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Recent Advances in Nonlinear Optimal Feedback Control Design

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Abstract: - This paper presents a detailed perspective of two constructive feedback strategies characterized by algebraic Riccati equations for solving continuous-time nonlinear deterministic optimal control problems. In each case, the system is first transformed into a *linear-like*, *state-dependent* structure. The first method is based on treating the adapted model as an *instantaneously linear* time-invariant system to approximate it around each point along a trajectory. The control applied at a particular point in state space is determined by solving an *infinite-time* linear-quadratic (LQ) regulator problem using the linear model for the particular point. The resultant control law, which is of feedback form and nonlinear in the state, involves finding the steady-state solution to the ***S**tate-**D**ependent **R**iccati **E**quation* (***SDRE***) at each point. The SDRE method has become exceptionally well-known within the control community for solving autonomous nonlinear regulator problems. Unfortunately, the undeveloped theory of the infinite-time LQ optimal tracking problem has hindered its application for solving nonlinear tracking problems. Therefore, an approximate approach is introduced in this paper for adapting the SDRE tracking methodology. The second method, still at its infancy, has been introduced recently for solving nonlinear optimal regulator and tracking problems. The algorithm has been inspired by the SDRE concept, and is characterized similarly by solving an ***A**pproximating **S**equence of **R**iccati **E**quations* (***ASRE***) associated with *finite-time* LQ theory. This paper presents the latest results on SDRE and ASRE theories developed in the literature, and focuses on illustrating the application, computational advantage and validity of each advanced control methodology on a realistic simulation example of a ducted fan engine model for high-performance thrust-vectored aircraft, to fill the gap between theory and practice. The proposed methods overcome many of the difficulties and shortcomings of existing methodologies, and deliver computationally simple, yet effective, algorithms for constructively synthesizing nonlinear optimal feedback controls.

Key-Words: - Nonlinear control systems; Optimal control; (Non)linear-(non)quadratic problems; Stabilization; Regulation; Tracking; Feedback control design; Continuous-time systems; Aircraft (flight) control systems

1 Introduction

During the 1950's and 1960's, aerospace engineering applications greatly stimulated the development of optimal control theory, where the objective is to derive the system states in such a way that some defined cost function is minimized. This turned out to have very useful applications in the design of *regulators* (where some steady state is to be maintained) and in *tracking* control strategies (where some predetermined state trajectory is to be followed). Among such applications was the problem of optimal flight trajectories for aircraft and space vehicles. Linear optimal control theory, in particular, has been very well documented and widely applied, where the plant that is controlled is assumed linear and the feedback controller is constrained to be linear with respect to its input. In recent years, however, the availability of powerful low-cost microprocessors has spurred great advantages in the theory and applications of nonlinear control. The competitive era of rapid technological change and aerospace exploration now demands stringent accuracy and cost requirements in nonlinear control systems. This has motivated the rapid development of nonlinear optimal control theory for application to challenging complex dynamical real-world problems, particularly those that bear major practical significance in the aerospace, marine and defense industries. Despite recent advances, however, there remain many unsolved problems, so much so that practicians often complain about the inapplicability of contemporary theories. For example, most of the techniques developed are, in fact, only local or have very limited applicability because of the strong conditions imposed on the system. The research in this paper represents an attack on this problem.

The nonlinear optimal control problem, associated with *autonomous* nonlinear *regulator* systems that are *affine* (linear) in the *controls* and with performance indices *quadratic* in the *controls*, has been studied by many authors. The mathematical tools for solving this control-affine nonlinear-quadratic optimization problem are well-known, but their application is usually a very tedious task. In general, with the exception of dynamic programming, the resultant control law is not in feedback form and some iterative technique is often employed for each set of initial conditions. Open-loop control is sensitive to random disturbances and requires that the initial state be on the optimal trajectory. Another difficulty in controlling the nonlinear dynamic system is related to the implementation of optimal control policies. A lengthy preliminary computation is often required, which presents quite unwieldy solutions for controller realization. The exact solution therefore becomes very complex and almost impossible to implement. As a consequence, the practicing engineer often seeks a control law which is close to optimal, with respect to the particular quadratic performance index, and which has attractive features such as feedback, small computations, etc. These problems have led to the study of approximately optimal (suboptimal) control laws which are easier to implement, but sacrifice some performance. Such suboptimal control laws are considered a trade-off between achieving true optimality, which is expensive and complicated to implement, and achieving a system performance which is not optimal but acceptable and inexpensive with ease to implement.

In the sequel, two constructive algorithms are discussed for solving nonlinear-nonquadratic optimal regulator and tracking control problems. The formulation of the problem is presented in Section 2. The results of SDRE theory for nonlinear-nonquadratic optimal regulation are summarized in Section 3. The approximate SDRE tracking counterpart is subsequently introduced. The ASRE algorithm is discussed in Section 4. The uses and shortcomings of any theory can only be appreciated by examining a realistic example. So, in Section 5, the practical use of the SDRE and ASRE strategies is demonstrated with a realistic problem arising in the context of thrust-vectored aircraft. Concluding remarks are given in Section 6.

