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The Matrix Analog of the Kneser-Süss Inequality

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Abstract: – The Brunn-Minkowski theory is a core part of convex geometry. At its foundation lies the Minkowski addition of convex bodies which led to the definition of mixed volume of convex bodies and to various notions and inequalities in convex geometry. Various matrix analogs of these notions and inequalities have been well known for a century. We present a few new analogs. The major theorem presented here is the matrix analog of the Kneser-Süss inequality.

Key-Words: – Elliptic Brunn-Minkowski theory, Minkowski inequality, Brunn-Minkowski inequality, Kneser-Süss inequality, Minkowski's determinant inequality, Blaschke summation, Matrix Blaschke summation, Mixed determinant, Matrix Kneser-Süss inequality.

1 Introduction

The Brunn-Minkowski theory is a core part of convex geometry. At its foundation lies the Minkowski addition of convex bodies which led to the definition of mixed volume of convex bodies and, implicitly, to the famous Brunn-Minkowski inequality. The latter dates back to 1887. Since then it has led to various notions and a series of inequalities in convex geometry. Various matrix analogs of these notions and inequalities have been well known for over a century and have been widely use in mathematical and engineering applications. Our purpose here is to develop an equivalent series of inequalities for positive definite symmetric matrices.

2 Materials and Methods

2.1 Mixed Determinant and Cofactors

A well known matrix analog of the convex geometry notion of mixed volume is called mixed determinant. Its definition is quoted here as follows:

Definition 1 (Mixed Determinant[a][1]). *Let $A_1, \ldots, A_r$ be $n \times n$ symmetric matrices, $\lambda_1, \ldots, \lambda_r$ be positive scalars. Then the determinant of $\lambda_1 A_1 + \cdots + \lambda_r A_r$ can be written as*

$$D(\lambda_1 A_1 + \cdots + \lambda_r A_r) = \sum \lambda_{i_1} \cdots \lambda_{i_n} D(A_{i_1}, \ldots, A_{i_n}),$$

where the sum is taken over all n-tuples of positive integers $(i_1, \ldots, i_n)$ whose entries do not exceed r. The coefficient $D(A_{i_1}, \ldots, A_{i_n})$, with A_{i_k}, $1 \leq k \leq n$ from the set $\{A_1, \ldots, A_r\}$, is called the **mixed determinant** *of the matrices $A_{i_1}, \ldots, A_{i_n}$.*

[a] The authors choose to quote this definition of mixed determinant in a way analogous to the definition of mixed volume in convex geometry [2, 3].

Properties of Mixed Determinants: Let $A_1, \ldots, A_n, A, B$ and B' be $n \times n$ matrices, $\lambda_1, \ldots, \lambda_n$ be positive scalars.

1.

$$\begin{aligned} D(\underbrace{A, \ldots, A}_{n-1}, B) &= D(\underbrace{A, \ldots, A}_{n-2}, B, A) \\ &= \cdots \\ &= D(A, B, \underbrace{A, \ldots, A}_{n-2}) \\ &= D(B, \underbrace{A, \ldots, A}_{n-1}). \end{aligned}$$

In fact, the mixed determinant is symmetric in its arguments, so in a larger generality one has:

(1)

$$\begin{aligned} D(\underbrace{A, \ldots, A}_{n-k}, \underbrace{B, \ldots, B}_{k}) &= \cdots \\ &= D(\underbrace{B, \ldots, B}_{k}, \underbrace{A, \ldots, A}_{n-k}). \end{aligned}$$

We use the notation $D(A, n-k; B, k)$ to represent any of $D(\underbrace{A, \ldots, A}_{n-k}, \underbrace{B, \ldots, B}_{k}), \ldots, D(\underbrace{B, \ldots, B}_{k}, \underbrace{A, \ldots, A}_{n-k})$ in (1).

2.

(2)

$$D(\lambda_1 A_1, \ldots, \lambda_n A_n) = \lambda_1 \cdots \lambda_n D(A_1, \ldots, A_n).$$

3.

(3)

$$\begin{aligned} D(A_1, \ldots, A_{n-1}, B + B') =& D(A_1, \ldots, A_{n-1}, B) \\ &+ D(A, \ldots, A_{n-1}, B'). \end{aligned}$$

In particular,

$$\begin{aligned} D(\underbrace{A, \ldots, A}_{n-1}, B + B') =& D(\underbrace{A, \ldots, A}_{n-1}, B) \\ &+ D(\underbrace{A, \ldots, A}_{n-1}, B'). \end{aligned}$$

The properties in (2) and (3) follow from the n-**linearity** of the mixed determinant.

