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Non-Linear Dynamics of Geared Systems Using the Asymptotic Method, Part 3: Computation of Instability Frontiers

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Abstract: **-** The increased interest for improved gear design has led to extensive research into the field of non-linear dynamics of such systems. The paper reveals a complex dynamic model to study the behavior in a gear-pair system taking into consideration backlash and time-dependent mesh stiffness and mesh damping. In many applications including turbo machinery, machine tools and diesel engines non-linearities are present due to tooth stiffness, damping and backlash that induced micro-vibrations of non-linear parametric type. In the mean time the input link of the driver ax and the output link of the driven ax induce non-linearities. The paper presents the use of asymptotic method in order to compute the frontiers of instability.

Keywords: Dynamics of geared systems, asymptotic method, instability frontiers

Nomenclature
c_i, i =1,2,3: calculus notations;
c_{ti}, i=1,2 : damping torsional coefficients of the driver and driven gear axes [Nms/rad];
c_v : the mesh damping of gear teeth[Ns/m];
c_{vo} : the mean value of mesh damping of gear teeth[Ns/m];
$\tilde{c}_v$: the amplitude of the variation of mesh damping of gear teeth[Ns/m];
$\bar{c}_v$: the mean value of the gear tooth damping[Ns/m];
e_i, i =1,2,3,4: calculus notations;
k_v : the mesh stiffness of gear teeth[N/m];
k_{vo} : the mean value of mesh stiffness of gear teeth[N/m];
$\tilde{k}_v$: the amplitude of the variation of mesh stiffness of gear teeth[N/m];
$\bar{k}_v$: the mean value of the gear tooth stiffness[N/m];
k_{ti}, i=1,2 : the torsional stiffness of the gear axes [Nm/rad];
n_i, i=1,2 : the rotation speed of gear wheels [rot/min];
r_i, $i = \overline{1,2}$: the gear radii of the driver and driven gear wheels[mm];
s : the relative displacement of the gears into the pitch plane[m];
$\bar{s}_1$: the amplitude of the dynamic response in the region of fundamental resonance;
$\bar{s}_2$: the amplitude of the dynamic response in the region of principal parametric resonance;
s_R : the clearance in the pitch plane[mm];
z_i, i=1,2 : the number of teeth of gear wheels;
J_1, J_2 : the mass moments of gear's inertia with respect to O_{z1}, O_{z2} axes[kg m^2];

$M_i(t),\ i=\overline{1,2}$:the external torque's loads [Nm];
M_{i0}, i=1,2 : the constant external torque's loads [Nm];
S : normalized excitation frequency;
α : the excitation parameter due to the mean value of mesh damping of gear teeth;
α_o: the excitation parameter of the amplitude of mesh damping variation of gear teeth;
α_n: the pressure angle [rad];
β: excitation parameter of the amplitude of mesh stiffness variation of gear teeth;
β_n: the helix angle [rad];
χ : the excitation parameter of the amplitude of mesh damping variation of gear teeth normalized by the excitation parameter due to the mean value of mesh damping of gear teeth α;
$\overline{\delta}$: the parameter of the gear clearance;
$\overline{\varepsilon}$: small positive parameter in asymptotic expansion, $0< \overline{\varepsilon} \leq 1$;
γ : excitation parameter due to cubic non-linearity's of the input-output linkages, defined by relation (6);
γ_o : linear excitation parameter due to the stiffness of the driver and driven axes, defined by relation (6);
γ_1: excitation parameter due to the damping of the driver and driven axes;
φ_i, $i=1,2$: rotation angle of gear wheels[rad];
μ : excitation parameter of the amplitude of mesh stiffness variation of gear teeth normalized by the Ω^2;
ν : the excitation frequency representing the frequency of entering the tooth into the pitch plane;
Δ: the logarithmic decrement of the damping;
Γ: the excitation parameter due to constant external torques $M_{i0},\ i=\overline{1,2}$;
$\tilde{\Gamma}$: the amplitude of the excitation parameter due to the harmonic variation of external torques $M_i(t),\ i=\overline{1,2}$;
Ω : the eigen frequency of the gear-pair system.

