right) \quad (10)$$

where,

$$\varepsilon_*=ka \quad (11)$$

$$D=\frac{\cosh(4kh)+8-2\tanh^2(kh)}{8\sinh^4(kh)}=\frac{9-12\tanh^2(kh)+13\tanh^4(kh)-2\tanh^6(kh)}{8\tanh^4(kh)} \quad (12)$$

$$f_1{}'(kh)=\tanh^5(kh),\; f_2{}'(kh)=\left(\frac{kh}{\sinh(kh)}\right)^4 \quad (13)$$

$$C=\frac{\sigma}{k} \quad \text{and} \quad C_g=\frac{d\sigma}{dk} \quad (14)$$

The nonlinear wave celerity and group velocity can be calculated using the dispersion relationship given in the equation (10) and the equations (11)-(14).

### 2.1 Boundary Conditions

There are two types boundary conditions used for wave transformations. One of them is partial reflection boundary condition, the other is given boundary condition.

$$\frac{\partial\phi}{\partial x}=\pm(-1)^{m'}\, i\alpha^* k_x\phi+2ik_x\phi_i \quad (15)$$

$$\frac{\partial\phi}{\partial y}=\pm(-1)^{m'}\, i\alpha^* k_y\phi+2ik_y\phi_i \quad (16)$$

where,

$$\alpha^*=\frac{1-R}{1+R},\; k_x=k\cos\theta,\; k_y=k\sin\theta \quad (17)$$

where $\alpha^*$ is the absorption coefficient; R is the reflection coefficient; $k_x$ and $k_y$ are wave numbers in x- direction and y- direction, respectively; θ is the wave incidence angle. The subscript 'i' denotes a physical quantity under specified conditions. (Hsu and Wen, 2001).

When there is total reflection condition:

$\phi_i=0$, m'=0, α=0 (18)

When there is partial reflection condition:

$\phi_i=0$, m'=0, 0<α≤1 (19)

When there is Radiation reflection condition:

$\phi_i=0$, m'=0, α=1 (20)

At a given boundary condition:

$$\phi_i=\frac{igH}{2\sigma}e^{is},\; m'=1, \alpha=1 \quad (21)$$

where

$$s=k_{xi}+k_{yi}-\sigma t=k_0\cos\theta_0 x-k_0\sin\theta_0 y-\sigma t \quad (22)$$

Wave number vector is calculated with the equation [23] and phase function can be found with the equation [24]. Re and Im are real part and imaginary part of complex potential function, respectively. Wave angle calculation is given in the equation [25].

$$\vec{k}=\nabla s \quad (23)$$

$$s=\tan^{-1}\left(\frac{\mathrm{Im}(\phi)}{\mathrm{Re}(\phi)}\right) \quad (24)$$

$$\theta=\tan^{-1}\left(\frac{\partial s/\partial y}{\partial s/\partial x}\right) \quad (25)$$

## 3 Numerical Method

Mac Cormack Method and Point Gauss Seidel Method are used together as numerical method. They have been applied to the equation [2] on an irregular mesh. Mac Cormack Method is a multi step method. In the predictor step, forward finite difference approximations are applied to first order derivatives and central finite difference approximations are applied to second order derivatives. In the corrector step, backward finite diffrence approximations are used for first order derivatives and central finite difference ,on $\pm x$ direction [21] approximations are applied to second order derivatives. Mac Cormack Method provides more accurate results especially in the solution of nonlinear equations. Numerical dispersion problem is reduced by using forward and backward finite difference methods together. Point Gauss Seidel method decreases the iteration number and increases the rate of convergence. The results of the linear theory are used as initial assumptions to begin the iteration.

$$a=a_0K_sK_r \quad (26)$$

$$K_s=\left[\frac{1}{\tanh kh(1+2kh/\sinh 2kh)}\right]^{1/2} \quad (27)$$

$$K_r=\left(\frac{\cos\theta_0}{\cos\theta}\right)^{1/2} \quad (28)$$

$a_0=H_0/2$ (29)

where $a_0$ is the deep water wave amplitude, $K_s$ is the shoaling coefficient and $K_r$ is the refraction coefficient.

