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Non-Linear Dynamics of Geared Systems Using the Asymptotic Method, Part 2: Computation of the Phase Angle

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Abstract: - The increased interest for improved gear design has led to extensive research into the field of non-linear dynamics of such systems. The paper reveals a complex dynamic model to study the behavior in a gear-pair system taking into consideration backlash and time-dependent mesh stiffness and mesh damping. In many applications including turbo machinery, machine tools and diesel engines non-linearities are present due to tooth stiffness, damping and backlash that induced micro-vibrations of non-linear parametric type. In the mean time the input link of the driver ax and the output link of the driven ax induce non-linearities. The paper presents the use of asymptotic method in order to compute the phase angle of steady state motion. By this way the paper reveals the phenomena's characteristics of multiple jumps specific to the non-linear dynamic behavior of gear-pair due to: non-linearities of the input-output linkages, backlash and self-induced parametric excitations, caused by the tooth stiffness and damping. It was highlighted the interaction between fundamental resonance and the principal parametric resonance.

Keywords: - Non-linear dynamics of geared systems, Asymptotic method, Phase angle

Nomenclature

c_i, i =1,2,3: calculus notations;
c_{ti}, i=1,2 : damping torsional coefficients of the driver and driven gear axes [Nms/rad];
c_v : the mesh damping of gear teeth[Ns/m];
c_{vo} : the mean value of mesh damping of gear teeth[Ns/m];
$\widetilde{c}_v$: the amplitude of the variation of mesh damping of gear teeth[Ns/m];
$\overline{c}_v$: the mean value of the gear tooth damping[Ns/m];
e_i, i =1,2,3,4: calculus notations;
k_v : the mesh stiffness of gear teeth[N/m];
k_{vo} : the mean value of mesh stiffness of gear teeth[N/m];
$\widetilde{k}_v$: the amplitude of the variation of mesh stiffness of gear teeth[N/m];
$\overline{k}_v$: the mean value of the gear tooth stiffness[N/m];
k_{ti}, i=1,2 : the torsional stiffness of the gear axes [Nm/rad];
n_i, i=1,2 : the rotation speed of gear wheels [rot/min];
$r_i, i=\overline{1,2}$: the gear radii of the driver and driven gear wheels[mm];
s : the relative displacement of the gears into the pitch plane[m];
$\overline{s}_1$: the amplitude of the dynamic response in the region of fundamental resonance;
$\overline{s}_2$: the amplitude of the dynamic response in the region of principal parametric resonance;
s_R : the clearance in the pitch plane[mm];
z_i, i=1,2 : the number of teeth of gear wheels;
A_i, B_i, i=1,2 : unknown function in asymptotic expansion, equations (21) and (22);
$H(\tau,\theta,s,\dot{s})$: perturbation term in equation (18);
J_1, J_2 : the mass moments of gear's inertia with respect to O_{z1}, O_{z2} axes[kg m^2];
LHS: left hand side in equations;

$M_i(t),\ i=\overline{1,2}$:the external torques [Nm];
M_{i0}, i=1,2 : the constant external torques [Nm];
RHS: right hand side in equations;
α : the excitation parameter due to the mean value of mesh damping of gear teeth, defined by relation (12);
α_o: the excitation parameter of the amplitude of mesh damping variation of gear teeth, defined by relation (12);
α_n: the pressure angle [rad];
β: excitation parameter of the amplitude of mesh stiffness variation of gear teeth, defined by relation (11);
β_n: the helix angle [rad];
$\bar{\delta}$: the parameter of the gear clearance;
$\bar{\varepsilon}$: small positive parameter in asymptotic expansion, 0< $\bar{\varepsilon}$ ≤ 1;
γ : excitation parameter due to cubic non-linearities of the input-output linkages, defined by relation (6);
γ_o : linear excitation parameter due to the stiffness of the driver and driven axes, defined by relation (6);
γ_1: excitation parameter due to the damping of the driver and driven axes, defined by relation (6);
φ_i, $i=1,2$: rotation angle of gear wheels[rad];
μ : excitation parameter of the amplitude of mesh stiffness variation of gear teeth normalized by the Ω^2, defined by relation (11);
τ : the *slowing* time, defined by relation (15);
ν : the excitation frequency representing the frequency of entering the tooth into the pitching plane, defined by relation (9) [rad/s];
ξ_1: the total phase angle of the dynamic response in the region of fundamental resonance, defined by relation (20);
ξ_2: the total phase angle of the dynamic response in the region of principal parametric resonance, defined by relation (20);
Δ: the logarithmic decrement of the damping;
Γ: excitation parameter due to constant external torques $M_{i0},\ i=\overline{1,2}$, defined by relation (6);
$\tilde{\Gamma}$: the amplitude of the excitation parameter due to the harmonic variation of external torques $M_i(t),\ i=\overline{1,2}$, defined by relation (6) ;
Ω : the eigen frequency of the gear-pair system, defined by relation (11) [Hz];
Ψ_1: the phase angle of the dynamic response in the region of fundamental resonance;
Ψ_2: the phase angle of the dynamic response in the region of principal parametric resonance.