2 Problem Formulation

To illustrate the development of the theories proposed in this paper, consider full-state observable, deterministic, autonomous, and input-affine nonlinear systems with state $\mathbf{x}(t) \in \mathbb{R}^n$, control $\mathbf{u}(t) \in \mathbb{R}^m$ and output $\mathbf{y}(t) \in \mathbb{R}^p$ ($m, p \le n$), associated with the systems dynamics

$$\left.\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{f}(\mathbf{x}) + \mathbf{B}(\mathbf{x})\mathbf{u}(t), \quad \mathbf{x}(t_0) = \mathbf{x}_0 \\ \mathbf{y}(t) &= \mathbf{g}(\mathbf{x}) \end{aligned}\right\} \quad (1)$$

where $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$, $\mathbf{B} : \mathbb{R}^n \to \mathbb{R}^{n\times m}$, $\mathbf{g} : \mathbb{R}^n \to \mathbb{R}^p$, and the origin $\mathbf{x} = \mathbf{0}$ is an *equilibrium point*, that is, $\mathbf{f}(\mathbf{0}) = \mathbf{0}$, without loss of generality. This can be satisfied by coordinate transformation provided that (1) has an equilibrium point. In this context, the optimization problem is considered as the minimization of a *finite*- or *infinite-time* performance functional, which is *nonquadratic* in $\mathbf{x}$ but *quadratic* in $\mathbf{u}$ such that, at an intermediate value of time $t \in [t_0, t_f]$, the *initial time* $t_0 \in [0, t_f)$, whereas the *final time* $t_f \in (t_0, \infty]$, which is *fixed*. Hence, written in a *quadratic-like* form, the cost is

$$\left.\begin{aligned} J &= \tfrac{1}{2}\mathbf{e}^T(t_f)\mathbf{F}(\mathbf{x}(t_f))\mathbf{e}(t_f) \\ &\quad + \tfrac{1}{2}\int_{t_0}^{t_f} \left\{\mathbf{e}^T(t)\mathbf{Q}(\mathbf{x})\mathbf{e}(t) + \mathbf{u}^T(t)\mathbf{R}(\mathbf{x})\mathbf{u}(t)\right\} dt, \\ \mathbf{e}(t) &\triangleq \mathbf{z}(t) - \mathbf{y}(t), \end{aligned}\right\} \quad (2)$$

where $\mathbf{e}(t)$ is the *error* being minimized, $\mathbf{z}(t) \in \mathbb{R}^p$ is a *desired* (*commanded*) *output* vector, $\mathbf{x}(t_f)$ is *free*, and $\mathbf{u}(\cdot)$ is *unconstrained*. Therefore, the objective is to control system (1) so that the output vector $\mathbf{y}(t)$ follows the commanded output vector $\mathbf{z}(t)$ as "close" as possible. Here, $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^{p\times p}$ and $\mathbf{Q} : \mathbb{R}^n \to \mathbb{R}^{p\times p}$ represent the *end-point* and *state* weighting matrices, respectively, and $\mathbf{R} : \mathbb{R}^n \to \mathbb{R}^{m\times m}$ is the *input* weighting matrix. These are assumed *state-dependent* (hence nonlinear), and can be defined quite generally to include either *stabilization* or *tracking* problems.

In contrast to nonlinear problems, notably in linear control problems the solution can be extremely simple. In the sequel, two methods are devised in an attempt to extend the advantages of this simplicity to the general class of nonlinear control problems (1) and (2).

3 SDRE Theory

Under the assumption $\mathbf{f}(\mathbf{0}) = \mathbf{0}$ and $\mathbf{g}(\mathbf{0}) = \mathbf{0}$, the nonlinear system (1) can be represented in a *linear-like*, *state-dependent* form

$$\left.\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}(\mathbf{x})\mathbf{x}(t) + \mathbf{B}(\mathbf{x})\mathbf{u}(t), \quad \mathbf{x}(t_0) = \mathbf{x}_0 \\ \mathbf{y}(t) &= \mathbf{C}(\mathbf{x})\mathbf{x}(t), \end{aligned}\right\} \quad (3)$$

without any loss of generality. Here $\mathbf{A} : \mathbb{R}^n \to \mathbb{R}^{n\times n}$, $\mathbf{B} : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ and $\mathbf{C} : \mathbb{R}^n \to \mathbb{R}^{l\times n}$ are general nonlinear matrix functions with continuous entries. Hence, analogous to the classical LTV problem, $\mathbf{A}$ becomes the *dynamic coefficient* matrix, $\mathbf{B}$ the *input coupling* matrix, and $\mathbf{C}$ the *measurement sensitivity* matrix of the nonlinear differential equations defining the dynamic and output systems (1). They are found by *mathematical factorization* and are, clearly, *nonunique*.

