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On a convergence result for sequences of functions with multiple scales

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Abstract: We introduce a version of two-scale convergence that deals with certain non-periodic cases still preserving some of the key properties of two-scale convergence. Examples different from those in traditional periodic two-scale convergence are demonstrated and the relationship with other generalizations of two-scale convergence is discussed.

Key-Words: Two-scale convergence, scale convergence, non-periodic oscillations.

1 Introduction

The two-scale convergence method ([1], [2], [9], [11], [12]) is the most efficient approach of today for periodic homogenization ([3], [4], [5], [8]) of partial differential equations and hence a powerful tool for the study of periodically arranged heterogeneous media. We show how the main compactness result for periodic two-scale convergence can be extended to certain non-periodic situations preserving the most essential properties of the traditional periodic case.

Notation 1 *Let $F(R^N)$ be some space of real valued function defined on R^N. Then $F_\#(Y)$ contains all functions in $F_{loc}(R^N)$ that are periodic with respect to $Y=(0,1)^N$. The space of functions in $F_\#(Y)$ with integral mean value zero over Y is denoted $F_\#(Y)/R$.*

2 Two-scale convergence

We define two-scale convergence in line with the definition in [9].

Definition 2 *A sequence $\{u^h\}$ in $L^2(\Omega)$ is said to two-scale converge to the limit $u_0 \in L^2(\Omega\times Y)$, where $Y=(0,1)^N$ is the unit cube in R^N and $\Omega\subset R^N$ is an open bounded set, if*

$$\lim_{h\to\infty}\int_\Omega u^h(x)v(x,hx)dx=\int_\Omega\int_Y u_0(x,y)v(x,y)dydx$$

with the same u_0 for any $v\in X=L^2(\Omega;C_\#(Y))$.

The following proposition associates two-scale limits with the usual weak limits.

Proposition 3 *If $\{u^h\}$ two-scale converges to u_0 and $v\in L^2(\Omega,C_\#(Y))$, then*

$$u^h \to \int_Y u_0(x,y)dy \text{ weakly in } L^2(\Omega)$$

and

$$v(x,hx)\to\int_Y v(x,y)dy \text{ weakly in } L^2(\Omega).$$

The compactness result below is the most essential feature of two-scale convergence.

Theorem 4 *Any bounded sequence in $L^2(\Omega)$ possesses a subsequence that two-scale converges.*

Proof See [9]. ■

3 Generalizations of two-scale convergence

If we introduce suitable sequences of maps τ^h it is possible to generalize two-scale convergence beyond the periodic setting.

Definition 5 *Let $\Omega, A\subset R^N$ be open, bounded sets, $X\subset L^2(\Omega\times A)$ a linear space and*

$$\tau^h\ :\ X\to L^2(\Omega)$$

linear maps. We say that $\{u^h\}$ two-scale converges to u_0 with respect to $\{\tau^h\}$ if

$$\lim_{h\to\infty}\int_\Omega u^h(x)\tau^h v(x)dx=\int_\Omega\int_A u_0(x,y)v(x,y)dydx$$

for all $v\in X$.

To establish a compactness result corresponding to Theorem 4 we introduce two conditions on $\{\tau^h\}$.

Definition 6 *We say that a sequence of operators*

$$\tau^h : X \to L^2(\Omega)$$

is two-scale compatible with respect to $X \subset L^2(\Omega \times A)$ *if there is a constant* $C > 0$ *such that*

$$\lim_{h\to\infty}\|\tau^h v\|_{L^2(\Omega)} \leq C\|v\|_{L^2(\Omega\times A)}$$

and

$$\|\tau^h v\|_{L^2(\Omega)} \leq C\|v\|_X.$$

We obtain the following compactness result.

Theorem 7 *Let* $\{u^h\}$ *be a bounded sequence in* $L^2(\Omega)$ *and assume that* $\{\tau^h\}$ *is two-scale compatible with respect to a separable Banach space X that is a dense subset of* $L^2(\Omega \times A)$.
Then there exists a subsequence such that for some $u_0 \in L^2(\Omega \times A)$

$$\lim_{h\to\infty}\int_\Omega u^h(x)\tau^h v(x)dx = \int_\Omega\int_A u_0(x,y)v(x,y)dydx$$

for all $v \in X$.
Proof See [6] and [7].

No connection with traditional week limits is contained in our definitions and results this far. Below we introduce such conditions inspired by the properties of periodic two-scale convergence found in Proposition 3.

Definition 8 *Let* $\{u^h\}$ *be any bounded sequence in* $L^2(\Omega)$ *that two-scale converges to* u_0 *with respect to a sequence* $\{\tau^h\}$ *that is two-scale compatible with respect to* $X \subset L^2(\Omega \times A)$. *We say that* $\{\tau^h\}$ *is strongly two-scale compatible if*

$$u^h \to \int_A u_0(x,y)dy \text{ weakly in } L^2(\Omega)$$

and

$$\tau^h v \to \int_A v(x,y)dy \text{ weakly in } L^2(\Omega)$$

for any $v \in X$.