## 4 Applications

The numerical model has been applied to the Fethiye Bay located in the Mediterranen coast of Turkey. The bathymetry of the Fethiye Bay is shown in the Fig.1. Deep water wave height is $H_0$=3m and wave period is T=8sec. The numerical model has been applied to the dominant direction W and SW, and wave height distributions are given in Fig.2. and in Fig.3 respectively. The grid size in x-direction and y- direction has been used as dx=dy=5m in numerical model.

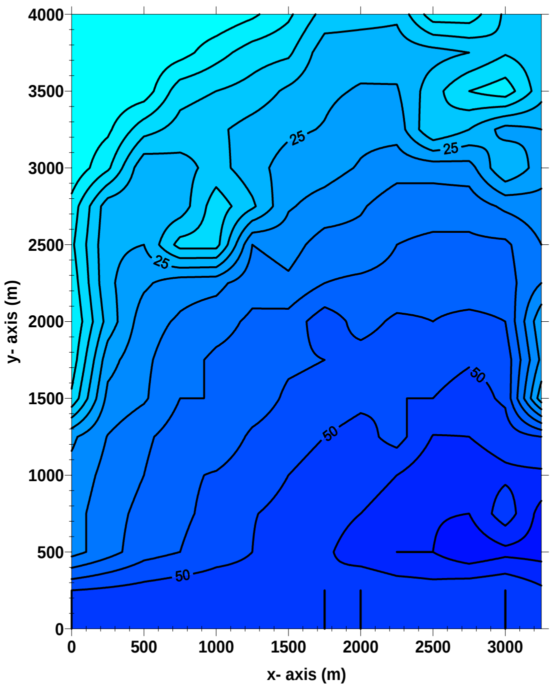


Fig. 1: Bathymetry of Fethiye Bay

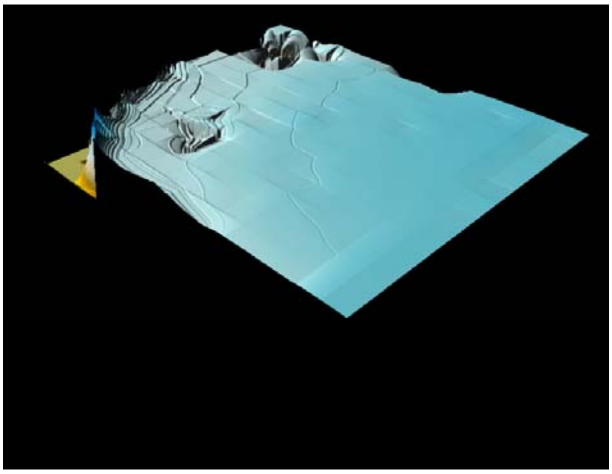

Figure 2: Wave height distribution for waves angle from W direction.

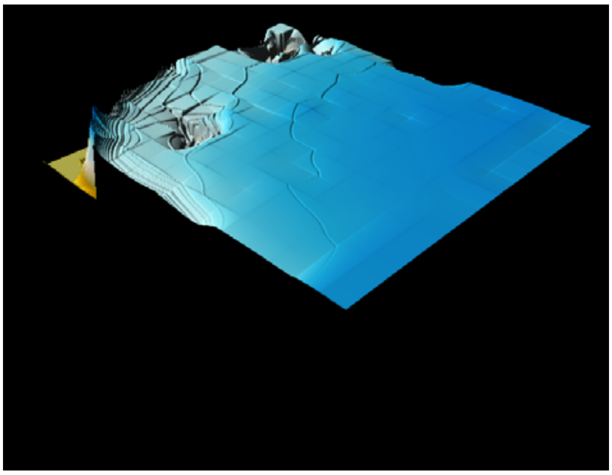

Figure 3: Wave height distribution for waves approaching from SW direction.

## 5 Conclusions

A numerical model is developed for the simulation of wave transformations, that is applicable to irregular bottom topographies. Model is based on extended mild slope equation and could simulate wave shoaling, refraction, diffraction, reflection and resonance. The use of nonlinear wave celerity and group velocity provides more accurate results in wave propagation solution. The importance increases especially in areas where there occur sudden changes in water depth causing strong refraction and also in shallow water regions.

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