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The Matrix Analogs of Firey's Extension of Minkowski Inequality and of Firey's Extension of Brunn-Minkowski Inequality

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Abstract: – The Brunn-Minkowski theory is a central part of convex geometry. At its foundation lies the Minkowski addition of convex bodies which led to the definition of mixed volume of convex bodies and to various notions and inequalities in convex geometry. Its origins were in Minkowski's joining his notion of mixed volumes with the Brunn-Minkowski inequality, which dated back to 1887. Since then it has led to a series of other inequalities in convex geometry. The existence is very useful and widely used in mathematical and engineering applications. Our purpose of this series was to develop an equivalent series of inequalities for positive definite symmetric matrices. The major theorems presented here are the matrix analogs of Firey's extension of Minkowski inequality and of Firey's extension of Brunn-Minkowski inequality.

Key-Words: – Aleksandrov inequality, Quermassintegral, Mixed Quermassintegral, Matrix Firey p-Sum, Mixed p-Quermassintegral, matrix analog of Firey's Extension of Brunn-Minkowski inequality, matrix analog of Firey's Extension of Minkowski inequality

1. Introduction

The Brunn-Minkowski theory is the heart of quantitative convexity. Its origins were in Minkowski's joining his notion of mixed volumes with the Brunn-Minkowski inequality, which dated back to 1887. Since then it has led to a series of other inequalities in convex geometry. The existence is very useful and widely used in mathematical and engineering applications. Our purpose of this series was to develop an equivalent series of inequalities for positive definite symmetric matrices. Some of these new matrix analogous notions and inequalities can be found in [9], among them the important ones are the matrix version of Blaschke summation [9], matrix version of parallel summation [3, 9, 12], matrix version of parallel Blaschke summation [9], and **the matrix analog of the Kneser-Süss inequality** [9]. In this paper, our goal here is to develop two new matrix analogs of the inequalities in convex geometry, namely, **the matrix analog of Firey's Extension of Minkowski inequality** and **the matrix analog of Firey's Extension of Brunn-Minkowski inequality**.

2. Materials and Methods

2.1 Matrix Quermassintegrals, Matrix Mixed Quermassintegrals, Firey's p Sum and Matrix Mixed p-Quermassintegrals

Let A be an $n \times n$ matrix with real entries and we denote its ijth entry by $(A)_{ij}$. We will denote by $|A|$ or $D(A)$ or $\det(A)$ the determinant of A. From this point on, unless stated otherwise, we assume that all matrices are symmetric. Let M_n^s denote the set of all $n \times n$ positive definite symmetric matrices. For $A_1, \ldots, A_n \in M_n^s$, let $D(A_1, \ldots, A_n)$ denote the mixed determinant [9] of $A_1, \ldots, A_n$.

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Remark 1 (Mixed Determinant [9]). *A mixed determinant $D(A_1, A_2, \ldots, A_n)$ of $n \times n$ matrices $A_1, A_2, \ldots, A_n$ can be regarded as the arithmetic mean of the determinants of all possible matrices which have exactly one row from the corresponding rows of $A_1, A_2, \ldots, A_n$.*

For example, for any $A, B \in M_n^s$, the mixed determinant $D(A, n-1; B, 1)$ is

$$D(\underbrace{A, \ldots, A}_{n-1}, B) = \frac{1}{n}\left(\begin{vmatrix} a_1 \\ \vdots \\ a_{n-1} \\ b_n \end{vmatrix} + \cdots + \begin{vmatrix} b_1 \\ a_2 \\ \vdots \\ a_n \end{vmatrix} \right). \tag{1}$$

We can easily prove that $nD_1(A, B) = nD(A, n-1; B, 1) = \lim_{\varepsilon \to 0^+} \frac{D(A+\varepsilon B) - D(A)}{\varepsilon} = \mathcal{C}A \cdot B$, where $A \cdot B := \sum_{i,j} (A)_{ij}(B)_{ij}$ and $\mathcal{C}A$ is the cofactor matrix of A, the transpose of the classical adjoint matrix of A.