1. Introduction

The failure of geared systems is due to undesired high-amplitude self-excited vibrations, which is also a cause of instability. As sources of perturbation, for a geared system, may be mentioned: the periodic variation of the tooth stiffness and damping[6-8]; the geometrical errors of the contact between teeth, that produces micro-impacts [11]; the variation of the external torque loads due to the non-linearities of input-output linkages. The first and the second type of excitations, which is present due to clearances and manufacturing tolerances, produce self-induced non-linear parametric vibrations [6], while the third type of excitation increases the non-linearity's effects of the dynamic behavior [9,10]. Many investigators have been working on the modeling of gear transmission systems using linear and non-linear parametric dynamical models, such as Bolinger [1], Bolotin [2], Bosch [4], Brauer [5] and Diekhans [16]. Other investigators followed with theoretical developments and experimental studies. Anyhow, most of them have been investigated the stability problems of geared systems, because such self-induced non-linear parametric vibrations represent a major problem for the stability of the gear transmission systems. All of the investigators started in their theoretical studies with models that cover the physical phenomena's that appear into a gear transmission. The aim of the paper is to improve such models for geared system and to enable the gear's designer to predict the dynamic behavior of non-linear self-induced vibrations for the geared systems.

2. The Physical Model of a Gear Transmission with One Stage

It was considered for the model that the investigation must carried out for the entire pitch plane [16]. This fact is due to the observation that for a cylindrical gear the force is acting on the teeth into a specific spatial direction because of the contact between two teeth. This cause torsional non-linear parametric vibrations. The gear wheels are helical therefore the mesh teeth stiffness $k_v(\varphi_i)$ and the mesh damping $c_v(\varphi_i)$ are influenced by the gear helix angle β_n and the pressure angle α_n [7,16]. This influence is given by the relations:

$$k_v = \overline{k}_v[\cos\alpha_n \cos\beta_n]^2,\quad c_v = \overline{c}_v[\cos\alpha_n \cos\beta_n]^2. \tag{1}$$

The forces will be transmitted through the teeth mesh stiffness $k_v(\varphi_i)$ and the mesh damping $c_v(\varphi_i)$ which depend upon the rotation angle of the wheel. Some authors developed dynamic models that take into consideration the teeth clearance s_R and therefore the influence of micro-impacts of the teeth [15]. The considered model takes into account the backlash s_R by modeling the gear clearance with a non-linear symmetric spring [15] and introduces all the components of the forces in the circumferential direction. The gear load is a function of relative movement between the driver gear and the driven gear.

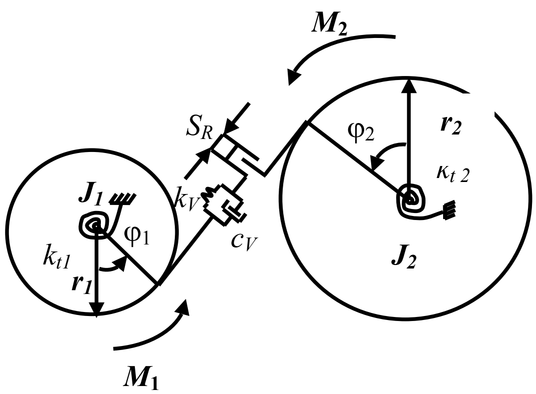

Figure 1. Physical model of a gear-pair system

The new model introduces the torsional stiffness of the gear axes as can be seen from figure 1, that presents a model with two degree of freedoms for a gear-pair system. Using the Lagrange's equations, the differential system of equations for the vibrating movement is

$$J_1\ddot{\varphi}_1 = -c_v \cdot r_1 \cdot \dot{s} - k_v \cdot r_1 \cdot s + M_1$$
$$J_2\ddot{\varphi}_2 = c_v \cdot r_2 \cdot \dot{s} + k_v \cdot r_2 \cdot s - M_2 , \quad (2)$$

where s is the relative displacement of the gears into the pitch plane:

$$s = s(t) = r_1 \cdot \varphi_1 - r_2\varphi_2 . \quad (3)$$

Multiplying the first equation of system (2) by r_1/J_1, the second one by r_2/J_2 and making the difference between them it was computed the differential equation of the non-linear parametric vibrations of the helical gear-pair system in the pitch plane:

$$\ddot{s} + c_v\left[\frac{r_1^2}{J_1} + \frac{r_2^2}{J_2}\right]\cdot\dot{s} + k_v\left[\frac{r_1^2}{J_1} + \frac{r_2^2}{J_2}\right]\cdot s = \frac{M_1 r_1}{J_1} + \frac{M_2 r_2}{J_2}. \quad (4)$$

