Proceedings of the 9th WSEAS International Conference on Applied Mathematics, Istanbul, Turkey, May 27-29, 2006 (pp152-156)

Spectral generalized inverses of Toeplitz-related matrices

M.C.GOUVEIA
Department of Mathematics
Institute of Telecommunications – University of Coimbra
Largo D.Dinis, 3000 COIMBRA
PORTUGAL
http://www.mat.uc.pt

Abstract: - A characterization of Drazin inverse of Toeplitz and Hankel matrices over a field is presented which provide an algorithm to compute this generalized inverse by a finite number of defined operations. The results are an application of the peculiar properties of those structured matrices together with recent results for matrices over arbitrary rings, showing that Drazin inverse preserves some spectral relations.

Key-Words: - Hankel and Toeplitz matrices; compound matrix; Drazin inverse.

1 Introduction

It is known from the literature that the Moore-Penrose inverse and other related pseudo-inverses provide a solution, or least square solution, for a system of linear algebraic equations , but they are not spectral inverses, i.e. they do not keep some spectral properties. However, it is known that Drazin inverses are spectral inverses which provide solutions for systems of linear differential [5] and linear difference [2] equations and which have many other applications [3]. This facts and the potential of structured matrices' properties played an important role in the motivation of this work.

Let K be a field and let $\tilde{A}$ be a linear transformation on K^n. We recall that the smallest non negative integer l such that

$K^n = R\left(\tilde{A}^l\right) + N\left(\tilde{A}^l\right)$ is called *the index of* $\tilde{A}$,

and we´ll denote it by $ind(\tilde{A})$. Hence , if A is the $n \times n$ matrix of $\tilde{A}$, we have the following equivalent algebraic definition.

1.1 Definition . A is said to have *Drazin index* l if l is the smallest natural number such that $rank\left(A^l\right) = rank(A^{l+1})$.

- It is clear that if $\tilde{A}$ is nonsingular then $ind(A) = 0$.

1.2 Definition. An $n \times n$ matrix over K is said *Drazin invertible* if there exists a (unique) solution A^D of the system of equations

$$A^l XA = A^l \qquad (1^l)$$

$$XAX = X \qquad (2)$$

$$AX = XA \quad . \qquad (3)$$

Then A^D is called the Drazin inverse of A .

- If l =1 then A^D is called the *group inverse* of A, and is denoted by $A^{\#}$.

We recall that if l=1, a solution of the equations (1^1), (2) and (3) is a commuting (1,2)-inverse of *A*. Moreover, if there exists *X* such that

$$AXA = A \qquad (1)$$

then A is *von Neumann regular,* and we write $X = A^{(1)}$ as an (1)*-inverse* of A .

It is clear that if *A* is von Neumann regular, then *A* is also (1,2)-invertible.

For the basic theory on generalized inverses we refer [1].

In the following we denote by $rank(A)$ the determinantal rank of A and by $A\left[i_1, \cdots, i_p / j_1, \cdots, j_q\right]$ the $p \times q$ submatrix of A taken from the $i_1 < \cdots < i_p$ rows and the $j_1 < \cdots < j_q$ columns. Other very common notations are omitted.

2 Drazin inverse of Hankel matrices

Let

$$H = (h_{ij}) = (h_{i+j}), \quad 0 \le i;\ j \le n-1;\ h_{ij} \in K,$$

be the $n \times n$ matrix of an Hankel form [7],

$$H=\begin{bmatrix} h_0 & h_1 & \cdots & h_{n-1} \\ h_1 & h_2 & \cdots & h_n \\ \vdots & ⋰ & ⋰ & \vdots \\ h_{n-1} & h_n & \cdots & h_{2n-2} \end{bmatrix}.$$

Let $rank(H)=\rho$ and let r be the order of the last principal minor

$$\det(H_\mu) := \det H[1,\cdots,\mu/1,\cdots,\mu]$$

which is different from zero.

Let $k := \rho - r$.

2.1 Definition r is called the *r-characteristic* of H, $0 \le r \le \rho$, and *(r,k)* is said to be the *(r,k)-characteristic* of H.