The generalization below is found in [10] and means that $\{hx\}$ is replaced by a fairly arbitrary sequence $\{\alpha^h(x)\}$ of functions defined on Ω.

Definition 9 *Let* Ω *be an open bounded subset of* R^N *and* A *a metrizable compact space,* μ *a Young measure and* $L^2_\mu(\Omega \times A)$ *the space of all* μ*-measurable functions with* μ*-integrable square. Further assume that we have a sequence of measurable functions*

$$\alpha^h : \Omega \to A.$$

We say that a sequence $\{u^h\}$ α^h*-converges to* $u_0 \in L^2_\mu(\Omega \times A)$ *if*

$$\lim_{h\to\infty}\int_\Omega u^h(x)v(x,\alpha^h(x))dx = \int_\Omega\int_A u_0(x,y)v(x,y)d\mu(x,y)$$

for all $v \in L^2(\Omega; C(A))$.

The kind of operators τ^h corresponding to the case introduced above are not in general two-scale compatible and hence the measure μ is not necesarily the Lebesgue measure.

Theorem 10 *Let* $\{u^h\}$ *be a bounded sequence in* $L^2(\Omega)$. *Then there exists a subsequence and an* α^h*-limit* $u_0 \in L^2_\mu(\Omega \times A)$ *such that this subsequence* α^h*-converges to* u_0 *for some Young measure* μ.
Proof See [10].

The main contribution of this paper is to identify conditions sufficient to achieve a kind of scale convergence based on choices of $\{\alpha^h\}$ and X making the corresponding sequence $\{\tau^h\}$ strongly two-scale compatible.

Definition 11 *A sequence* $\{u^h\}$ *in* $L^2(\Omega)$ *is said to* (α^h, λ)*-scale converge to* $u_0 \in L^2(\Omega \times Y)$, *where* $Y = (0,1)^N$ *is the unit cube in* R^N *and* $\Omega \subset \mathrm{R}^N_+$ *is an open bounded set, if*

$$\lim_{h\to\infty}\int_\Omega u^h(x)v(x,\alpha^h(x))dx = \int_\Omega\int_Y u_0(x,y)v(x,y)dydx$$

with the same u_0 *for any* $v \in X = L^2(\Omega; C_\#(Y))$.

To obtain strong two-scale compatibility we introduce sequences of functions $\{\alpha^h\}$ and a space X with suitable characteristics. Let

$$\alpha^h : \mathrm{R}^N_+ \to \mathrm{R}^N_+$$

be a continuous bijection and $\{Y^j\}_{j=1}^{\infty}$ a covering of R^N_+ by unit cubes and $\{Y^j_k\}_{k=1}^{n^N}$ a covering of Y^j with cubes of sidelength n^{-1}, and define

$$\Omega^h_j = (\alpha^h)^{-1}(Y^j) \cap \Omega.$$

Hence, for some $q(h)$

$$\Omega = \cup_{j=1}^{q(h)} \Omega^h_j,$$

and in a similar way we introduce

$$\Omega^h_{j,k} = (\alpha^h)^{-1}(Y^j_k) \cap \Omega$$

where Y^j_k are Y-periodic repetitions of Y_k.
Finally, we assume that $\Omega^h_j \subset N_{r(h)}(x^{h,j})$, where $N_{r(h)}(x^{h,j})$ is a ball centered at $x^{h,j} \in \Omega^h_j$ with radius $r(h) \to 0$ for $h \to \infty$.

Definition 12 W*e say that* $\{\alpha^h\}$ *is asymptotically uniformly distributed if*

$$\lambda((\alpha^h)^{-1}(Y^j_k)) / \lambda((\alpha^h)^{-1}(Y^j)) - \lambda(Y_k) \to 0$$

for any open set $Y_k \subset Y$ *when* $h \to \infty$.

Remark 13 *The definition can be extended to averaging over clusters of cells* Y^j. *For the sake of briefness and lucidity of this paper we use the more transparent definition above.*

This definition enables us to prove the following crucial lemma. χ_{Y_k} is the characteristic function for $\cup_{j=1}^{\infty} Y^j_k$.