1. Introduction

The failure of geared systems is due to undesired high-amplitude self-excited vibrations, which is also a cause of instability. As sources of perturbation, for a geared system, may be mentioned: the periodic variation of the tooth stiffness and damping[6-8]; the geometrical errors of the contact between teeth, that produces micro-impacts [11]; the variation of the external torque loads due to the non-linearities of input-output linkages. The first and the second type of excitations, which is present due to clearances and manufacturing tolerances, produce self-induced non-linear parametric vibrations [6], while the third type of excitation increases the non-linearity's effects of the dynamic behavior [9,10]. Many investigators have been working on the modeling of gear transmission systems using linear and non-linear parametric dynamical models, such as Bolinger [1], Bolotin [2], Bosch [4], Brauer [5] and Diekhans [16]. Other investigators followed with theoretical developments and experimental studies. Anyhow, most of them have been investigated the stability problems of geared systems, because such self-induced non-linear parametric vibrations represent a major problem for the stability of the gear transmission systems. All of the investigators started in their theoretical studies with models that cover the physical phenomena's that appear into a gear transmission. The aim of the paper is to improve such models for geared system and to enable the gear's designer to predict the dynamic behavior of non-linear self-induced vibrations for the geared systems.

2. The Physical Model of a Gear Transmission with One Stage

It was considered for the model that the investigation must carried out for the entire pitch plane [16]. This fact is due to the observation that for a cylindrical gear the force is acting on the teeth into a specific spatial direction because of the contact between two teeth. This cause torsional non-linear parametric vibrations. The gear wheels are helical therefore the mesh teeth stiffness $k_v(\varphi_i)$ and the mesh damping $c_v(\varphi_i)$ are influenced by the gear helix angle β_n and the pressure angle α_n [7,16]. This influence is given by the relations:

$$\begin{aligned} k_v &= \bar{k}_v[\cos\alpha_n \cos\beta_n]^2, \\ c_v &= \bar{c}_v[\cos\alpha_n \cos\beta_n]^2. \end{aligned} \tag{1}$$

The forces will be transmitted through the teeth mesh stiffness $k_v(\varphi_i)$ and the mesh damping $c_v(\varphi_i)$ which depend upon the rotation angle of the wheel. Some authors developed dynamic models that take into consideration the teeth clearance s_R

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and therefore the influence of micro-impacts of the teeth [15]. The considered model takes into account the backlash s_R by modeling the gear clearance with a non-linear symmetric spring [15] and introduces all the components of the forces in the circumferential direction. The gear load is a function of relative movement between the driver gear and the driven gear.

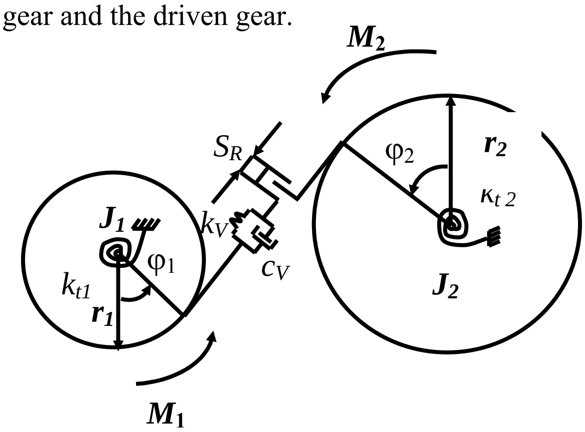

Figure 1. Physical model of a gear-pair system

The new model introduces the torsional stiffness of the gear axes as can be seen from figure 1, that presents a model with two degree of freedoms for a gear-pair system. Using the Lagrange's equations, the differential system of equations for the vibrating movement is