2.2 Remark To compute r with Matlab , do

$$\begin{aligned} &function\, r = rCaracteristica(A),\\ &\quad for\, i = 1 : size(H,1),\\ &\quad r(i,2) = \det(H(1:i,1:i));\\ &\quad r(i,1) = i;\\ &end \end{aligned}$$

Moreover, from Frobenius´Theorem [6] it follows that

$$D_{r+k} = \det\begin{bmatrix} H_r & H_{r,k} \\ H_{k,r} & H_{k,k} \end{bmatrix} \neq 0,$$

where

$$H_{r,k} = H[1,\cdots,r/n-k+1,\cdots,n] = H_{k,r}^T$$

and

$$H_{k,k} = H[n-k+1,\cdots,n/n-k+1,\cdots,n].$$

Consequently, H has a full rank decomposition $H := EF$ such as

$$H = \begin{bmatrix} I_r & O \\ X & Y \\ O & I_k \end{bmatrix} \begin{bmatrix} H[1,\cdots,r/1,\cdots,n] \\ H[n-k+1,\cdots,n/1,\cdots,n] \end{bmatrix} \qquad (4)$$

where

$$[X \quad Y] := H[r+1,\cdots,n-k/1,\cdots,n]W_r^{-1},$$

with W_r^{-1} as a right inverse of F.

In the following we´ll call the full rank factorization (4) a *Frobenius´ full rank factorization* .

2.3 Proposition *Let H be a singular $n \times n$ Hankel matrix with rank ρ and with r- characteristic.*

(i) If $r = \rho$, then ind(H)=1 if and only if $I_r + X^T X$ is non singular.

(ii) ind(H)= ind[$H_r(I_r + X^T X)$]+1.

***Proof*.** (i) If $\rho = r$, then

$$H = \begin{bmatrix} I_r \\ X \end{bmatrix} H[1,\cdots,r/1,\cdots,n] = EF,$$

with

$$X := H[n-r+1,\cdots,n/1,\cdots,n]\begin{bmatrix} H_r^{-1}+T \\ Z \end{bmatrix},$$

$$T := -H_r^{-1}H[1,\cdots,r/n-r+1,\cdots,n]Z,$$

Z arbitrary.

Since E has full column rank and F has full row rank, there are E_{li}^{-1}, F_r^{-1}, respectively, a left and a right inverse, such that

$$E_l^{-1}H^2F_r^{-1} = FE.$$

Consequently, *rank* (H^2) = *rank (FE)*.

Therefore we have *ind*(H)=1 if and only if *rank(H)= rank(FE)*.

Finally , since

$$\begin{aligned} FE &= H_r + H[1,\cdots,r/n-r+1,\cdots,n]X \\ &= H_r(I + X^T X), \end{aligned}$$

it is obvious that *ind*(H) =1 if and only if $I_r + X^T X$ is invertible. In that case ,

$$H^\# = E(FE)^{-2}F.$$

(ii) Otherwise, with

$$F = H_r[I \quad X^T],$$

it is clear that

$$H^i = \begin{bmatrix} I_r \\ X \end{bmatrix}[H_r(I + X^T X)]^{i-1}H_r[I_r \quad X^T],$$

for a positive integer i, and also that

$$rank(H^i) = rank(H_r(I + X^T X))^{i-1}.$$

Thus, $rank(H^i) = rank(H^{i+1})$ if and only if $rank(H_r(I+X^TX))^{i-1} = rank(H_r(I+X^TX))^i$, and, consequently,

$$l = ind\,[\ H_r(I_r + X^T X)\ ]+1.$$

In this case, an easy computation shows that the following matrix satisfies (1^l), (2) and (3),

$$H^D = \begin{bmatrix} I_r \\ X \end{bmatrix}\left[H_r(I + X^T X)^2\right]^D H_r[I_r \quad X^T].$$

■

As it is well known, over a field every matrix is von Neumann regular. If we consider an Hankel matrix in the form (4), it is clear that

$$X = F_r^{-1}E_l^{-1} \qquad (5)$$

is a (1,2)-inverse of H .