Lemma 14 *Assume that* $\{\alpha^h\}$ *is asymptotically uniformly distributed. Then*

$$\chi_{Y_k}(\alpha^h(x)) \to \lambda(Y_k) \text{ weakly in } L^2(\Omega).$$

Proof We have assumed that

$$\left|\lambda((\alpha^h)^{-1}(Y^j_k)) / \lambda((\alpha^h)^{-1}(Y^j)) - \lambda(Y_k)\right| < \varepsilon_h$$

and hence

$$\left|\lambda((\alpha^h)^{-1}(Y^j_k)) - \lambda(Y_k)\lambda((\alpha^h)^{-1}(Y^j))\right| <$$
$$\varepsilon_h \lambda((\alpha^h)^{-1}(Y^j)).$$

We obtain

$$\int_\Omega \chi_{Y_k}(\alpha^h(x)) v(x) dx =$$
$$\sum_{j=1}^{q(h)} \int_{\Omega^h_j} \chi_{Y_k}(\alpha^h(x)) v(x) dx =$$
$$\sum_{j=1}^{q(h)} \int_{\Omega^h_{j,k}} v(x) dx = \sum_{j=1}^{q(h)} \lambda(\Omega^h_{j,k}) v(x^{h,j}),$$

where $x^{h,j} \in \Omega^h_{j,k}$. Obviously

$$\left|\sum_{j=1}^{q(h)} \lambda(\Omega^h_{j,k}) v(x^{h,j}) - \lambda(Y_k) \lambda(\Omega^h_j) v(x^{h,j})\right| \le$$
$$\left|\sum_{j=1}^{q(h)} \varepsilon_h \lambda(\Omega^h_j) v(x^{h,j})\right| \to 0$$

and thus

$$\int_\Omega \chi_{Y_k}(\alpha^h(x)) v(x) dx \to \int_\Omega \lambda(Y_k) v(x) dx$$

for all $v \in D(\Omega)$ and hence for all $v \in L^2(\Omega)$.
We have proven that

$$\chi_{Y_k}(\alpha^h(x)) \to \lambda(Y_k) \text{ weakly in } L^2(\Omega)$$

holds true.■

We are now ready to prove that we have found sufficient conditions on $\{\alpha^h\}$ to make scale convergence strongly two-scale compatible for an appropriate choice of the admissible space *X*.

Proposition 15 *Let* $v \in L^2(\Omega; C_\#(Y))$ *and assume that* $\{\alpha^h\}$ *is a sequence of functions that are asymptotically uniformly distributed. Then*

$$v(x, \alpha^h(x)) \to \int_Y v(x,y) dy \text{ weakly in } L^2(\Omega)$$

and

$$u^h \to \int_Y u_0(x,y) dy \text{ weakly in } L^2(\Omega).$$

Further,

$$\left\|v(x, \alpha^h(x))\right\|_{L^2(\Omega)} \to \|v\|_{L^2(\Omega \times Y)}$$

and

$$\left\|v(x, \alpha^h(x))\right\|_{L^2(\Omega)} \to \|v\|_{L^2(\Omega; C_\#(Y))}$$

This means that $\{\tau^h\}$ *is strongly two-scale compatible.*

Proof Let χ_{Y_k} be the characteristic function for $\cup_{j=1}^{\infty} Y_k^j$ and define

$$v_n(x,y) = \sum_{k=1}^{n^N} v(x,y_k)\chi_{Y_k}(y).$$

If $\{\alpha^h\}$ is asymptotically uniformly distributed it holds for any fixed $y_k \in Y_k$ that

$$\int_\Omega v(x,y_k)\chi_{Y_k}(\alpha^h(x))dx \to \int_\Omega v(x,y_k)\lambda(Y_k)dx.$$

Summing over k we obtain

$$\int_\Omega v_n(x,\alpha^h(x))dx \to \int_\Omega\int_Y v_n(x,y)dydx.$$

The rest of the proof follows exactly along the lines of the second half of the proof of Lemma 2 in [1] if we replace $\{hx\}$ with $\{\alpha^h(x)\}$.■

In the same way as in usual two-scale convergence the second scale vanishes in the limit if the sequence $\{u^h\}$ is strongly convergent.

Corollary 16 *Let $\{u^h\}$ be a strongly convergent sequence in $L^2(\Omega)$ with limit u and assume that $\{\alpha^h\}$ is asymptotically uniformly distributed. Then, up to a subsequence and for all $v \in L^2(\Omega; C_\#(Y))$,*

$$\lim_{h\to\infty}\int_\Omega u^h(x)v(x,\alpha^h(x))dx = \int_\Omega\int_Y u(x)v(x,y)dydx$$

Proof We combine the assumption of strong convergence of $\{u^h\}$ with Proposition 15.

Finally, we give a few simple examples of cases contained in our concept of strongly two-scale compatible λ-scale convergence.