Hypotheses 1.

(A1) $\mathbf{A}(\cdot)$ and $\mathbf{B}(\cdot)$ are $C^1(\mathbb{R}^n)$ functions.

(A2) $\mathbf{Q}(\mathbf{x})$ is positive-semidefinite and $\mathbf{R}(\mathbf{x})$ is positive-definite $\forall \mathbf{x} \in \mathbb{R}^n$.

(A3) The final time $t_f = \infty$, such that there is no terminal cost, that is, $\mathbf{F}(\mathbf{x}) = \mathbf{0}$.

(A4) The triple $(\mathbf{A}(\mathbf{x}), \mathbf{B}(\mathbf{x}), \mathbf{Q}^{1/2}(\mathbf{x}))$ is pointwise stabilizable and detectable for each $\mathbf{x} \in \mathbb{R}^n$ in the linear-system sense.

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3.1 The Nonlinear State-Regulator Problem

Let us first consider the *infinite-time* nonlinear optimal *stabilization* (*regulation*) problem (2) subject to (3), where $t_0 = 0$, $t_f = \infty$, $\mathbf{z}(t) = \mathbf{0}$ and $\mathbf{C}(\mathbf{x}) = \mathbf{I}_{n\times n}$.

Algorithm 1 (Nonlinear optimal stabilizing SDRE feedback control law [1],[2]).

1. Starting at time $t_0 = 0$, with the initial state $\mathbf{x}(t_0) = \mathbf{x}_0$, evaluate $\mathbf{A}(\mathbf{x})$, $\mathbf{B}(\mathbf{x})$, $\mathbf{Q}(\mathbf{x})$ and $\mathbf{R}(\mathbf{x})$ at the current state $\mathbf{x} = \bar{\mathbf{x}}$ at each sampling instant.
2. At each point $\bar{\mathbf{x}}$ along the trajectory $\mathbf{x}(t)$, evaluate the positive-definite solution of the infinite-time algebraic "*State-Dependent Riccati Equation*" (or *SDRE*)

$$\mathbf{Q}(\mathbf{x}) + \mathbf{P}(\mathbf{x})\mathbf{A}(\mathbf{x}) + \mathbf{A}^T(\mathbf{x})\mathbf{P}(\mathbf{x}) - \mathbf{P}(\mathbf{x})\mathbf{S}(\mathbf{x})\mathbf{P}(\mathbf{x}) = \mathbf{0} \quad (4)$$

with $\mathbf{S}(\mathbf{x}) \triangleq \mathbf{B}(\mathbf{x})\mathbf{R}^{-1}(\mathbf{x})\mathbf{B}^T(\mathbf{x})$.

3. At $\mathbf{x} = \bar{\mathbf{x}}$, apply the nonlinear feedback control

$$\mathbf{u}(\mathbf{x}) = -\mathbf{R}^{-1}(\mathbf{x})\mathbf{B}^T(\mathbf{x})\mathbf{P}(\mathbf{x})\mathbf{x}, \quad (5)$$

so that the SDRE-controlled trajectory is the solution of

$$\dot{\mathbf{x}}(t) = [\mathbf{A}(\mathbf{x}) - \mathbf{S}(\mathbf{x})\mathbf{P}(\mathbf{x})]\mathbf{x}(t). \quad (6)$$

Theorem 1 (SDRE Stability [2]). Consider system (1) with feedback control (5) applied, where $\mathbf{x} \in \mathbb{R}^n$ ($n > 1$) and $\mathbf{P}(\mathbf{x})$ is the unique, symmetric, positive-semidefinite, pointwise-stabilizing solution of the SDRE (4). Then the origin of the resulting closed-loop SDRE-controlled system is *locally asymptotically stable*.

Theorem 2 (SDRE Optimality [2]). Under stability of the nonlinear multivariable system (1) by SDRE feedback (4) and (5), the necessary optimality condition $\frac{\partial H}{\partial \mathbf{u}} = 0$ is always satisfied, whereas $\dot{\boldsymbol{\lambda}} = -\frac{\partial H}{\partial \mathbf{x}}$ is *asymptotically* satisfied. If the optimal cost (value) function $V(\mathbf{x})$ has a gradient of the form $\mathbf{P}(\mathbf{x})\mathbf{x}$, then $\dot{\boldsymbol{\lambda}} = -\frac{\partial H}{\partial \mathbf{x}}$ is also satisfied if, and only if, the matrix $\frac{\partial}{\partial \mathbf{x}}[\mathbf{P}(\mathbf{x})\mathbf{x}]$ is symmetric. The control (5) then generates the *global optimal feedback control* with respect to (2).

Theorem 3 (Scalar Problem [2]). In the *scalar* case ($n = 1$), the origin is globally asymptotically stable. In addition, the symmetry of $\frac{\partial}{\partial \mathbf{x}}[\mathbf{P}(\mathbf{x})\mathbf{x}]$ for the necessary optimality conditions (4) is always satisfied, and so the globally asymptotically stabilizing SDRE feedback solution is also (*globally*) *optimal* on $\mathbb{R}^1$.