One can show that for $n \times n$ matrices A and B:

(4)

$$D(\underbrace{A, \ldots, A}_{n-1}, B) = \frac{1}{n}\left(\begin{vmatrix} a_1 \\ \vdots \\ a_{n-1} \\ b_n \end{vmatrix} + \cdots + \begin{vmatrix} b_1 \\ a_2 \\ \vdots \\ a_n \end{vmatrix} \right),$$

of which the generalization gives an alternative definition of the mixed determinant as in the following remark:

Remark 2 ([1]). *A mixed determinant $D(A_1, A_2, \ldots, A_n)$ of $n \times n$ matrices $A_1, A_2, \ldots, A_n$ can be regarded as the arithmetic mean of the determinants of all possible matrices which have exactly one row from the corresponding rows of $A_1, A_2, \ldots, A_n$.*

Definition 3 (Cofactor Matrix [1]). *The cofactor matrix, $\mathcal{C}A$, of an $n \times n$ matrix A, is the transpose of the well known classical adjoint of A, thus it is defined by*

(5) $$(\mathcal{C}A)_{ij} := (-1)^{i+j} D(A(i|j))$$

where $A(i|j)$ denotes the $(n-1) \times (n-1)$ matrix obtained by deleting the i-th row and the j-th column of the matrix A.

We use a similar notation in matrix theory to represent an analog of the mixed volume $V_1(K, L)$, where K and L are convex bodies, as follows:

Definition 4 ([1]). *$D_1(A, B)$ is the following mixed determinant of $n \times n$ matrices A and B:*

(6) $$D_1(A, B) := D(\underbrace{A, \ldots, A}_{n-1}, B)$$

2.2 Matrix Version of Blaschke Summation and Matrix Analogs of Mixed Volume between Two Convex Bodies

Learning the properties of the notion of Blaschke summation of convex bodies in convex geometry, we introduce its analog in matrix theory as follows:

Definition 5 (Matrix Blaschke Summation [1]). *The Blaschke Summation of the $n \times n$ matrices A and B, denoted by $A \boldsymbol{+} B$, is defined as the matrix whose cofactor matrix is the sum of the cofactor matrices of A and B; that is, it satisfies the following equality:*

(7) $$\mathcal{C}(A \boldsymbol{+} B) = \mathcal{C}A + \mathcal{C}B.$$

Theorem 6 ([1]). *Let $A = [a_{ij}]_{n \times n}, B = [b_{ij}]_{n \times n}$. If*

$$A \cdot B := \sum_{i,j} a_{ij} b_{ij},$$

then, for any positive scalar ε,

(8)

$$nD_1(A, B) = \mathcal{C}A \cdot B = \lim_{\varepsilon \to 0} \frac{D(A + \varepsilon B) - D(A)}{\varepsilon}.$$

It is natural to regard the product $\mathcal{C}A \cdot B$ see the proof in [1] as an equivalent of the mixed volume between two convex bodies A and B. The previous theorem was proved by the asymptotic expansion of the determinant of $A + \varepsilon B$ which is similar to the Steiner's polynomial for the volume of $A+\varepsilon B$, where A and B are convex bodies.

One can easily show that $D(A+B)$ can be expanded as (see [1])

(9)
$$D(A+B) = \left\{ \sum_{i=0}^{n} \binom{n}{i} D(\mathcal{C}A, n-i; \mathcal{C}B, i) \right\}^{1/(n-1)}.$$

Also $D(A+\varepsilon \cdot B)$ can be expanded as

(10)
$$D(A+\varepsilon \cdot B) = \left\{ \sum_{i=0}^{n} \binom{n}{i} \varepsilon^i D(\mathcal{C}A, n-i; \mathcal{C}B, i) \right\}^{1/(n-1)},$$

where $\varepsilon \cdot B = \varepsilon^{1/(n-1)} B$. Then as ε is close to 0, we get

$$D(A+\varepsilon \cdot B) \approx \left\{ D^{n-1}(A) + \varepsilon n D(\mathcal{C}A, n-1; \mathcal{C}B) \right\}^{1/(n-1)},$$

and we have the linear approximation

(11)
$$D(A+\varepsilon \cdot B) \approx D(A) + \varepsilon \frac{n}{n-1} \frac{D(\mathcal{C}A, n-1; \mathcal{C}B)}{D^{n-2}(A)}.$$

Therefore we have the following equality:

Theorem 7 ([1]). *Let A, B be $n \times n$ positive definite symmetric matrices, ε be a positive scalar. Then*

$$(12) \quad \frac{1}{n-1} \mathcal{C}B \cdot A = \lim_{\varepsilon \to 0} \frac{D(A+\varepsilon \cdot B) - D(A)}{\varepsilon},$$

where $\varepsilon \cdot B = \varepsilon^{1/(n-1)} B$.

2.3 The Matrix Analogs of the Brunn-Minkowski, the Minkowski, the Kneser-Süss Inequalities

The following theorem is a well known inequality proved by Minkowski.