Example 17 *We start with the case corresponding to usual periodic two-scale convergence. Let*

$$\alpha^h(x) = hx,\ x \in \Omega = (1,5).$$

Then

$$Y^j = (j, j+1), j \geq h$$

and

$$Y_k^j = (a_k^j, b_k^j), a_k^j, b_k^j \in (j, j+1)$$

with

$$b_k^j - a_k^j = \lambda(Y_k).$$

(All the assumptions above will be used also in the next two examples.)
We obtain

$$(\alpha^h)^{-1}(Y^j) = (h^{-1}j, h^{-1}(j+1))$$

and

$$(\alpha^h)^{-1}(Y_k^j) = (h^{-1}a_k^j, h^{-1}b_k^j)$$

and hence it is obvious that

$$\frac{\lambda((\alpha^h)^{-1}(Y_k^j))}{\lambda((\alpha^h)^{-1}(Y^j))} \to \lambda(Y_k).$$

Example 18 *In our second example α^h is still continuous with $\alpha^h(1) = h$ but grows with different constant speeds while its values passes through different periods Y^j. We have*

$$\frac{d}{dx}\alpha^h(x) = h_j,\ x \in (\alpha^h)^{-1}(Y^j),$$

where $h_j(h) \to \infty$. Then

$$(\alpha^h)^{-1}(Y^j) = ((\alpha^h)^{-1}(j), (\alpha^h)^{-1}(j) + h_j^{-1})$$

and

$$(\alpha^h)^{-1}(Y_k^j) = \left((\alpha^h)^{-1}(a_k^j), (\alpha^h)^{-1}(a_k^j) + h_j^{-1}(b_k^j - a_k^j)\right).$$

Hence

$$\frac{\lambda((\alpha^h)^{-1}(Y_k^j))}{\lambda((\alpha^h)^{-1}(Y^j))} \to \lambda(Y_k)$$

and it is thus clear that $\{\alpha^h\}$ is asymptotically uniformly distributed.

Example 19 *Finally, we let α^h be non-linear. For*

$$\alpha^h(x) = hx^2$$

we obtain

$$(\alpha^h)^{-1}(Y^j) = (\sqrt{h^{-1}j}, \sqrt{h^{-1}(j+1)})$$

and

$$(\alpha^h)^{-1}(Y_k^j) = (\sqrt{h^{-1}a_k^j}, \sqrt{h^{-1}b_k^j}).$$

For large values of h also j will be large. Hence

$$\frac{\lambda((\alpha^h)^{-1}(Y_k^j))}{\lambda((\alpha^h)^{-1}(Y^j))} = \frac{\sqrt{b_k^j} - \sqrt{a_k^j}}{\sqrt{j+1} - \sqrt{j}} \to \lambda(Y_k)$$

and thus $\{\alpha^h\}$ is asymptotically uniformly distributed.

4 Conclusions

The results in this report demonstrates that, under certain restrictions on the sequences $\{\alpha^h\}$ of functions, it is possible to make meaningful generalizations of two-scale convergence along the lines of scale convergence without involving any other measure than the Lebesgue measure.

References:

[1] G. Allaire, Homogenization and two-scale convergence, *SIAM J. Math. Anal.*, Vol. 23, No. 6, 1992, pp 1482-1518.

[2] G. Allaire, M. Briane, Multi-scale convergence and reiterated homogenization, *Proc. Roy. Soc. Edinburgh Sect. A*, Vol. 126, No. 2, 1996.

[3] N. Bakhvalov, G. Panasenko, *Homogenization: Averaging processes in periodic media*, Kluwer Academic Publishers, London, 1989.

[4] A. Bensoussan, J. L. Lions, G. Papanicolau, *Asymptotic analysis for periodic structures. Studies in mathematics and its applications*, North-Holland, 1978.

[5] D. Cioranescu, P. Donato, *An introduction to homogenization*, Oxford University Press Inc., New York, 1999.

[6] A. Holmbom, J. Silfver, N. Svanstedt, N. Wellander, On the relationship between some weak compactnesses with different numbers of scales, *Chalmers Finite Element Center,* Preprint 2003-25, Göteborg, 2003.

[7] A. Holmbom, J. Silfver, N. Svanstedt, N. Wellander, On two-scale convergence and related sequential compactness topics, To appear.

[8] V. Jikov, S. Kozlov, O. Oleinik, *Homogeniztaion of differential operators and variational problems*, Springer-Verlag, Berlin, 1994. Longman Sci. Tech., Harlow, 1994.

[9] D. Lukkassen, G. Nguetseng and P. Wall, Two-scale convergence, *Int. J. Pure and Appl. Math.*, Vol. 2, No. 1 2002, pp. 35-86.

[10] M. L. Mascarenhas, A-M Toader, Scale convergence in homogenization, *Numer. Funct. Anal. Optim.,* 22 2001, No. 1-2, pp. 127--158.

[11] G. Nguetseng; A general convergence result for a functional related to the theory of homogenization, *SIAM J. Math. Anal.*, Vol. 20, No. 3, 1989, pp. 608-623.

[12] G. Nguetseng; Homogenization structures and applications I, *Journal for Analysis and its Applications,* Vol. 22 No. 1, 2003, pp. 73-107.