Proceedings of the 9th WSEAS International Conference on Applied Mathematics, Istanbul, Turkey, May 27-29, 2006 (pp256-262)

Quantization with Fractional Calculus

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Abstract: As a continuation of Riewe's pioneering work [*Phys. Rev. E* **55**, 3581(1997)], the canonical quantization with fractional drivatives is carried out according to the Dirac method. The canonical conjugate-momentum coordinates are defined and turned into operators that satisfy the commutation relations, corresponding to the Poisson-bracket relations of the classical theory. These are generalized and the equations of motion are redefined in terms of the generalized brackets. A generalized Heisenberg equation of motion containing fractional derivatives is introduced.
Key-Words:- Hamiltonian Formulation, Canonical Quantization, Fractional Calculus, Non-conservative systems.

1. Introduction

Most advanced methods of classical mechanics deal only with conservative systems, although all natural processes in the physical world are nonconservative. Classically or quantum-mechanically treated, macroscopically or microscopically viewed, the physical world shows different kinds of dissipation and irreversibility. Mostly ignored in analytical techniques, this dissipation appears in friction, Brownian motion, inelastic scattering, electrical resistance, and many other processes in nature.

Many attempts have been made to incorporate nonconservative forces into Lagrangian and Hamiltonian formulations; but those attempts could not give a completely consistent physical interpretation of these forces. The Rayleigh dissipation function, invoked when the frictional force is proportional to the velocity [1], was the first to be used to describe frictional forces in the Lagrangian. However, in that case, another scalar function was needed, in addition to the Lagrangian, to specify the equations of motion. At the same time, this function does not appear in the Hamiltonian. Accordingly, the whole process is of no use when it is attempted to quantize nonconservative systems.

The most substantive work in this context was that of Riewe[2,3] who used fractional derivatives to study nonconservative systems and was able to generalize the Lagrangian and other classical functions to take into account nonconservative effects.

As a sequel to Riewe's work, Rabei *et al.* [4] used Laplace transforms of fractional integrals and fractional derivatives to develop a general formula for the potential of any arbitrary force, conservative or nonconservative. This led directly to the consideration of the dissipative effects in Lagrangian and Hamiltonian formulations.

Nonconservative systems can be incorporated easily into the equation of motion using the Newtonian procedure; but it is difficult to quantize systems with this procedure. The only scheme for the quantization of dissipative systems seems to be the stochastic quantization procedure [5]. This procedure leads to the nonlinear Schrödinger-Langevin equation. The reason for the impossibility of the direct quantization of nonconservative systems is the absence of the proper Lagrangian or Hamiltonian. Riewe has used fractional calculus to construct the Lagrangian and the Hamiltonian for such systems [2, 3]. In particular, he has shown that using fractional derivatives it is possible to construct a complete mechanical description of nonconservative systems, including Lagrangian and Hamiltonian mechanics, canonical transformations, Hamilton-Jacobi theory, and quantum mechanics. But the wave function for the damped harmonic oscillator is written in terms of three coordinates x, $x_{1/2}$, and $x_{-1/2}$; while we have two canonical conjugate momenta. Thus, one of the coordinates is not physical. In addition, Riewe has mentioned neither Poisson's brackets nor the commutators. Riewe also did not consider the causality so a mistake has appeared when he apply his theorem on the example which he has introduced as an illustration[8, 9].

In this paper we will show how to quantize nonconservative, or dissipative, systems using fractional calculus. The correct canonical conjugate variables will be determined. The Poisson brackets and the quantum commutators will be generalized to include fractional derivatives. Besides, the equation of motion in terms of Poisson brackets will be introduced, and the wave function for the damped harmonic oscillator will be obtained in terms of two canonical coordinates.

The paper is arranged as follows. In Section 2, we introduce some concepts of fractional calculus. In Section 3, Riewe's fractional Hamiltonian mechanics is reviewed. In Section 4, the canonical conjugate variables are determined. This leads to Poisson's brackets, the generalized Hamilton's equation in terms of these brackets, and the commutation relations. In addition, we introduce a generalized form of Heisenberg's equation of motion. An illustrative example, given by Riewe[2,3], is discussed according to our quantization procedure in Section 5. Some concluding remarks follow in Section 6.