$$\begin{aligned} J_1\ddot{\varphi}_1 &= -c_v \cdot r_1 \cdot \dot{s} - k_v \cdot r_1 \cdot s + M_1 \\ J_2\ddot{\varphi}_2 &= c_v \cdot r_2 \cdot \dot{s} + k_v \cdot r_2 \cdot s - M_2 \end{aligned}, \qquad (2)$$

where s is the relative displacement of the gears into the pitching plane:

$$s = s(t) = r_1 \cdot \varphi_1 - r_2\varphi_2 \,. \qquad (3)$$

Multiplying the first equation of system (2) by r_1/J_1, the second one by r_2/J_2 and making the difference between them it was computed the differential equation of the non-linear parametric vibrations of the helical gear-pair system in the pitch plane:

$$\begin{aligned} &\ddot{s} + c_v\left[\frac{r_1^2}{J_1}+\frac{r_2^2}{J_2}\right]\cdot\dot{s} + k_v\left[\frac{r_1^2}{J_1}+\frac{r_2^2}{J_2}\right]\cdot s \\ &= \frac{M_1 r_1}{J_1} + \frac{M_2 r_2}{J_2}. \end{aligned} \qquad (4)$$

The right hand term of equation (4) can be expressed as:

$$\frac{M_1 r_1}{J_1} + \frac{M_2 r_2}{J_2} = f(\Gamma, t) + \gamma_0 \cdot s + \gamma_1 \dot{s} - \gamma \cdot s^3 \qquad (5)$$

where $f(\Gamma, t)$ produces static deflection of the gear teeth and induces the harmonic external forced excitation due to the harmonic variation of the external torques. The other terms induce the effects of torsional stiffness of the gear axes γ_o, the torsional damping of the gear axes γ_1 and the non-linearities of the input-output linkages γ in order to transmit the external torques M_1 and M_2. This terms are defined by the relations:

$$f(\Gamma,t) = \Gamma + \tilde{\Gamma}\cos\nu t,\ \Gamma = \frac{M_{10}\, r_1}{J_1} + \frac{M_{20}\, r_2}{J_2},\ \tilde{\Gamma} = 0.1\Gamma$$

$$\gamma_0 = \frac{\gamma_{10}\gamma_{20}}{\gamma_{10}+\gamma_{20}},\ \gamma_{10} = \frac{k_{t1}}{J_1},\ \gamma_{20} = \frac{k_{t2}}{J_2},$$

$$\gamma = 0.001\frac{\gamma_{10}\gamma_{20}}{r_1^2 r_2^2\left[\dfrac{\gamma_{10}}{r_1^2}+\dfrac{\gamma_{20}}{r_2^2}\right]},$$

$$\gamma_1 = \frac{\gamma_{11}\gamma_{12}}{\gamma_{11}+\gamma_{12}}, \gamma_{11} = \frac{c_{t1}}{J_1},\ \gamma_{12} = \frac{c_{t2}}{J_2}. \qquad (6)$$

considering that the cubic non-linearities are very small and the variation of external torque loads is 10%. In equation (4) can be neglect the term Γ can be neglect because it produces only the static deflection and the paper deals only with the dynamic behavior of the system. Equation (4) is a bi-parametric one because of the variation of $k_v(t)$ and also because the damping $c_v(t)$ that depends upon the stiffness [7,8,16]. Taking into consideration the experimental data [7,8,16], the mesh stiffness $k_v(t)$ and the mesh damping $c_v(t)$ for helical gears can be expressed by the mathematical relations

$$\begin{aligned} k_v(t) &= k_{vo} + \tilde{k}_v \cos\nu t, \\ c_v(t) &= c_{vo} + \tilde{c}_v \cos\nu t. \end{aligned} \qquad (7)$$