Theorem 8 (Minkowski,"the Brunn-Minkowski inequality " [4, 6, 7, 8]). *Let A, B be $n \times n$ positive definite symmetric matrices. Then*

$$(13) \quad D(A+B)^{1/n} \geq D(A)^{1/n} + D(B)^{1/n},$$

with equality if and only if $A = cB$. □

It is called Minkowski's determinant inequality [4, 6, 8], and is a matrix analog of the Brunn-Minkowski inequality in convex geometry. And here are couple of others.

Theorem 9 (Matrix analog of the Minkowski inequality [a][1]). *Let A, B be $n \times n$ positive definite symmetric matrices. Then*

$$(14) \quad D_1(A, B) \geq D(A)^{\frac{n-1}{n}} D(B)^{\frac{1}{n}},$$

with equality if and only if $A = cB$.

Proof. *Using AM-GM inequality: $\frac{1}{n} \operatorname{tr} Q \geq D(Q)^{1/n}$ for any matrix Q with positive eigenvalues, and $A \cdot B := \sum_{i,j} a_{ij} b_{ij}$, it can be easily proved that for $n \times n$ positive definite symmetric matrices A and B,*

$$(15) \quad \operatorname{tr}(AB) = A \cdot B \geq n D(A)^{1/n} D(B)^{1/n},$$

and the equality holds if and only if $AB = cI$, or A is a multiple of B^{-1}; that is, $A = cB^{-1}$. Note that the eigenvalues of the product of two positive definite matrices are positive, since $\lambda(AB) = \lambda(A^{1/2} B A^{1/2})$. Then it follows directly from (15) that

$$\begin{aligned} \mathcal{C}A \cdot B &\geq n D(\mathcal{C}A)^{\frac{1}{n}} D(B)^{\frac{1}{n}} \\ &= n D(A)^{\frac{n-1}{n}} D(B)^{\frac{1}{n}}, \end{aligned}$$

and equality holds if and only if $c_1 A^{-1} = \mathcal{C}A = c_2 B^{-1}$ or $A = cB$, where c_1, c_2, c are constants. □

This inequality is a matrix version of the Minkowski inequality in convex geometry. It can also be shown that the analog of the Brunn-Minkowski inequality (13) is equivalent to the analog of the Minkowski inequality (14). First we shows (14) implies (13). For any positive definite symmetric matrix Q, it follows from (14) that

$$(16) \quad \begin{aligned} \mathcal{C}Q \cdot Q &= n D(Q)^{(n-1)/n} D(Q)^{1/n} \\ &= n D(\mathcal{C}Q)^{1/n} D(Q)^{1/n}. \end{aligned}$$

Letting $Q = A + B$, where A, B are positive definite symmetric matrices, we have

$$\begin{aligned} D(A+B)^{1/n} &= \frac{\mathcal{C}Q \cdot (A+B)}{n D(\mathcal{C}Q)^{1/n}} \\ &= \frac{\mathcal{C}Q \cdot A}{n D(\mathcal{C}Q)^{1/n}} + \frac{\mathcal{C}Q \cdot B}{n D(\mathcal{C}Q)^{1/n}} \\ &= \frac{\mathcal{C}Q \cdot A}{n D(Q)^{\frac{n-1}{n}}} + \frac{\mathcal{C}Q \cdot B}{n D(Q)^{\frac{n-1}{n}}} \\ &\geq D(A)^{1/n} + D(B)^{1/n}. \end{aligned}$$

The last inequality follows from (14). This concludes that (14) implies (13). We will now show (13) implies (14). By (13) and with ε being a positive scalar, we have

[a] Despite lacking of reference literature, the authors believe that this theorem is a well known theorem in matrix theory.

$$
\begin{aligned}
&\frac{D(A+\varepsilon B)-D(A)}{\varepsilon} \\
&= \frac{\left(D(A+\varepsilon B)^{1/n}\right)^n - D(A)}{\varepsilon} \\
&\geq \frac{\left(D(A)^{1/n}+D(\varepsilon B)^{1/n}\right)^n - D(A)}{\varepsilon} \\
&= \frac{\left(D(A)^{1/n}+\varepsilon D(B)^{1/n}\right)^n - D(A)}{\varepsilon} \\
&= \frac{1}{\varepsilon}\Bigg[\Bigg(D(A)+\binom{n}{1}D(A)^{(n-1)/n}\varepsilon D(B)^{1/n} \\
&\quad + \binom{n}{2}D(A)^{(n-2)/n}\varepsilon^2 D(B)^{2/n}+\cdots\Bigg) \\
&\quad - D(A)\Bigg],
\end{aligned}
$$

and as ε approaches 0, we infer that

$$\lim_{\varepsilon\to 0}\frac{D(A+\varepsilon B)-D(A)}{\varepsilon} \geq \binom{n}{1}D(A)^{\frac{n-1}{n}}D(B)^{\frac{1}{n}},$$

which is

$$\mathcal{C}A\cdot B \geq nD(A)^{\frac{n-1}{n}}D(B)^{\frac{1}{n}}$$

or

$$D_1(A,B) \geq D(A)^{\frac{n-1}{n}}D(B)^{\frac{1}{n}}.$$

This concludes the proof that (13) implies (14).