2.4 Proposition *Let H be a singular $n \times n$ Hankel*

Proceedings of the 9th WSEAS International Conference on Applied Mathematics, Istanbul, Turkey, May 27-29, 2006 (pp152-156)

matrix with rank ρ and with r- characteristic such that $0 \le r \le \rho$. Let H=EF be a Frobenius' full rank factorization of H, and

$$H_i := H^{(1,2)} H^i H^{(1,2)},$$
$$U_l := H_l H^2 H_l H_l^{(1,2)} + I_n - H_l H_l^{(1,2)}$$
$$V_l := H_l^{(1,2)} H_l H^2 H_l + I_n - H_l^{(1,2)} H_l .$$

Then the following are equivalent
(i) ind(H)=l
(ii) l is the smallest natural number such that U_l is invertible.
(iii) l is the smallest natural number such that V_l is invertible.
In that case,

$$H^D := H^l U_l^{-1} H_l V_l^{-1} H .$$

***Proof*.** The group inverse of an Hankel matrix over a field was already characterized in [6].
So we know that *ind(H)*=1 if and only if, for an (1)-inverse of *H,*

$$U_1 := H^2 H^{(1)} + I_n - HH^{(1)}$$
$$V_1 := H^{(1)} H^2 + I_n - H^{(1)} H$$

are invertible. In that case ,

$$H^{\#} = U_1^{-1} H \ V_1^{-1} .$$

Obviously, H and $H^{\#}$ are equivalent matrices.
Therefore, this is the case of the theorem when *ind*(*H*)=1, $H^{(1)} = F_r^{-1} E_l^{-1}$ such as in (5) , and

$$U_1 := HEE_l^{-1} + I_n - EE_l^{-1}$$
$$V_1 := F_r^{-1} FH + I_n - F_r^{-1} F .$$

The rest of the proof for an arbitrary Drazin index *l* is a consequence of the Corollary in [8].

■

2.5 Definition The $\rho - th$ *compound matrix* of a square matrix A is the matrix $C_\rho(A) = (c_{\alpha,\beta})$ whose $(\alpha,\beta)th$ entry is equal to $\det A[\alpha,\beta]$, where $\alpha = \{ i_1,\cdots,i_\rho \}$, $\beta = \{ j_1,\cdots,j_\rho \}$, and $1 \le i_1 < \cdots < i_\rho \le n ; 1 \le j_1 < \cdots < j_\rho \le n$.

Hence, applying the Cauchy-Binet formula to (4), we have

$$\begin{aligned} C_\rho(H) &= C_\rho(E) C_\rho(F) \\ &= \left(\det(E^\alpha)\right)_{\binom{n}{\rho}\times 1} \left(\det(F_\beta)\right)_{1\times\binom{n}{\rho}} \\ &:= e\ f^* \qquad (6) \end{aligned}$$

From the definition it is clear that

- $rank(C_\rho(H)) = 1$
- $C_\rho(H^T) = (C_\rho(H))^T$

As a consequence of Frobenius full rank factorization

- Every line L_i of $C_\rho(H)$ is such that $L_i = \lambda_i L_s$, where L_s is the row of the $\rho \times \rho$ minors taken from the first *r* and the last *k* rows of *H.*
- If $m - \rho \ge \rho$, $C_\rho(H)$ has at least $\binom{m-k}{\rho}$ null rows .

2.6 Theorem *Let H be a $n \times n$ Hankel matrix with rank $\rho < n$ and with r- characteristic .Then*
(i) H has Drazin index 1 if and only if

$$Trace \ \left(C_\rho(H)\right) \ne 0 .$$

(ii) H has Drazin index l if and only if rank(H^l)=ξ and

$$Trace \ \left(C_\xi\left(H^l\right)\right) \ne 0 .$$

Proof. (i) Since $rank\left(C_\rho(H)\right) = 1$, it follows from (6) that

$$Trace\left(C_\rho(H)\right) = Trace(ef^*) = Trace(f^* e) = f^* e$$

Hence, $Trace \ \left(C_\rho(H)\right)^2 = Trace(C_\rho H)(C_\rho H)$.
Thus, the compound is Drazin index 1 if and only if it has an invertible trace .
Let us consider the three different situations:
1. If $r = \rho$, let $c_{ss} = \det(H_r)$. Then $c_{ss} \ne 0$.
From the structure and properties of $C_\rho(H)$ we obtain

$$c_{ii} = \alpha_i \ c_{si} \ , c_{is} = \alpha_i c_{ss} \ , i \ne s \ ; \ \alpha_s = 1 .$$