Definition 2 (Matrix Quermassintegrals). *For $A \in M_n^s$, let $W_0(A), W_1(A), \ldots, W_n(A)$ denote the matrix Quermassintegrals of A defined by*

$$W_i(A) = D(A, n-i; I, i), \tag{2}$$

the mixed determinant with $n-i$ copies of A and i copies of I.

Thus, $W_0(A) = D(A)$, the determinant of A, $W_1(A) = D(A, n-1; I, 1) = D_1(A, I) = \frac{\mathcal{C}A \cdot I}{n} = \frac{1}{n}\operatorname{tr}(\mathcal{C}A) = \frac{1}{n}D(A)\operatorname{tr}(A^{-1}) = \frac{1}{n}D(A)\sum_{i=1}^{n} \lambda^{-1}$, and $W_{n-1}(A) = D(A, 1; I, n-1) = D(I, n-1; A, 1) = D_1(I, A) = \frac{\mathcal{C}I \cdot A}{n} = \frac{I \cdot A}{n} = \frac{1}{n}\operatorname{tr} A = \frac{1}{n}\sum_{i=1}^{n} \lambda_i$ – the mean of eigenvalues of A, where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $A \in M_n^s$.

Definition 3 (Matrix Mixed Quermassintegrals). *The matrix mixed Quermassintegrals $W_0(A, B), W_1(A, B), \ldots, W_{n-1}(A, B)$ of $A, B \in M_n^s$ are defined by*

$$(n-i)W_i(A, B) = \lim_{\varepsilon \to 0^+} \frac{W_i(A+\varepsilon B) - W_i(A)}{\varepsilon}.$$

Since $W_i(cA) = c^{n-i}W_i(A)$, it follows that $W_i(A, A) = W_i(A)$, for all i.

Since the Quermassintegral W_{n-1} is Minkowski linear, it follows that $W_{n-1}(A, B) = W_{n-1}(B)$, $\forall A$. The mixed Quermassintegral $W_0(A, B)$ will usually be written as $D_1(A, B)$.

The fundamental inequality for mixed Quermassintegrals in convexity theory states that: For $K, L \in \mathcal{K}^n$ and $0 \le i < n-1$,

$$W_i(K, L)^{n-i} \ge W_i(K)^{n-i-1}W_i(L), \tag{3}$$

with equality if and only if K and L are homothetic. Good general references for this material are Busemann [2] and Leichtweiß [8]. Its matrix version states as follows: For $A, B \in M_n^s$ and $0 \le i < n-1$,

$$W_i(A, B)^{n-i} \ge W_i(A)^{n-i-1}W_i(B), \tag{4}$$

with equality if and only if A and B are scalar multiple of each other. This matrix inequality can be proved easily using Aleksandrov inequality [1, 11], which is the matrix version of the Aleksandrov-Fenchel inequality for mixed volumes.

Definition 4 (Matrix Firey p Summation). *Let $A, B \in M_n^s$. Then the matrix Firey p sum of A and B, denoted $A +_p B$, is defined by*

$$A +_p B := (A^p + B^p)^{1/p}. \tag{5}$$

The commutativity and the associativity of $+_p$ are obvious. Mixed Quermassintegrals are, of course, the first variation of the ordinary Quermassintegrals, with respect to Minkowski addition. The first variation of the ordinary Quermassintegrals with respect to Firey addition is as follows:

Definition 5 (Matrix Mixed p-Quermassintegral). *Define the mixed p-Quermassintegrals $W_{p,0}(A, B), W_{p,1}(A, B), \ldots, W_{p,n-1}(A, B)$ as the first variation of the ordinary Quermassintegrals, with respect to Firey addition: For $A, B \in M_n^s$, and $p \ge 1$, $0 \le i \le n-1$, the mixed p-Quermassintegrals of A, B, denoted $W_{p,i}(A, B)$, is defined by*

$$\frac{n-i}{p} W_{p,i}(A, B) = \lim_{\varepsilon \to 0^+} \frac{W_i(A +_p \varepsilon \cdot B) - W_i(A)}{\varepsilon}$$

where $\varepsilon \cdot B = \varepsilon^{1/p}B$ expresses the relation between Firey scalar multiplication $(\cdot)$ and Minkowski scalar multiplication.