The relation between the mesh stiffness and the mesh damping is given by[7,16]

$$c_v = \frac{\Delta k_v}{\pi\nu}, \qquad (8)$$

where the excitation frequency is

$$\nu = \frac{\pi z_1 n_1}{30} = \frac{\pi z_2 n_2}{30}. \qquad (9)$$

Equation (4) is a non-linear differential equation because of the non-linear term expressed by (6) that is due to the link flange of the gear transmission [6]. Taking into account that the perturbation in equation (4) has a non-linear variation expressed by relations (5),(6) and also equation (4) has a bi-parametric mathematical form given by the expressions (7)-(9), this can be put into the mathematical form

$$\ddot{s} + (\alpha + \alpha_0\cos\nu t)\dot{s} + \Omega^2(1+\mu\cos\nu t)s + \gamma s^3 = \tilde{\Gamma}\cos\nu t. \qquad (10)$$

In equation (10), the eigen frequency Ω and the excitation parameter μ of the mesh stiffness are given by the expressions

$$\Omega = \sqrt{k_{\mathrm{V}_0}\left(\frac{r_1^2}{J_1} + \frac{r_2^2}{J_2}\right) - \gamma_0}\ ,$$

$$\mu = \frac{\beta}{k_{\mathrm{V}_0}\left(\frac{r_1^2}{J_1} + \frac{r_2^2}{J_2}\right) - \gamma_0},\ \beta = \tilde{k}_{\mathrm{V}_0}\left(\frac{r_1^2}{J_1} + \frac{r_2^2}{J_2}\right), \qquad (11)$$

where α and α_0 are the excitation parameter due to the mean value of mesh damping and the excitation parameter of the amplitude of mesh damping variation expressed by the relations

$$\alpha = c_{v0}\left[\frac{r_1^2}{J_1} + \frac{r_2^2}{J_2}\right] - \gamma_1 = \frac{\Delta\, k_{v0}}{\pi\nu}\left[\frac{r_1^2}{J_1} + \frac{r_2^2}{J_2}\right] - \gamma_1,$$
$$\alpha_0 = \tilde{c}_v\left[\frac{r_1^2}{J_1} + \frac{r_2^2}{J_2}\right] = \frac{\Delta\, \tilde{k}_v}{\pi\nu}\left[\frac{r_1^2}{J_1} + \frac{r_2^2}{J_2}\right]. \qquad (12)$$

In expressions (12) the relations (7) and (8) were used. From the equation (10) is missing only the term that introduces the backlash effect given by the gear clearances, which induces micro-impacts between the gear teeth flanks. Modeling the gear clearance with a non-linear symmetric spring [15] equation (10) becomes

$$\ddot{s} + (\alpha + \alpha_0 \cos\nu t)\dot{s} + \Omega^2(1 + \mu\cos\nu t)(s + \bar{\delta}s^3) +$$
$$+ \gamma s^3 = \tilde{\Gamma}\cos\nu t. \qquad (13)$$

Equation (13) is a second-order non-homogeneous bi-parametric non-linear differential equation and has a mathematical form of a modified non-linear Mathieu-Duffing-Hill equation. Equation (13) represents the equation of motion of the relative displacement between the gear flanks in the pitch plane taking into account the following phenomena's of the dynamics of helical gear-pair system:

- the time-dependent variation of the gear mesh stiffness and gear mesh damping [5-8],
- the backlash that induces micro-impacts on the teeth flanks [15],
- the interaction between gear mesh stiffness and gear mesh damping [7,8,11],
- the torsional stiffness and the torsional damping of the driver and driven gear axes,
- the non-linearities of the input-output linkages [8],
- the time-dependent variation of the external torques loads [11,15],
- the influence of the gear helix angle and the pressure angle on the mesh teeth stiffness and the mesh damping [6,16].

3. Asymptotic Solution of the Motion Equation

A rigorous investigation of non-linear differential equations in the general case leads to serious mathematical difficulties. However, a broad class of differential equations exists that can be solved by effective approximate methods. In particular, the equations describing the vibrations of geared systems belong to this class of equations. The author had used the asymptotic method [14] in previous analytical investigations [9,10], therefore this method is considered one of the most effective tool for analyzing non-linear vibrations of geared systems. For equation (13) *a first order approximation of asymptotic method* shall be determined, with the assumption that the non-linear vibrating geared system has *slowly* varying parameters. Also it is assumed that the excitation frequency varies slowly with the time, i.e.,