Theorem 10 (Matrix analog of the Kneser-Süss inequality [1]). *Let A, B be $n\times n$ positive definite symmetric matrices. Then*

$$D(A+B)^{\frac{n-1}{n}} \geq D(A)^{\frac{n-1}{n}} + D(B)^{\frac{n-1}{n}}, \tag{17}$$

with equality if and only if $A = cB$.

Proof. *To prove this matrix version of Knesser-Süss inequality, it suffices to show that it is equivalent to the analog of the Brunn-Minkowski inequality* (13). *Using* (13) *we have*

$$
\begin{aligned}
D(A+B)^{\frac{n-1}{n}} &= D(\mathcal{C}A+\mathcal{C}B)^{1/n} \\
&\geq D(\mathcal{C}A)^{1/n}+D(\mathcal{C}B)^{1/n} \\
&= D(A)^{\frac{n-1}{n}}+D(B)^{\frac{n-1}{n}}.
\end{aligned}
\tag{18}
$$

This shows that (13) *implies* (17).

One can easily verifies that an $n\times n$ matrix A is positive definite symmetric if and only if its cofactor matrix $\mathcal{C}A$ is a positive definite symmetric. Let $X = \mathcal{C}A$, $Y = \mathcal{C}B$. Since A and B are positive definite symmetric then so are X and Y.

Using the definition of Blaschke addition and (17)*, we obtain*

$$
\begin{aligned}
D(X+Y)^{1/n} &= D(\mathcal{C}A+\mathcal{C}B)^{1/n} \\
&= D(\mathcal{C}(A+B))^{1/n} \\
&= D(A+B)^{\frac{n-1}{n}} \\
&\geq D(A)^{\frac{n-1}{n}}+D(B)^{\frac{n-1}{n}} \\
&= D(\mathcal{C}A)^{1/n}+D(\mathcal{C}B)^{1/n} \\
&= D(X)^{1/n}+D(Y)^{1/n}.
\end{aligned}
\tag{19}
$$

This shows that (17) *implies* (13)*, and the theorem is proved.* □

The last inequality was unknown in matrix theory. One may recognize the equivalent of this inequality in convex geometry, where volumes replace the determinants and convex bodies replace positive definite symmetric matrices. The convexity version of the last two theorems are given in Appendix A.

3. Conclusion

The matrix Blaschke summation and the AM-GM inequality, $\frac{1}{n}\operatorname{tr} Q \geq D(Q)^{1/n}$ as in the proof of Theorem 9, play important roles in the derivation of matrix analogs of notions and inequalities in convex geometry. These analogs look very similar to their convex geometry version ones. The author believes that a plethora of other matrix inequalities can be obtained by choosing strategic positive definite matrices Q in the AM-GM inequality.

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Appendix A. The Brunn-Minkowski Inequality, the Minkowski Inequality and the Kneser-Süss Inequality in Convex Geometry

Theorem 11 (The Brunn-Minkowski inequality [2, 3, 8]). *Let K, L be convex bodies in $\mathbb{R}^n$. Then*

$$V(K+L)^{1/n} \geq V(K)^{1/n} + V(L)^{1/n}, \tag{20}$$

with equality if and only if K and L are homothetic. □

The theorem now named after Brunn and Minkowski was discovered (for dimensions ≤ 3) by Brunn (1887, 1889) [9, 10]. Its importance was recognized by Minkowski, who gave an analytic proof for the n-dimensional case (Minkowski 1910 [11]) and characterized the equality case; for the latter, see also Brunn (1894) [12].

Theorem 12 (The Minkowski inequality [2, 3, 8]). *Let K, L be convex bodies in $\mathbb{R}^n$. Then*

$$V_1(K, L) \geq V(K)^{\frac{n-1}{n}} V(L)^{\frac{1}{n}}, \tag{21}$$

with equality if and only if K and L are homothetic. □

Theorem 13 (The Kneser-Süss inequality [3]). *Let K, L be convex bodies in $\mathbb{R}^n$. Then*

$$V(K+L)^{\frac{n-1}{n}} \geq V(K)^{\frac{n-1}{n}} + V(L)^{\frac{n-1}{n}}, \tag{22}$$

with equality if and only if K and L are homothetic. □

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