3.2 The Nonlinear Tracking Problem

Regardless of the undeveloped theory of *infinite-time* LQ optimal tracking control, a good approximation can be developed for *excessively large* t_f. The SDRE tracking algorithm can then be stated based on this approximate relation as follows:

Algorithm 2 (Nonlinear optimal tracking SDRE feedback control law).

1. Starting at time $t_0 = 0$, with the initial state $\mathbf{x}(t_0) = \mathbf{x}_0$, evaluate $\mathbf{A}(\mathbf{x})$, $\mathbf{B}(\mathbf{x})$, $\mathbf{Q}(\mathbf{x})$ and $\mathbf{R}(\mathbf{x})$ at the current state $\mathbf{x} = \bar{\mathbf{x}}$ at each sampling instant.
2. At each point $\bar{\mathbf{x}}$ along the trajectory $\mathbf{x}(t)$, find the positive-definite solution of the infinite-time *SDRE*

$$\mathbf{C}^T(\mathbf{x})\mathbf{Q}(\mathbf{x})\mathbf{C}(\mathbf{x}) + \mathbf{P}(\mathbf{x})\mathbf{A}(\mathbf{x}) + \mathbf{A}^T(\mathbf{x})\mathbf{P}(\mathbf{x}) - \mathbf{P}(\mathbf{x})\mathbf{S}(\mathbf{x})\mathbf{P}(\mathbf{x}) = \mathbf{0} \quad (7)$$

and the linear vector differential equation

$$\dot{\mathbf{s}}(t) = -[\mathbf{A}(\mathbf{x}) - \mathbf{S}(\mathbf{x})\mathbf{P}(\mathbf{x})]^T \mathbf{s}(t) - \mathbf{C}^T(\mathbf{x})\mathbf{Q}(\mathbf{x})\mathbf{z}(t). \quad (8)$$

3. At $\mathbf{x} = \bar{\mathbf{x}}$, apply the nonlinear feedback control

$$\mathbf{u}(\mathbf{x}) = -\mathbf{R}^{-1}(\mathbf{x})\mathbf{B}^T(\mathbf{x})[\mathbf{P}(\mathbf{x})\mathbf{x}(t) - \mathbf{s}(t)], \quad (9)$$

so that the SDRE-controlled trajectory is the solution of

$$\dot{\mathbf{x}}(t) = [\mathbf{A}(\mathbf{x}) - \mathbf{S}(\mathbf{x})\mathbf{P}(\mathbf{x})]\mathbf{x}(t) + \mathbf{S}(\mathbf{x})\mathbf{s}(t). \quad (10)$$

Remark 1. In the special case when $\mathbf{z}(t) = \mathbf{0}$ in Algorithm 2, the cost functional (2) (with $\mathbf{F} = \mathbf{0}$) simplifies to the *LQ output-regulator* problem, which requires bringing and keeping the output $\mathbf{y}(t)$ "near" zero. Additionally with $\mathbf{C} = \mathbf{I}_{n\times n}$, the tracking problem (2) reduces to the *LQ state-regulator* problem. Hence, the corresponding optimal regulator system, obtained by setting $\mathbf{s}(t) = \mathbf{0}$ in the set of equations (8)-(10), is equivalent to (6) in Algorithm 1.

4 ASRE Theory

The proposed ASRE theory is formulated using classical results of the *finite-time* LQ optimal control problem, by generalizing these classical results to the nonlinear-nonquadratic optimization problem given by (1) and (2). Synonymous to SDRE theory, the application of the ASRE algorithm first requires representing the general nonlinear system (1) in the *linear-like*, *state-dependent* form (3). Let us now formally state the basic conditions required for the ASRE methodology.

Hypotheses 2.

(B1) $\mathbf{A}(\cdot)$, $\mathbf{B}(\cdot)$ and $\mathbf{C}(\cdot)$ are *locally Lipschitz continuous* in their arguments (in this case, only $\mathbf{x}$).

(B2) $\mathbf{F}(\mathbf{x})$ and $\mathbf{Q}(\mathbf{x})$ are positive-semidefinite, and $\mathbf{R}(\mathbf{x})$ is positive-definite $\forall \mathbf{x} \in \mathbb{R}^n$.

Eqs. (3) can now be replaced with the corresponding sequence of LTV problems

$$\left.\begin{aligned} \dot{\mathbf{x}}^{[i]}(t) &= \mathbf{A}(\mathbf{x}^{[i-1]}(t))\mathbf{x}^{[i]}(t) + \mathbf{B}(\mathbf{x}^{[i-1]}(t))\mathbf{u}^{[i]}(t), \mathbf{x}^{[i]}(t_0) = \mathbf{x}_0 \\ \mathbf{y}^{[i]}(t) &= \mathbf{C}(\mathbf{x}^{[i-1]}(t))\mathbf{x}^{[i]}(t) \end{aligned}\right\} \quad (11)$$

for $i = 0, 1, \ldots$, where the sequence is initiated at $i = 0$ with the *initial guess* $\mathbf{x}^{[i-1]}(t) = \mathbf{x}_0$ for the *first iteration*.