The right hand term of equation (4) can be expressed as:

$$\frac{M_1 r_1}{J_1} + \frac{M_2 r_2}{J_2} = f(\Gamma,t) + \gamma_0 \cdot s + \gamma_1\dot{s} - \gamma \cdot s^3 \quad (5)$$

where $f(\Gamma, t)$ produces static deflection of the gear teeth and induces the harmonic external forced excitation due to the harmonic variation of the external torques. The other terms induce the effects of torsional stiffness of the gear axes γ_o, the torsional damping of the gear axes γ_1 and the non-linearities of the input-output linkages γ in order to transmit the external torques M_1 and M_2. This terms are defined by the relations:

$$f(\Gamma,t) = \Gamma + \widetilde{\Gamma}\cos\nu t,\ \Gamma = \frac{M_{10}\, r_1}{J_1} + \frac{M_{20}\, r_2}{J_2},\ \widetilde{\Gamma} = 0.1\Gamma$$

$$\gamma_0 = \frac{\gamma_{10}\gamma_{20}}{\gamma_{10} + \gamma_{20}},\ \gamma_{10} = \frac{k_{t1}}{J_1},\ \gamma_{20} = \frac{k_{t2}}{J_2},$$

$$\gamma = 0.001\frac{\gamma_{10}\gamma_{20}}{r_1^2 r_2^2\left[\frac{\gamma_{10}}{r_1^2} + \frac{\gamma_{20}}{r_2^2}\right]},$$

$$\gamma_1 = \frac{\gamma_{11}\gamma_{12}}{\gamma_{11} + \gamma_{12}}, \gamma_{11} = \frac{c_{t1}}{J_1},\ \gamma_{12} = \frac{c_{t2}}{J_2}. \quad (6)$$

considering that the cubic non-linearities are very small and the variation of external torque loads is 10%. In equation (4) can be neglect the term Γ can be neglect because it produces only the static deflection and the paper deals only with the dynamic behavior of the system. Equation (4) is a bi-parametric one because of the variation of $k_v(t)$ and also because the damping $c_v(t)$ that depends upon the stiffness [7,8,16]. Taking into consideration the experimental data [7,8,16], the mesh stiffness $k_v(t)$ and the mesh damping $c_v(t)$ for helical gears can be expressed by the mathematical relations

$$k_v(t) = k_{vo} + \widetilde{k}_v \cos\nu t,$$
$$c_v(t) = c_{vo} + \widetilde{c}_v \cos\nu t. \quad (7)$$

The relation between the mesh stiffness and the mesh damping is given by[7,16]

$$c_v = \frac{\Delta k_v}{\pi\nu}, \quad (8)$$

where the excitation frequency is

$$\nu = \frac{\pi z_1 n_1}{30} = \frac{\pi z_2 n_2}{30}. \quad (9)$$

Equation (4) is a non-linear differential equation because of the non-linear term expressed by (6) that is due to the link flange of the gear transmission [6]. Taking into account that the perturbation in equation (4) has a non-linear variation expressed by relations (5),(6) and also equation (4) has a bi-parametric mathematical form given by the expressions (7)-(9), this can be put into the mathematical form

$$\ddot{s} + (\alpha + \alpha_0 \cos\nu t)\dot{s} + \Omega^2(1 + \mu\cos\nu t)s + \gamma s^3 = \widetilde{\Gamma}\cos\nu t. \quad (10)$$