Of course for $p = 1$, the mixed p-Quermassintegral $W_{p,i}(A, B)$ is just $W_i(A, B)$. Obviously, $W_{p,i}(A, A) = W_i(A)$, for all $p \ge 1$.

Lutwak [7] states that for these new mixed Quermassintegrals, there is an extension of inequality (3): If $K, L \in \mathcal{K}_0^n$, $0 \le i \le n-1$, and $p > 1$, then

$$W_{p,i}^{n-i}(K, L) \ge W_i^{n-i-p}(K) W_i^p(L), \qquad (6)$$

with equality if and only if K and L are dilates. We will prove its matrix version: If $A, B \in M_n^s$, $0 \le i \le n-1$, and $p > 1$, then

$$W_{p,i}^{n-i}(A, B) \ge W_i^{n-i-p}(A) W_i^p(B), \qquad (7)$$

with equality if and only if $B = c{\cdot}A$, $c > 0$ in Theorem 18.

Theorem 6.

(a) *For $A, B \in M_n^s$ and $\alpha, \beta > 0, W_{p,i}(\alpha A, \beta B) = \alpha^{n-i-p}\beta^p W_{p,i}(A, B)$, and when $p = n-i$ and $\beta = 1, W_{p,i}(\alpha A, B) = W_{p,i}(A, B)$ for all $\alpha > 0$.*

(b) *For all $Q, A, B \in M_n^s, W_{p,i}(Q, A +_p B) = W_{p,i}(Q, A) + W_{p,i}(Q, B)$.* □

This result shows that the mixed p-Quermassintegral is linear, with respect to Firey p-sums, in its second argument.

2.2 Variational Characterizations of Eigenvalues of Symmetric Matrices

For a general matrix $A \in M_n$, about the only characterization of the eigenvalues is the fact that they are the roots of the characteristic equation $p_A(t) = 0$. For symmetric matrices, however, the eigenvalues can be characterized as the solutions of a series of optimization problems.

Since the eigenvalues of a symmetric matrix $A \in M_n$ are real, we shall adopt the convention that they are labeled according to increasing (non-decreasing) size:

$$\lambda_{min} = \lambda_1 \le \lambda_2 \le \cdots \le \lambda_{n-1} \le \lambda_n = \lambda_{max}. \qquad (8)$$

The smallest and largest eigenvalues are easily characterized as the solutions to a constrained minimum and maximum problem. The characterization theorem bears the names of two British physicists, and the expression $\dfrac{x^TAx}{x^Tx}$ is known as a *Rayleigh-Ritz* ratio.

Theorem 7 (Rayleigh-Ritz [4]). *Let $A \in M_n$ be symmetric, and let the eigenvalues of A be ordered as in (8). Then*

$$\lambda_1 x^Tx \le x^TAx \le \lambda_n x^Tx \quad \forall x \in \mathbb{R}^n,$$

$$\lambda_{max} = \lambda_n = \max_{x \ne 0} \frac{x^TAx}{x^Tx} = \max_{x^Tx=1} x^TAx,$$

$$\lambda_{min} = \lambda_1 = \min_{x \ne 0} \frac{x^TAx}{x^Tx} = \min_{x^Tx=1} x^TAx. \quad \square$$

The Rayleigh-Ritz theorem provides a variational characterization of the largest and smallest eigenvalues of a symmetric matrix A, but what about the rest of the eigenvalues? This question is answered in the following Courant-Fischer "min-max theorem."