Theorem 4 ([3]). Under (B1) of Hypotheses 2, the sequences (11) *converge uniformly* on $[t_0, t]$ to the unique finite bounded solution of (3).

Remark 2. The remarkable fact is that (11) provides a *universal* representation, and is equivalent to the nonlinear dynamics (3) (that is, (1)) in that the proposed LTV problems provide a *global linearization* to the original nonlinear problem, as opposed to the usual linear methods which are only *local* in their applicability. Therefore, assuming that the control input $\mathbf{u}(t)$ is known in advance, the solution to the nonlinear problem (1) is given by the limit of the LTV sequences in (11). For *engineering purposes*, however, the systems given in (11) would actually achieve convergence when the sequence is *near enough* to the exact solution, which merely requires that the *error* as measured by some norm should become small, say $\|\mathbf{x}^{[i]}(t)-\mathbf{x}^{[i-1]}(t)\|\le\sigma$, where σ is some predefined error bound (a constant).

Instead of assuming that $\mathbf{u}(t)$ is an open-loop input, which is known *a priori*, let us now present the ASRE theory for constructing optimal feedback controls for the nonlinear problem (1), where the optimization problem is chosen to minimize the *finite-time* horizon Bolza cost functional (2). In [4], *stabilization* (or *regulation*) was set as the target problem, whereas the *tracking* problem was synthesized in [5]. For ASRE control synthesis presented in this paper, *tracking control* is set as the target problem, since it reduces to the problem of *output regulation* in the special case when the desired output trajectory $\mathbf{z}(t)\equiv\mathbf{0}$, which further simplifies to the problem of *state regulation* if, in addition, the measurement sensitivity matrix $\mathbf{C}\equiv\mathbf{I}_{n\times n}$. The corresponding sequence of LQ costs, from (2), are

$$\left.\begin{aligned}&J=\tfrac{1}{2}\mathbf{e}^{[i]T}(t_f)\mathbf{F}(\mathbf{x}^{[i-1]}(t_f))\mathbf{e}^{[i]}(t_f)\\&\quad+\tfrac{1}{2}\int_{t_0}^{t_f}\left\{\mathbf{e}^{[i]T}(t)\mathbf{Q}(\mathbf{x}^{[i-1]}(t))\mathbf{e}^{[i]}(t)+\mathbf{u}^{[i]T}(t)\mathbf{R}(\mathbf{x}^{[i-1]}(t))\mathbf{u}^{[i]}(t)\,dt\right\},\\&\mathbf{e}^{[i]}(t)=\mathbf{z}(t)-\mathbf{C}(\mathbf{x}^{[i-1]}(t))\mathbf{x}^{[i]}(t).\end{aligned}\right\}\tag{12}$$

Since the sequence of optimal control problems (11) and (12) are each LTV and quadratic, the solution can be approached by classical finite-time LQ optimal control theory. Therefore, the ASRE methodology for the nonlinear-nonquadratic optimal control problem (1) and (2) can be stated based upon this formulation, where the well-established results of finite-time LQ optimal control theory are generalized to give the proposed line of attack. Hence, the optimal tracking control law for the nonlinear-nonquadratic problem is given as follows:

Algorithm 3 (Nonlinear optimal tracking ASRE feedback control law [5]).

1. Given t_f, start at $i=0$ with $\mathbf{x}^{[i-1]}(t)=\mathbf{x}_0$, and use standard numerical procedures to integrate the following respective *Approximating Sequence of Riccati Equations* (*ASRE*) and sequence of linear vector differential equations from $t=t_f$ backwards in time by taking negative time-steps:

$$\begin{aligned}\dot{\mathbf{P}}^{[i]}(t)=&-\mathbf{C}^T(\mathbf{x}^{[i-1]}(t))\mathbf{Q}(\mathbf{x}^{[i-1]}(t))\mathbf{C}(\mathbf{x}^{[i-1]}(t))-\mathbf{P}^{[i]}(t)\mathbf{A}(\mathbf{x}^{[i-1]}(t))\\&-\mathbf{A}^T(\mathbf{x}^{[i-1]}(t))\mathbf{P}^{[i]}(t)+\mathbf{P}^{[i]}(t)\mathbf{S}(\mathbf{x}^{[i-1]}(t))\mathbf{P}^{[i]}(t),\end{aligned}\tag{13}$$

$$\begin{aligned}\dot{\mathbf{s}}^{[i]}(t)=&-[\mathbf{A}(\mathbf{x}^{[i-1]}(t))-\mathbf{S}(\mathbf{x}^{[i-1]}(t))\mathbf{P}^{[i]}(t)]^T\mathbf{s}^{[i]}(t)\\&-\mathbf{C}^T(\mathbf{x}^{[i-1]}(t))\mathbf{Q}(\mathbf{x}^{[i-1]}(t))\mathbf{z}(t),\end{aligned}\tag{14}$$

where $\mathbf{S}(\mathbf{x}^{[i-1]}(t))\triangleq\mathbf{B}(\mathbf{x}^{[i-1]}(t))\mathbf{R}^{-1}(\mathbf{x}^{[i-1]}(t))\mathbf{B}^T(\mathbf{x}^{[i-1]}(t))$, and the respective boundary conditions of (13), (14) are