In equation (10), the eigen frequency Ω and the excitation parameter μ of the mesh stiffness are given by the expressions

$$\Omega = \sqrt{k_{V_0}\left(\frac{r_1^2}{J_1}+\frac{r_2^2}{J_2}\right)-\gamma_0}\,,$$

$$\mu = \frac{\beta}{k_{V_0}\left(\frac{r_1^2}{J_1}+\frac{r_2^2}{J_2}\right)-\gamma_0},\ \beta = \widetilde{k}_{V_0}\left(\frac{r_1^2}{J_1}+\frac{r_2^2}{J_2}\right), \qquad (11)$$

where α and α_0 are the excitation parameter due to the mean value of mesh damping and the excitation parameter of the amplitude of mesh damping variation expressed by the relations

$$\alpha = c_{v0}\left[\frac{r_1^2}{J_1}+\frac{r_2^2}{J_2}\right]-\gamma_1 = \frac{\Delta\, k_{v0}}{\pi\nu}\left[\frac{r_1^2}{J_1}+\frac{r_2^2}{J_2}\right]-\gamma_1,$$

$$\alpha_0 = \widetilde{c}_v\left[\frac{r_1^2}{J_1}+\frac{r_2^2}{J_2}\right] = \frac{\Delta\, \widetilde{k}_v}{\pi\nu}\left[\frac{r_1^2}{J_1}+\frac{r_2^2}{J_2}\right]. \qquad (12)$$

In expressions (12) the relations (7) and (8) were used. From the equation (10) is missing only the term that introduces the backlash effect given by the gear clearances, which induces micro-impacts between the gear teeth flanks. Modeling the gear clearance with a non-linear symmetric spring [15] equation (10) becomes

$$\ddot{s} + (\alpha + \alpha_0 \cos\nu t)\dot{s} + \Omega^2(1+\mu\cos\nu t)(s+\overline{\delta}s^3) + \\ + \gamma s^3 = \widetilde{\Gamma}\cos\nu t. \qquad (13)$$

Equation (13) is a second-order non-homogeneous bi-parametric non-linear differential equation and has a mathematical form of a modified non-linear Mathieu-Duffing-Hill equation. Equation (13) represents the equation of motion of the relative displacement between the gear flanks in the pitch plane taking into account the following phenomena's of the dynamics of helical gear-pair system:

- the time-dependent variation of the gear mesh stiffness and gear mesh damping [5-8],
- the backlash that induces micro-impacts on the teeth flanks [15],
- the interaction between gear mesh stiffness and gear mesh damping [7,8,11],
- the torsional stiffness and the torsional damping of the driver and driven gear axes,
- the non-linearities of the input-output linkages [8],
- the time-dependent variation of the external torques loads [11,15],
- the influence of the gear helix angle and the pressure angle on the mesh teeth stiffness and the mesh damping [6,16].

The stationary response has amplitude in the region of fundamental resonance $\overline{s}_1$ and the amplitude in the region of principal parametric resonance $\overline{s}_2$ given by the expressions

$$c_1^2\,\overline{s}_1^6 + 2c_1c_2\,\overline{s}_1^4 + (c_2^2+c_3^2)\,\overline{s}_1^2 - 1 = 0, \qquad (14)$$

$$\overline{s}_2 = \sqrt{-\frac{e_1e_2+e_3e_4}{e_2^2+e_4^2} \pm \sqrt{\left(\frac{e_1e_2+e_3e_4}{e_2^2+e_4^2}\right)^2 - \frac{e_1^2+e_3^2-1}{e_2^2+e_4^2}} - 2\,\overline{s}_1^2}\,, \qquad (15)$$

after the use of the asymptotic method as can be seen from the part 1 of this paper.