Theorem 8 (Courant-Fischer [4]). *Let $A \in M_n$ be a symmetric matrix with eigenvalues $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$, and let k be a given integer with $1 \le k \le n$. Then*

$$\min_{w_1, w_2, \ldots, w_{n-k} \in \mathbb{R}^n} \max_{\substack{x \ne 0,\, x \in \mathbb{R}^n \\ x \perp w_1, w_2, \ldots, w_{n-k}}} \frac{x^TAx}{x^Tx} = \lambda_k \qquad (9)$$

and

$$\max_{w_1, w_2, \ldots, w_{k-1} \in \mathbb{R}^n} \min_{\substack{x \ne 0,\, x \in \mathbb{R}^n \\ x \perp w_1, w_2, \ldots, w_{k-1}}} \frac{x^TAx}{x^Tx} = \lambda_k. \qquad (10)$$

R*emark: If $k = n$ in (9) or $k = 1$ in (10), we agree to omit the outer optimization, as the set over which the optimization takes place is empty. In the two cases the assertions reduce to the Rayleigh-Ritz theorem (Theorem 7).* □

2.3 Some Applications of the Variational Characterizations

Among the many important applications of the Courant-Fischer theorem, one of the simplest is to the problem of comparing the eigenvalues of $A + B$ with those of A. We denote the eigenvalues of a matrix A by $\lambda_i(A)$.

Theorem 9 (Weyl [4]). *Let $A, B \in M_n$ be symmetric and let the eigenvalues $\lambda_i(A)$, $\lambda_i(B)$, and $\lambda_i(A+B)$ be arranged in increasing order (8). For each $k = 1, 2, \ldots, n$ we have*

$$\lambda_k(A) + \lambda_1(B) \le \lambda_k(A + B) \le \lambda_k(A) + \lambda_n(B). (11)$$

Proof. *For any nonzero $x \in \mathbb{R}^n$ we have the bound*

$$\lambda_1(B) \le \frac{x^TBx}{x^Tx} \le \lambda_n(B)$$

and hence for any $k = 1, 2, \ldots, n$ we have

$$\begin{aligned}
\lambda_k(A + B) &= \min_{w_1, \ldots, w_{n-k} \in \mathbb{R}^n} \max_{\substack{x \ne 0 \\ x \perp w_1, \ldots, w_{n-k}}} \frac{x^T(A + B)x}{x^Tx} \\
&= \min_{w_1, \ldots, w_{n-k} \in \mathbb{R}^n} \max_{\substack{x \ne 0 \\ x \perp w_1, \ldots, w_{n-k}}} \left[\frac{x^TAx}{x^Tx} + \frac{x^TBx}{x^Tx}\right] \\
&\ge \min_{w_1, \ldots, w_{n-k} \in \mathbb{R}^n} \max_{\substack{x \ne 0 \\ x \perp w_1, \ldots, w_{n-k}}} \left[\frac{x^TAx}{x^Tx} + \lambda_1(B)\right] \\
&= \lambda_k(A) + \lambda_1(B).
\end{aligned}$$

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A similar argument establishes the upper bound as well. □

Weyl's theorem gives two-sided bounds for the eigenvalues of $A+B$ for any symmetric matrices A and B. Further refinements can be obtained by restricting B to have a special form – for example, positive definite, rank 1, rank k, or bodering matrix.

Recall that a matrix $B \in M_n$ such that $x^T Bx \geq 0$ for all $x \in \mathbb{R}^n$ is said to be *positive semidefinite*; an equivalent condition is that B is symmetric and have all eigenvalues nonnegative. For symmetric matrices A, B we write $B \leq A$ or $A \geq B$ to mean that $A - B$ is positive semidefinite. In particular, $A \geq 0$ indicates that A is positive semidefinite. This is known as the Löwner partial order. If A is positive definite, that is, positive semidefinite and invertible, we write $A > 0$.

The following result, an immediate corollary of Weyl's theorem known as the *monotonicity theorem*, says that all the eigenvalues of a symmetric matrix increase if a positive semidefinite matrix is added to it.