$\mathbf{P}^{[i]}(t_f)=\mathbf{C}^T(\mathbf{x}^{[i-1]}(t_f))\mathbf{F}(\mathbf{x}^{[i-1]}(t_f))\mathbf{C}(\mathbf{x}^{[i-1]}(t_f))$, and

$\mathbf{s}^{[i]}(t_f)=\mathbf{C}^T(\mathbf{x}^{[i-1]}(t_f))\mathbf{F}(\mathbf{x}^{[i-1]}(t_f))\mathbf{z}(t_f)$.

2. At each sampling instant, evaluate the respective LTV state and control trajectories

$$\begin{aligned}\dot{\mathbf{x}}^{[i]}(t)=&[\mathbf{A}(\mathbf{x}^{[i-1]}(t))-\mathbf{S}(\mathbf{x}^{[i-1]}(t))\mathbf{P}^{[i]}(t)]\mathbf{x}^{[i]}(t)\\&+\mathbf{S}(\mathbf{x}^{[i-1]}(t))\mathbf{s}^{[i]}(t),\end{aligned}\tag{15}$$

$$\mathbf{u}^{[i]}(t)=-\mathbf{R}^{-1}(\mathbf{x}^{[i-1]}(t))\mathbf{B}^T(\mathbf{x}^{[i-1]}(t))\{\mathbf{P}^{[i]}(t)\mathbf{x}^{[i]}(t)-\mathbf{s}^{[i]}(t)\}\tag{16}$$

forwards in time, starting at $t=t_0$ with $\mathbf{x}^{[i]}(t_0)=\mathbf{x}_0$.

3. Repeat steps 1 and 2 for $i=i+1$, where $\mathbf{x}^{[i-1]}(t)$ for $i>0$ presume the corresponding values of $\mathbf{x}(t)$ from the preceding sequence at each sampling instant.

4. If $\|\mathbf{x}^{[k]}(t)-\mathbf{x}^{[k-1]}(t)\|\le\sigma$ for some desired error bound $\sigma>0$, which is defined *a priori*, stop the iteration and apply the ASRE feedback gain

$$\mathbf{K}(\mathbf{x}(t),t)\triangleq-\mathbf{R}^{-1}(\mathbf{x})\mathbf{B}^T(\mathbf{x})\mathbf{P}^{[k]}(t),$$

together with $\mathbf{s}^{[k]}(t)$, to the actual nonlinear system (1).

Theorem 5 ([4],[5]). Given the nonlinear optimal control problem (1) and (2), the sequence of LQ and time-varying optimal control problems (11) and (12) can be introduced. If the control is chosen to minimize (16), that is, the corresponding LQ tracking solution associated with each LTV quadratic problem then, by Hypotheses 2, the sequences of LTV feedback control systems (13)-(16) *converge uniformly* on $[t_0,t]$, which follows from the fact that $\lim_{i\to\infty}\|\mathbf{x}^{[i]}(t)-\mathbf{x}^{[i-1]}(t)\|=0$ and $\lim_{i\to\infty}\|\mathbf{u}^{[i]}(t)-\mathbf{u}^{[i-1]}(t)\|=0$.

Remark 3. In solving the ASRE (13) for $\dot{\mathbf{P}}^{[i]}(t)$ in Algorithm 3, note that $\mathbf{P}^{[i]}(t)=\mathbf{P}(\mathbf{x}^{[i-1]}(t),t)$. Therefore, the limit of the ASRE controls (16) converges to the nonlinear, nonautonomous *feedback* control law

$$\mathbf{u}(\mathbf{x}(t),t)=-\mathbf{R}^{-1}(\mathbf{x})\mathbf{B}^T(\mathbf{x})\{\mathbf{P}(\mathbf{x}(t),t)\mathbf{x}(t)-\mathbf{s}(\mathbf{x}(t),t)\},$$

which is *unique*. The local Lipschitz assumption (B1) assures the existence of a continuous solution $\mathbf{u}$ here as shown in [4]. Therefore, $\dot{\mathbf{x}}$ as given by the ASRE controlled system is a continuous function of $\mathbf{x}$ and t.