3. Boundaries of the Regions of Principal Parametric Instability

The base width of the stationary dynamic response in the region of principal parametric resonance is the region in which principal parametric vibrations may normally be initiated. By setting the amplitudes $\overline{s}_1$, $\overline{s}_2$ to zero in the region of principal parametric resonance the boundaries of the principal region of instability can be obtained from equation (15). The normalized excitation frequency S and the excitation parameter of the amplitude of mesh damping variation of gear teeth χ are

$$S = \frac{\nu}{2\Omega}, \qquad \chi = \frac{\alpha_0}{\alpha}. \qquad (16)$$

Setting to zero the amplitudes $\overline{s}_1$, $\overline{s}_2$ in equation (15) yields the equation

$$S^4\left[\mu^2\left(\frac{\Omega}{\alpha}\right)^4 + \chi^2\left(\frac{\Omega}{\alpha}\right)^2\right] - \\ -2S^3\left[\mu^2\left(\frac{\Omega}{\alpha}\right)^4 + \chi^2\left(\frac{\Omega}{\alpha}\right)^2\right] + \\ + S^2\left[\chi^2\left(1+\left(\frac{\Omega}{\alpha}\right)^2\right) + \mu^2\left(\left(\frac{\Omega}{\alpha}\right)^2 + \left(\frac{\Omega}{\alpha}\right)^4\right)\right] - \\ -\frac{1}{4}\left[\chi^2 + \mu^2\left(\frac{\Omega}{\alpha}\right)^2\right]^2 = 0 \qquad (17)$$

Equation (17) of fourth degree in S makes it possible to compute the boundaries of the principal region of instability in the parameter normalized space (S, μ, χ).

4.Results and Conclusions

The numerical results for the instability frontiers, based on the equation (17), are presented in the figures 2,3 and 4. The instability region is closed between the surface branches in the parametric space (ν, χ, μ). As can be seen at the increase in the values of Δ the volume of the instability region increases, so that we can conclude that in fact the damping is an excitation factor as expected from the equation (13).

The paper highlights all of the qualitative connections between the values of a wide range of parameters that occur in order to establish the instability frontiers of a helical gear-pair system with backlash parametrically excited.

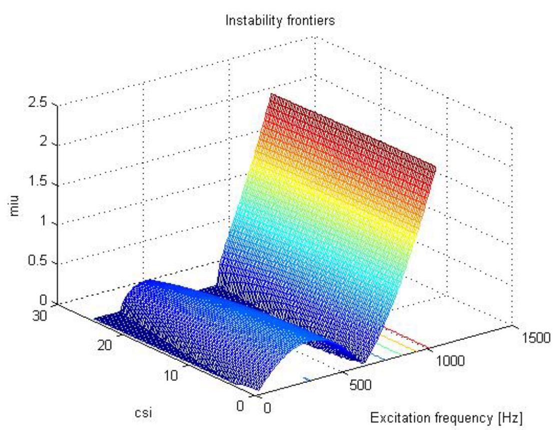

Figure 2. Instability frontiers for Δ=0.05.

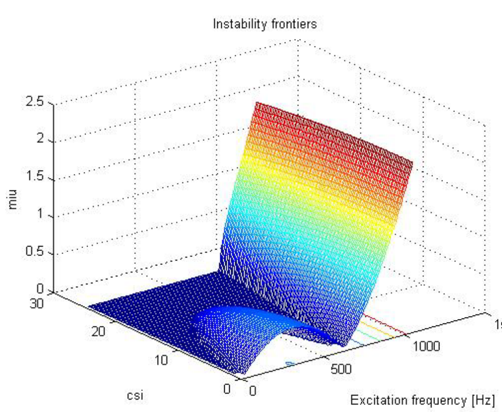

Figure 3. Instability frontiers for Δ=0.1.

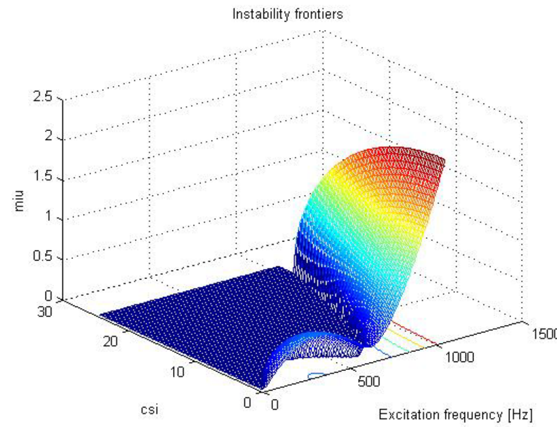

Figure 4. Instability frontiers for Δ=0.3.

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