Corollary 10 ([4]). *Let $A, B \in M_n$ be symmetric. Assume that B is positive semidefinite and that the eigenvalues of A and $A+B$ are arranged in increasing order* (8)*. Then*

$$\lambda_k(A) \leq \lambda_k(A+B), \quad \forall k = 1, 2, \ldots, n. \quad \Box$$

Corollary 11 ([4]). *If $A, B \in M_n$ are positive definite symmetric, then if $A \geq B$, then* $\det A \geq \det B$ *and* $\operatorname{tr} A \geq \operatorname{tr} B$*; and more generally, if $A \geq B$, then $\lambda_k(A) \geq \lambda_k(B)$ for all $k = 1, 2, \ldots, n$ if the respective eigenvalues of A and B are arranged in the same (increasing or decreasing) order.* □

2.4 Some Useful Results from Löwner Partial Order

A map $\Phi : M_m \to M_n$ is called *positive* if it maps positive semidefinite matrices to positive semidefinite matrices: $A \geq 0 \Rightarrow \Phi(A) \geq 0$. Φ is called *unital* if $\Phi(I_m) = I_n$.

Given $A = [a_{ij}] \in M_m, B = [b_{ij}] \in M_n$, then the right *Kronecker* (or *direct*, or *tensor*) *product* of A and B, written $A \otimes B$, is defined to be the partitioned matrix

$$A \otimes B = [a_{ij}B]_{i,j=1}^m \in M_{mn}.$$

Given $A = [a_{ij}] \in M_m, B = [b_{ij}] \in M_n$, the Hadamard product of A and B is defined as the entry-wise product: $A \circ B \equiv [a_{ij}b_{ij}] \in M_n$. We denote by $A[\alpha]$ the principal submatrix of A indexed by α. The following observation is very useful.

Lemma 12 ([5, 12]). *For any $A, B \in M_n$, $A \circ B = (A \otimes B)[\alpha]$ where $\alpha = \{1, n+2, 2n+3, \ldots, n^2\}$. Consequently there is a unital positive linear map Φ from M_{n^2} to M_n such that $\Phi(A \otimes B) = A \circ B$ for all $A, B \in M_n$.*

As an illustration of the usefulness of this lemma, consider the following reasoning: If $A, B \geq 0$, then evidently $A \otimes B \geq 0$. Since $A \circ B$ is a principal submatrix of $A \otimes B$, $A \circ B \geq 0$. Similarly $A \circ B > 0$ for the case when both A and B are positive definite. In other words, the Hadamard product of positive semidefinite (definite) matrices is positive semidefinite (definite). This important fact is known as the Schur product theorem. Let P_n be the set of positive semidefinite matrices in M_n. A map $\Psi : P_n \times P_n \to P_m$ is called *jointly concave* if

$$\begin{aligned}\Psi(\lambda A + (1-\lambda)B, \lambda C + (1-\lambda)D) \\ \geq \lambda\Psi(A, C) + (1-\lambda)\Psi(B, D).\end{aligned}$$

for all $A, B, C, D \geq 0$ and $0 < \lambda < 1$.

For $A, B > 0 \in M_n^s$ the *parallel sum* of A and B is defined as

$$A \tilde{+} B := \left(A^{-1} + B^{-1}\right)^{-1}.$$

We have the extremal representation for the *parallel sum* of $A, B \in P_n$:

$$A \tilde{+} B = \max\left\{X \geq 0 : \begin{bmatrix} A+B & A \\ A & A-X \end{bmatrix} \geq 0\right\}$$

where the maximum is with respect to the Löwner partial order. From this extremal representation we can show that the map $(A, B) \mapsto A \tilde{+} B$ is jointly concave [10, 12].

Lemma 13 ([10, 12]). *For $0 < r < 1$ the map*

$$(A, B) \mapsto A^r \circ B^{1-r}$$

is jointly concave in $A, B \geq 0$.

Theorem 14. *For $A, B, C, D \geq 0$ and $p, q > 1$ with $1/p + 1/q = 1$,*

$$A \circ B + C \circ D \leq (A +_p C) \circ (B +_q D).$$

Proof. *This is just the mid-point joint concavity case $\lambda = 1/2$ of Lemma 13 with $r = 1/p$.* □

2.5 Main Results

Theorem 15. *For $X, Y > 0 \in M_n^s$ and $\varepsilon \in [0,1]$ we have*

$$\begin{aligned}(1-\varepsilon)X + \varepsilon Y &\leq ((1-\varepsilon)X^p + \varepsilon Y^p)^{1/p} \\ &=: (1-\varepsilon) \cdot X +_p \varepsilon \cdot Y\end{aligned}$$