Remark 4. The ASRE algorithm for the *state-regulator* problem can be generated from Algorithm 3 by setting $\mathbf{z}(t)=\mathbf{0}$ and $\mathbf{C}(\mathbf{x})=\mathbf{I}_{n\times n}$, thus canceling the tracking features of the method, since $\mathbf{s}$ becomes zero.

5 Thrust-Vectored Ducted Fan Aircraft

The effectiveness of the SDRE and ASRE optimal controllers will now be illustrated in a realistic flight control problem, where the model is representative of the longitudinal dynamics of a *thrust-vectored aircraft*. This example is concerned with high-performance jet aircraft performing *aggressive maneuvers*, where the nonlinearities must be exploited to enhance performance. Very few design methods are capable of achieving efficient control for this class of systems. Vectored propulsion systems provide the ideal platform for improving high-performance capabilities of modern jet aircraft, such as performing rapid transition between hover, forward flight and reverse flight, as well as other aggressive flight maneuvers. Fig. 1 depicts a simple planar model of a *ducted fan engine* for controlling either a *Harier* in hover mode or a *thrust-vectored aircraft* such as F18-HARV or X-31 in forward flight. If (x, y, θ) denote the horizontal, vertical, and angular position (with the vertical), respectively, of a point on the main axis of the fan, and if the aerodynamic effects of lift are ignored, the pitch dynamics corresponding to various flight modes of the system are given by (see [7] for details)

$$m_x\ddot{x} = -d\dot{x} + f_1\cos\theta - f_2\sin\theta$$
$$m_y\ddot{y} = -d\dot{y} + f_1\sin\theta + f_2\cos\theta - m_g g$$
$$J\ddot{\theta} = rf_1.$$

The forces f_1 and f_2 act perpendicular and parallel to the axis of the fan, respectively, with f_1 acting at a distance r from the center of mass. Typically f_2 is much larger than f_1. The parameters (m_x, m_y) are inertial masses of the fan in the (x, y) direction, $m_g g$ is the weight of the fan, J its moment of inertia, and g is the gravitational constant. The drag terms are modeled as viscous friction with d as the viscous friction coefficient.

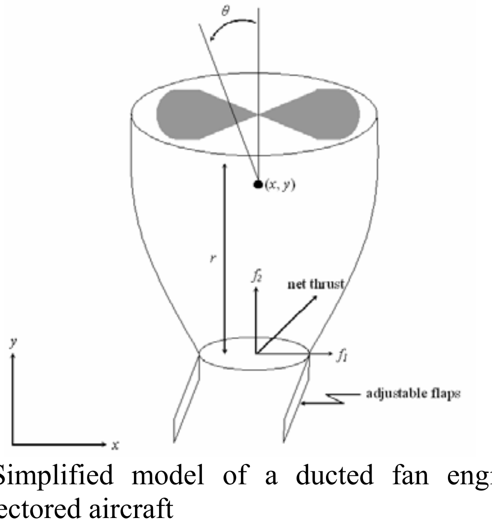

Fig.1. Simplified model of a ducted fan engine for thrust-vectored aircraft

Following [7], the inputs to the system can be defined as $u_1 \triangleq f_1$ and $u_2 \triangleq f_2 - m_g g$, where the control u_2 has been shifted to compensate for gravity, so that the origin of the system is an equilibrium point when there is zero input. Therefore, with u_1 and u_2 as control inputs, the equations of motion of the *planar ducted fan* become

$$m_x\ddot{x} = -m_g g\sin\theta - d\dot{x} + u_1\cos\theta - u_2\sin\theta$$
$$m_y\ddot{y} = m_g g\left(\cos\theta - 1\right) - d\dot{y} + u_1\sin\theta + u_2\cos\theta$$
$$J\ddot{\theta} = ru_1,$$

which have nonlinearities similar to those found on thrust-vectored aircraft, sharing many of the basic nonlinear characteristics of more complicated flight control systems. This particular model has been used for several studies in *nonlinear stabilization*, where numerous design tools that emerged relatively recently have been utilized and compared for addressing optimal performance and stability of nonlinear systems (see [8] and [9] for instance). Popular design techniques include Jacobian linearization, feedback linearization, the use of control Lyapunov functions, receding horizon control, linear parameter varying methods, recursive backstepping, and other hybrid approaches. Although these methods provide powerful tools of designing stabilizing controllers for nonlinear systems, in general there are no guarantees on the performance of the resulting closed-loop systems, which is highly problem dependent and range widely from near optimal to very poor for any given method, as illustrated in [8] and [9] for the problem of stabilizing the thrust-vectored aircraft. However, nonlinear theory directed at *stabilization* about a single operating point may sometimes turn out to be a waste of effort since linear controllers, which generally have a great domain of attraction for stabilizing a single equilibrium point of nonlinear plants, can typically achieve the same objective. For nonlinear systems, and particularly motion control systems, the problem of *tracking* is considerably harder and, while reducing control effort, gives a more aggressive response than point stabilization around a changing output. Therefore, the application of the SDRE and ASRE tracking methodologies (Algorithms 2 and 3) will now be demonstrated on the ducted fan model.