2. Riewe's Fractional Hamiltonian Mechanics

Riewe [2, 3] started with the Lagrangian $L(q_{r,s(i)},t)$ which is a function of time t and the set of all $q_{r,s(i)}$, where $r=1,\cdots,R$ indicates the particular coordinate (forexampl $x_1=x,\ x_2=y,\ x_3=z$) and $s(i)$ indicates the order of the ith derivative, $i=1,\cdots,N$. He then used the conventional calculus of variations in classical mechanics to obtain the following generalized Euler-Lagrange equation:

$$\sum_{i=0}^{N}(-1)^{s(i)}\frac{d^{s(i)}}{d(t-a)^{s(i)}}\frac{\partial L}{\partial q_{r,s(i)}}=0, \quad (11)$$

where, for each order of derivative in the Lagrangian, the generalized coordinates $q_{r,s(i)}$ are defined as

$$q_{r,s(i)}=q_{r,s(i),b}=\frac{d^{s(i)}x_r}{d(t-b)^{s(i)}}. \quad (12)$$

Here $s(i)$ can be any non-negative real number. We define $s(0)$ to be 0; so that $q_{r,s(0)}$ denotes the coordinate x_r [2, 3]. In Eq.s (11) and (12) Riewe used left hand differentiations on right handed coordinates, we think that this what causes the mistake appeared in his illustration[2,3,8]. To go over this conflict we will use the left handed coordinates, i.e.,

$$q_{r,s(i)}=q_{r,s(i),a}=\frac{d^{s(i)}x_r}{d(t-a)^{s(i)}}$$

over the whole present work, in Riewe's an in ours. This will introduce the causal appearance of our work. If in any case the Lagrangian is an anticausal or a mixed one, then Riewe's original equations and definitions may be used with a correction of use the left operation with the left coordinates, and so the right operation with the right coordinates.

In order to derive the generalized Hamilton's equations, Riewe [2, 3] defined the generalized momenta as follows:

$$p_{r,s(i)} = p_{r,s(i),a}$$
$$= \sum_{k=0}^{N-i-1} (-1)^{s(k+i+1)-s(i+1)} \frac{d^{s(k+i+1)-s(i+1)}}{d(t-a)^{s(k+i+1)-s(i+1)}}$$
$$\times \left\{ \frac{\partial L}{\partial q_{r,s(k+i+1)}} \right\}$$
(13)

Thus, the Hamiltonian reads

$$H = \sum_{i=1}^{N} q_{r,s(i)} p_{r,s(i-1)} - L \,, \qquad (14)$$

and the Hamilton's equations of motion are defined as [2, 3]

$$\frac{\partial H}{\partial q_{r,s(i)}} = (-1)^{s(i+1)-s(i)} \frac{d^{s(i+1)-s(i)}}{d(t-a)^{s(i+1)-s(i)}} p_{r,s(i)} \qquad (15)$$

$$\frac{\partial H}{\partial p_{r,s(i)}} = q_{r,s(i+1)} \,; \qquad (16)$$

and

$$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} \,. \qquad (17)$$

4. Quantization with Fractional Calculus

4.1 Canonical Conjugate Variables and Poisson Brackets

The process of quantizing the Hamiltonian starts with changing the coordinates $q_{r,s(i)}$ and momenta $p_{r,s(i)}$ into operators satisfying commutation relations which correspond to the Poisson-bracket relations of the classical theory [11]. But the first step in our work is to determine which of the $p_{r,s(i)}$ and $q_{r,s(i)}$ are the canonical conjugate variables.

This canonical-conjugate relation could be obtained directly from Hamilton's equation defined by Riewe [2, 3], Eq. (16), as follows:

$$\frac{\partial H}{\partial p_{r,s(i)}} = q_{r,s(i+1)}$$
$$= \frac{d^{s(i+1)-s(i)}}{d(t\text{-}a)^{s(i+1)-s(i)}} \left(\frac{d^{s(i)-s(i+1)}}{d(t-a)^{s(i)-s(i+1)}} \right) q_{r,s(i+1)}$$

$$= \frac{d^{s(i+1)-s(i)}}{d(t\text{-}a)^{s(i+1)-s(i)}} q_{r,s(i)} \;, \quad 0 \le i \le N\text{-}1$$
. (18)

We conclude that $p_{r,s(i)}$ is the canonical conjugate of $q_{r,s(i)}$.

We can then introduce the Hamiltonian in the form

$$H = \sum_{i=0}^{N-1} \frac{d^{s(i+1)-s(i)}}{d(t-a)^{s(i+1)-s(i)}} q_{r,s(i)} p_{r,s(i)} - L \,,$$
$0 \le i \le N$-1

$$= \sum_{i=0}^{N-1} q_{r,s(i+1)} p_{r,s(i)} - L \,. \qquad (19)$$

This is equivalent to Riewe's Hamiltonian. It is applicable to higher-order Lagrangians with integral derivatives obtained by Pimental and Teixeira [12].