Proof. *This is just Theorem 14 with $A = (1-\varepsilon)^{1/p}X$, $B = (1-\varepsilon)^{1/q}1_n$, $C = \varepsilon^{1/p}Y$ and $D = \varepsilon^{1/q}1_n$, where $1_n = [1]_{n\times n}$, that is 1_n is the $n \times n$ matrix that has all of its entries equal to 1.* □

Applying Corollary 11 to the inequality in Theorem 15 and the fact that for any matrix $A \in M_n$, $W_0(A) = \lambda_1\lambda_2 \cdots \lambda_n$, $W_1(A) = \sum \lambda_{i_1} \cdots \lambda_{i_{n-1}}$, ..., $W_{n-3}(A) = \sum \lambda_{i_1}\lambda_{i_2}\lambda_{i_3}$, $W_{n-2}(A) = \sum \lambda_{i_1}\lambda_{i_2}$, $W_{n-1}(A) = \sum \lambda_i$. We can easily see that for any $A, B > 0$

$$\begin{aligned} &W_i((1-\varepsilon) \cdot A +_p \varepsilon \cdot B) \\ &\quad \geq W_i((1-\varepsilon)A + \varepsilon B) \\ &\quad \geq ((1-\varepsilon)W_i(A)^{\frac{1}{n-i}} + \varepsilon W_i(B)^{\frac{1}{n-i}})^{n-i}, \\ &\qquad 0 \leq i \leq n-1, \end{aligned} \tag{12}$$

where the last inequality is by applying the Aleksandrov-Fenchel inequality.

Theorem 16. *If $A_0, B_0 > 0 \in M_n^s$ with $W_i(A_0) = W_i(B_0) = 1$ then*

$$W_i(\alpha \cdot A_0 +_p (1-\alpha) \cdot B_0) \geq 1 \quad \forall \alpha \in [0,1], \quad 0 \leq i \leq n-1.$$

Proof. *Apply* (12) *to A_0, B_0 with $\varepsilon = 1 - \alpha$.* □

The following two theorems are analogs of Firey's Extension of Brunn-Minkowski and Minkowski theorems (see Lutwak [7] for the original theorems in convex geometry).

Theorem 17 (Matrix Analog of Firey's Extension of Brunn-Minkowski Inequality)**.** *If $A, B \in M_n^s$, $0 \leq i \leq n-1$, $p > 1$, then*

$$W_i^{\frac{p}{n-i}}(A +_p B) \geq W_i^{\frac{p}{n-i}}(A) + W_i^{\frac{p}{n-i}}(B) \tag{13}$$

with equality if and only if $A = c \cdot B, c > 0$.

Proof. *We apply Theorem 16 with*

$$\begin{aligned} A_0 &= \frac{1}{W_i(A)^{\frac{p}{n-i}}} \cdot A, \\ B_0 &= \frac{1}{W_i(B)^{\frac{p}{n-i}}} \cdot B, \\ \alpha &= \frac{W_i(A)^{\frac{p}{n-i}}}{W_i(A)^{\frac{p}{n-i}} + W_i(B)^{\frac{p}{n-i}}} \end{aligned}$$

to obtain

$$W_i\left(\frac{1}{W_i(A)^{\frac{p}{n-i}} + W_i(B)^{\frac{p}{n-i}}} \cdot A \right.$$

$$\left. +_p \frac{1}{W_i(A)^{\frac{p}{n-i}} + W_i(B)^{\frac{p}{n-i}}} \cdot B\right)^{\frac{p}{n-i}} \geq 1$$

$$W_i\left[\frac{A^p + B^p}{\left(W_i(A)^{\frac{p}{n-i}} + W_i(B)^{\frac{p}{n-i}}\right)}\right]^{\frac{1}{n-i}} \geq 1$$

$$\frac{1}{\left(W_i(A)^{\frac{p}{n-i}} + W_i(B)^{\frac{p}{n-i}}\right)} W_i\left(A^p + B^p\right)^{\frac{1}{n-i}} \geq 1$$

$$W_i(A^p + B^p)^{\frac{1}{n-i}} \geq W_i(A)^{\frac{p}{n-i}} + W_i(B)^{\frac{p}{n-i}}$$

or

$$W_i((A^p + B^p)^{\frac{1}{p}})^{\frac{p}{n-i}} \geq W_i(A)^{\frac{p}{n-i}} + W_i(B)^{\frac{p}{n-i}},$$

that is,

$$W_i(A +_p B)^{\frac{p}{n-i}} \geq W_i(A)^{\frac{p}{n-i}} + W_i(B)^{\frac{p}{n-i}}.$$