Since $Trace\, C_\rho(H) = Trace(e\ f^*)$, it follows that

$$\begin{aligned} Trace\ (C_\rho(H)) &= \sum_i \alpha_i^2 c_{ss} \\ &= (1 + \sum_i \alpha_i^2) c_{ss} . \end{aligned}$$

Then $Trace\ (C_\rho(H)) \ne 0$.
Moreover, if $ind\ (C_\rho(H))=1$, then,

$$\begin{aligned} (C_\rho(H))^D &= (C_\rho(H))^{\#} \\ &= \frac{1}{(Trace(C_\rho(H)))^2} (C_\rho(H)) \\ &= \left(\sum_i \alpha_i^2 c_{ss} \right)^{-2} (e\ f^*) . \end{aligned}$$

If $Trace\ (C_\rho(H))= 0$, $(C_\rho(H))^D = O$.

.

2. If $0 < r < \rho$, let L_s be the row which contains $D_{r+k} := c_{sj}$. Then, since every 2×2 minor of $C_\rho(H)$ equals zero, it follows

$$c_{ll} = \alpha_l c_{sl},\ l \neq s;\ c_{ss}c_{jj} = c_{sj}^{\ 2},$$
$$Trace\ (C_\rho(H) = c_{ss} + \sum_{l\neq s} \alpha_l c_{sl}\ .$$

So, *ind*$(C_\rho(H))$=1 if $\sum_{l\neq s} \alpha_l c_{sl} \neq -c_{ss}$, and $(C_\rho(H))^D$ is the null matrix , otherwise.

3. If $0 = r < \rho$, $k := \rho$, then H is a lower (right) triangular matrix, i.e., $h_{ij} = 0$, *if* $i + j \leq n-1$. Then

$$H = \begin{bmatrix} X \\ I_k \end{bmatrix} H[n-k,\cdots,n/1,\cdots,n] = EF\ ,$$

and a similar reasoning to that of **1.** can be done , with $c_{ss} = \det H[n-k,\cdots,n/n-k,\cdots,n]$.

(ii) By definition, H has Drazin index l if and only if $rank(H^l) = rank(H^{l+1}) = \xi$, or yet, since $H^{l+1} = E(FE)^l F$, H has Drazin index l if and only if FE has Drazin index l-1. In this case $index(C_\xi(H^l)) = 1$.

The rest of the proof follows from **1**. just above. ■

3 Drazin inverse of Toeplitz matrices

Let

$$T = (t_{ij}) = (t_{i-j}),\quad 0 \leq i;\ j \leq n-1;\ t_{ij} \in F$$

be the $n\times n$ matrix of a *Toeplitz* form [4],

$$T = \begin{bmatrix} t_0 & t_{-1} & \cdots & t_{-n+1} \\ t_1 & t_0 & \cdots & t_{-n+2} \\ \vdots & \ddots & \ddots & \vdots \\ t_{n-1} & t_{n-2} & \cdots & t_0 \end{bmatrix}.$$

It is clear that there exist Hankel matrices *H*, *H'* such that

$$T = HJ = JH'$$

where *J* is the nonsingular matrix

$$J = \left(\delta_{i,n-j+1}\right),\ i, j = 1,\cdots,n\ .$$

So, *rank*(*T*)= *rank*(*H*).