Consider the problem of following a commanded trajectory $\mathbf{z}(t)$ while minimizing a quadratic performance index in form (2), where

$$\mathbf{x} = [x \quad y \quad \theta \quad \dot{x} \quad \dot{y} \quad \dot{\theta}]^T \text{ and } \mathbf{u} = [u_1 \quad u_2]^T.$$

The control objective is to track both x and y trajectories. Therefore, the tracking output $\mathbf{y} = [x \quad y]^T$, where the weighting matrices are chosen as $\mathbf{F} = \mathbf{0}$, $\mathbf{R} = \mathbf{I}_{2\times 2}$ and $\mathbf{Q} = \mathrm{diag}\{1, 10^3\}$. Clearly, with $\frac{\sin\theta}{\theta}$ and $\frac{\cos\theta - 1}{\theta}$ well-defined, the factorization of the planar ducted fan state-space model in form (3) becomes obvious. The specific numerical values for model parameters are given by $m_g = 0.46\,\text{kg}$, $m_x = 4.9\,\text{kg}$, $m_y = 8.5\,\text{kg}$, $r = 0.12\,\text{m}$, $J = 0.05\,\text{kg m}^2$, $d = 1.2$ and, of course, $g = 9.81\,\text{m s}^{-2}$. Simulation results of the closed-loop system in Fig. 2 illustrate the performance achieved using the SDRE and ASRE nonlinear tracking algorithms with a desired tracking output $\mathbf{z}(t) = [0.5t \quad 0.1\sin(0.1t)]^T$, which corresponds to a complex trajectory of commanding the fan to fly rapidly in the horizontal direction at varying altitude, starting from a challenging initial condition

$$\mathbf{x}(0) = [0 \quad 0 \quad -\pi/4 \quad 0 \quad 0 \quad 0]^T .$$

This complex maneuver demonstrates the effectiveness of the proposed controllers, where the commanded trajectory takes the fan far from hover, which is the point linear controllers are usually designed. Therefore, for aggressive trajectories over a wide operating envelope, these controllers perform extremely well, albeit the SDRE-controlled trajectory appears to lag behind the commanded input. Nevertheless, the on-line computation of the SDRE scheme makes this technique ideal for real-time trajectory generation, that is, when the desired trajectory is not known ahead of time, so that the controller must perform all operations in "real-time".

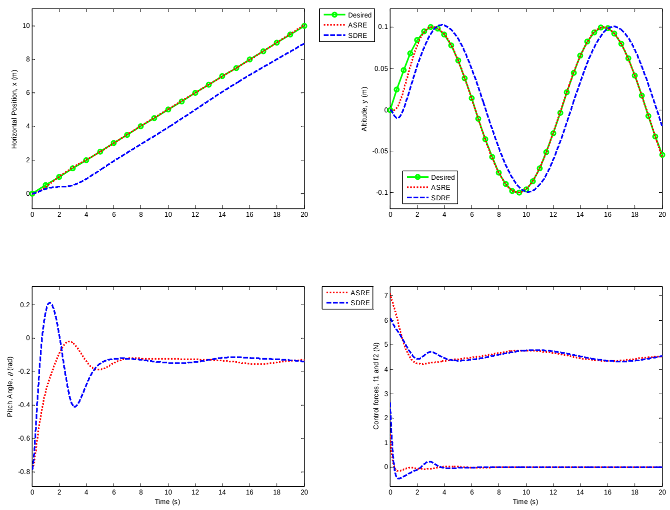

Fig.2. State trajectories and control effort

6 Conclusion

This paper presented a review of synthesizing feedback controllers for nonlinear optimal control problems using two highly effective frameworks, namely,

a) *State-Dependent Riccati Equation*, and

b) *Approximating Sequence of Riccati Equations*.

Both methods are based upon recasting the nonlinear plant into a *linear-like, state-dependent* form, with the solutions characterized by solving algebraic Riccati equations. The SDRE technique requires determination of nonlinear optimal feedback controls *on-line* (at each point) as the solution proceeds. This renders the SDRE algorithm ideal for real-time implementation. The basic idea behind ASRE methodology, on the other hand, is to use an *iteration technique* to transform the associated deterministic nonlinear-nonquadratic optimal control problem into a convergent sequence of time-varying LQ optimal control problems. These can then be solved by classical methods, leading to an optimal control for the nonlinear system. Consequently, feedback controls by ASRE strategy need to be determined *off-line*. Each method provides an effective algorithm based on reliable mechanized procedures of only solving algebraic Riccati equations, which are then used for constructing nonlinear optimal state-feedback controllers. This is achieved by extending the simple, but valuable, design tools of classical LQ optimal control theory to nonlinear systems, which thereby results in computationally simple nonlinear design schemes that overcome several difficulties and shortcomings of LQ designs.

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