The equality part can be seen by directly substituting $A = c \cdot B$. This completes the proof. □

Theorem 18 (Matrix Analog of Firey's Extension of Minkowski Inequality)**.** *If $A, B \in M_n^s, 0 \leq i \leq n-1, p > 1$, then*

$$W_{p,i}^{n-i}(A,B) \geq W_i^{n-i-p}(A)W_i^p(B) \tag{14}$$

with equality if and only if $A = c \cdot B, c > 0$.

Proof. *Theorem 17 implies $W_i^{\frac{p}{n-i}}((1-\varepsilon) \cdot A +_p \varepsilon \cdot B) \geq (1-\varepsilon)W_i^{\frac{p}{n-i}}(A) + \varepsilon W_i^{\frac{p}{n-i}}(B)$, and since*

$$\begin{aligned} &\lim_{\varepsilon \to 0} \frac{W_i\left((1-\varepsilon) \cdot A +_p \varepsilon \cdot B\right) - W_i(A)}{\varepsilon} \\ &\qquad = 1\,\frac{n-i}{p} W_{p,i}(A,B) + \frac{i-n}{p} W_i(A). \end{aligned}$$

Then

$$\begin{aligned} W_{p,i}(A,B) &= W_i(A) + \frac{p}{n-i} \lim_{\varepsilon \to 0} \frac{W_i\left((1-\varepsilon) \cdot A +_p \varepsilon \cdot B\right) - W_i(A)}{\varepsilon} \\ &\geq W_i(A) \\ &\quad + \frac{p}{n-i} \lim_{\varepsilon \to 0} \frac{\left[(1-\varepsilon)W_i^{\frac{p}{n-i}}(A) + \varepsilon W_i^{\frac{p}{n-i}}(B)\right]^{(n-i)/p} - W_i(A)}{\varepsilon} \\ &= W_i(A) \\ &\quad + \frac{p}{n-i}\left[\frac{n-i}{p}\right]\left[W_i(A)^{p/(n-i)[(n-i)/p-1]}W_i(B)^{p/(n-i)} - W_i(A)\right] \end{aligned}$$

$$= W_i(A)^{1-p/(n-i)} W_i(B)^{p/(n-i)}.$$

This completes the proof. □

Furthermore, we can also show that the inequalities (13) and (14) are equivalent. Since we already show that (13) implies (14), it suffices to show that (14) implies (13):

Since $W_{p,i}(A,B) \geq W_i^{\frac{n-i-p}{n-i}}(A) W_i^{\frac{p}{n-i}}(B)$, for $A, B \in M_n^s, 0 \leq i < n-1, p > 1$, and $W_{p,i}(Q, A +_p B) = W_{p,i}(Q, A) + W_{p,i}(Q, B)$ then $W_{p,i}(Q, A +_p B) \geq W_i^{\frac{n-i-p}{n-i}}(Q)[W_i^{\frac{p}{n-i}}(A) + W_i^{\frac{p}{n-i}}(B)]$. We now set $A +_p B$ equal to Q and use the fact that $W_{p,i}(Q,Q) = W_i(Q)$ to obtain

$$W_i(A +_p B) \geq W_i^{\frac{n-i-p}{n-i}}(A +_p B) \left[W_i^{\frac{p}{n-i}}(A) + W_i^{\frac{p}{n-i}}(B) \right],$$

or

$$W_i^{\frac{p}{n-i}}(A +_p B) \geq W_i^{\frac{p}{n-i}}(A) + W_i^{\frac{p}{n-i}}(B)$$

which is (13). □

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