Without loss of generality, let *T* =*HJ.* Then if *H*=*EF* such as (4), then

$$T = EFJ := EG \qquad (8)$$

is a full rank factorization of *T.*

3.1 Proposition *Let T be a $n\times n$ Toeplitz matrix with rank $\rho < n$. Let H be an Hankel matrix such that T =HJ and let H=EF be a Frobenius' full rank factorization of H. Then T has group inverse if and only if FJE is invertible.*

In that case,

$$T^{\#} = E(FJE)^{-2} FJ\ .$$

Proof. From (8) it is clear that $rank(T^2) = rank(T)$ if and only if $rank(EFJEF) = rank(EF)$ ′, or yet, if and only if FJE is invertible. In this case , a simple computation shows that $X = E(FJE)^{-2} FJ$ satisfies the algebraic definition of group inverse of *T.* ■

3.2 Remark We observe that , since $J^2 = I$, then $X = JH^{(1,2)}$ is a (1,2)- inverse of T for *T* =*EFJ.* Then a (1,2)-inverse of T is given by $T^{(1,2)} = JF_r^{-1}E_l^{-1}$.

However, we can not say that " T^D *exists if and only if* H^D *exists"*. For example, over the complexes, let

$$T = \begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix} = \begin{bmatrix} -1 & i \\ i & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Then T has Drazin index 1, but $rank(H^2) \neq rank(H)$.

3.3 Proposition *Let T be an $n\times n$ Toeplitz matrix with rank $\rho < n$. Let H be an Hankel matrix such that T=HJ, and let H=EF be a Frobenius'decomposition of H. If $F_1 := F$ and for all natural numbers i>1,*

$$F_i := FJT^{i-2}EF\ ,$$
$$U_l := F_l JEF_l F_l^{(1)} + I_n - F_l F_l^{(1)}$$
$$V_l := F_l^{(1)} F_l JEF_l + I_n - F_l^{(1)} F_l$$

then the following are equivalent

(i) ind(T)=l.

(ii) l is the smallest natural number such that U_l is invertible.

(iii) l is the smallest natural number such that V_l is invertible.

In that case,

$$T^D = T^{l-1}(EU_l^{-1}F_l V_l^{-1}J)$$
$$= T^{l-1}(EU_l^{-2}F_l J)$$
$$= T^{l-1}(EF_l V_l^{-2}J)$$

Proof. Since

$$T = PAQ,\ P = E,\ A = F,\ Q = J\ ,$$

there exist $P' = E_l^{-1}$, $Q' = J$, such that

$$P'PA = A = AQ\,Q'\ .$$

Moreover,

$$T^i = PA_iQ = EF_iJ$$

and , by definition, $ind(T)=l$ if $ind(T^l)=1$. Since

$$P'PF_i = F_i = F_iQ\,Q'\ ,$$

then

$$(T^i)^{(1)} = JF_i^{(1)}E_l^{-1}\ .$$

The rest of the proof follows from the Theorem in [8]. ■

Finally we observe that not all of the foregoing results are true if the matrices are defined over a ring even if it is an integral domain . This research is not included in this text due to page limitations, but it will be part of further publication.

Acknowledgement
We want to thank João P. Alves , ISEC, Coimbra, who helped us with Matlab syntax.

References:

[1] A.Ben-Israel, T.N.E.Greville, *Generalized Inverses- Theory and Applications,* Canadian Math. Soc., 2nd edition, 2000.

[2] S.L.Campbell, Limit behaviour of solutions of singular difference equations, *Linear Algebra and its Applications,* 23 (1979), 167-178.
__________ , *Recent applications of Generalized Inverses,* Boston, M.A, Pitman, 1982.

[3] S.L.Campbell, C.D.Meyer, *Generalized Inverses of Linear Transformations,* Dover Publications, Inc., USA, 1991.

[4] S.L.Campbell, C.D.Meyer ,N.J.Rose, Applications of the Drazin inverse to linear systems of differential equations, *SIAM J. Appl.Math.,* 31(1976), 411-425.

[5] M.C.Gouveia, Generalized Invertibility of Hankel and Toeplitz Matrices, *Linear Algebra and its Applications,* 193 (1993), 95-106.

[6] I.S.Iohvidov, *Hankel and Toeplitz Matrices and Forms, Algebraic Theory,* Birkhauser, Boston, 1982.

[7] Puystjens, M.C.Gouveia, Drazin invertibility for matrices over an arbitrary ring, *Linear Algebra and its Applications,* 385 (2004) ,105-116.