Now, let us define the most general classical Poisson bracket for any two functions, F and G, in phase space:

$$\{F,G\}=\sum_r\sum_{k=0}^{N-1}\frac{\partial F}{\partial q_{r,s(k)}}\frac{\partial G}{\partial p_{r,s(k)}}-\frac{\partial F}{\partial p_{r,s(k)}}\frac{\partial G}{\partial q_{r,s(k)}}$$

(20)

The fundamental Poisson brackets read

$$\{q_{r,s(i)},p_{l,s(j)}\}=\sum_m\sum_{k=0}^{N-1}\frac{\partial q_{r,s(i)}}{\partial q_{m,s(k)}}\frac{\partial p_{l,s(j)}}{\partial p_{m,s(k)}}-\frac{\partial q_{r,s(i)}}{\partial p_{m,s(k)}}\frac{\partial p_{l,s(j)}}{\partial q_{m,s(k)}},\quad 0\le i,j\le N\text{-}1$$

$$=\delta_{ij}\delta_{rl}\,. \qquad (21)$$

Substituting integral derivatives, one can recover the well-known definition of Poisson brackets.

According to our definition of the Hamiltonian, Hamilton's equations of motion can be written in terms of Poisson brackets as

$$\frac{d^{s(i+1)-s(i)}}{d(t-a)^{s(i+1)-s(i)}}q_{r,s(i)}=q_{r,s(i+1)}=\{q_{r,s(i)},H\} \qquad (22)$$

$$(-1)^{s(i+1)-s(i)}\frac{d^{s(i+1)-s(i)}}{d(t-a)^{s(i+1)-s(i)}}p_{r,s(i)}=-\{p_{r,s(i)},H\}$$

(23)

These two definitions are valid for higher-order Lagrangians with integral derivatives and lead to the same definitions given by Pimental and Teixeira [12]. This means that our generalized definitions are applicable for fractional and integral systems as well.

4.2 Quantum Mechanical Operator Brackets

We can now connect the canonical conjugate variables quantum mechanically by defining the momentum operator as

$$p_{s(i)}=\frac{\eta}{i}\frac{\partial}{\partial q_{s(i)}},\quad i=0,1,\ldots,N-1. \quad (24)$$

The correspondence between the quantum-mechanical operator bracket and the classical Poisson bracket is

$$[q_{r,s(i)},p_{r,s(i)}]\Psi=[q_{r,s(i)}p_{r,s(i)}-p_{r,s(i)}q_{r,s(i)}]\Psi \qquad (25)$$

$$=\frac{\eta}{i}\left[q_{s(i)}\frac{\partial}{\partial q_{s(i)}}-\frac{\partial}{\partial q_{s(i)}}q_{s(i)}\right]\Psi$$

$$=i\eta\Psi\,; \quad (26)$$

and the Schrödinger equation reads

$$H\Psi=i\eta\frac{\partial}{\partial t}\Psi\,. \quad (27)$$

Thus, the commutators of the quantum-mechanical operators are proportional to the corresponding classical Poisson brackets:

$$[q_{r,s(i)},p_{r,s(i)}]\leftrightarrow i\eta\{q_{r,s(i)},p_{r,s(i)}\}. \quad (28)$$

4.3 Generalization of Heisenberg's Equation of Motion

For any operator Q, Heisenberg's equation of motion states that [13, 14]

$$\frac{d}{dt}\hat{Q} = \frac{1}{i\eta}\left[\hat{Q},\hat{H}\right]. \qquad (29)$$

This equation can be generalized for coordinate operators as

$$\frac{d^{s(i+1)-s(i)}}{d(t-a)^{s(i+1)-s(i)}}\hat{q}_{r,s(i)} = \frac{1}{i\eta}\left[\hat{q}_{r,s(i)},\hat{H}\right],$$

(30)

and for momentum operators as

$$(-1)^{s(i+1)-s(i)}\frac{d^{s(i+1)-s(i)}}{d(t-a)^{s(i+1)-s(i)}}\hat{p}_{r,s(i)} = -\frac{1}{i\eta}\left[\hat{p}_{r,s(i)},\hat{H}\right]$$

. (31)

Equations (30) and (31) are valid for integer-order derivatives as well as non-integer order.

6. Conclusion

We have demonstrated that the canonical quantization procedure can be applied to nonconservative systems using fractional derivatives.

This procedure should be very helpful in quantizing nonconservative systems related to many important physical problems: either where the ordinary quantum-mechanical treatment leads to an incomplete description, such as the energy loss by charged particles when passing through matter; or where it leads to complicated nonlinear equations such as Brownian motion.

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