$$\frac{\mathrm{d}\,\theta}{\mathrm{d}\,t} = \nu(\tau), \theta = \nu\, t, \qquad (14)$$

where τ is the *slowing* time defined by

$$\tau = \bar{\varepsilon}t. \qquad (15)$$

All the parameters of the system are formally incorporated in the asymptotic method by representing these quantities in the form

$$\alpha = \bar{\varepsilon}\alpha,\ \alpha_0 = \bar{\varepsilon}\ \alpha_0,\ \mu = \bar{\varepsilon}\ \mu,\ \bar{\delta} = \bar{\varepsilon}\bar{\delta},\ \gamma = \bar{\varepsilon}\ \gamma,\ \tilde{\Gamma} = \bar{\varepsilon}\ \tilde{\Gamma}, \qquad (16)$$

that is an expansion in terms of the small positive parameter $\bar{\varepsilon}$. Use of exactly the same notation for each is made for convenience. Equation (13) becomes, after terms of first order in $\bar{\varepsilon}$ are considered and transferred to the right-hand side (RHS),

$$\ddot{s} + \Omega^2\, s = \bar{\varepsilon}[-\Omega^2\mu\cos\theta\, s - (\alpha + \alpha_0\cos\theta)\dot{s} -$$
$$-(\Omega^2\bar{\delta} + \gamma)s^3 + \tilde{\Gamma}\cos\theta]. \qquad (17)$$

Equation (17) may be symbolically represented as

$$\ddot{s} + \Omega^2\, s = \bar{\varepsilon}\, H(\tau, \theta, s, \dot{s}), \qquad (18)$$

where the perturbation $H(\tau,\theta,s,\dot{s})$ on the RHS is a periodic function in θ with period 2π. The left-hand side (LHS) of equation (18) represents a linear oscillator. When the perturbation is absent, $\bar{\varepsilon}$=0, and τ is constant, the solution of equation (18) will be expressed by a sinusoidal function, term with an amplitude and phase of oscillation defined by initial conditions. In the presence of a perturbation ($\bar{\varepsilon} \neq 0$), overtones may occur and various resonances may take place. The presence of the slowly varying time τ also give rises, in the system, to a number of additional phenomena not observable in oscillating systems described by an equation with constant parameters. Resonance phenomena in nonlinear vibrating systems under the action of external periodic forces may ensue upon the fulfillment of the condition

$$\nu \approx \frac{k}{l}\Omega, \qquad (19)$$

where k and l are mutually prime integers, usually small. Taking into account all these physical considerations and confining our attention to the investigation of the fundamental resonance (k =1, l =1) and principal parametric resonance (k =2, l =1), the solution of equation (18) is sought in the following form (to the first order of approximation in ε)

$$s = \bar{s}_1 \cos(\theta + \Psi_1) + \bar{s}_2 \cos(\frac{1}{2}\theta + \Psi_2) = \bar{s}_1 \cos\xi_1 + \bar{s}_2 \cos\xi_2 \tag{20}$$

where the amplitudes $\bar{s}_1, \bar{s}_2$ and the phase angles Ψ_1, Ψ_2 are functions of time defined by the systems of differential equations

$$\frac{d\,\bar{s}_1}{dt} = \varepsilon\, A_1(\tau, \bar{s}_1, \Psi_1),\ \frac{d\,\Psi_1}{dt} = \Omega - \nu + \varepsilon\, B_1(\tau, \bar{s}_1, \Psi_1), \tag{21}$$

$$\begin{aligned} &\frac{d\,\bar{s}_2}{dt} = \varepsilon\, A_2(\tau, \bar{s}_1, \bar{s}_2, \Psi_1), \frac{d\,\Psi_2}{dt} = \\ &= \Omega - \frac{1}{2}\nu + \varepsilon\, B_2(\tau, \bar{s}_1, \bar{s}_2, \Psi_1). \end{aligned} \tag{22}$$

The functions A_i, B_i are selected in such a way that equation (20), after replacing $\bar{s}_1, \bar{s}_2, \Psi_1$ and Ψ_2 by the functions defined in equations (21) and (22), would represent a solution of equation (17). As can be seen from the systems of differential equations (21) and (22) the interaction between the fundamental resonance and the principal parametric resonance is taken into account. It is instructive to keep in mind that if Z is a function of τ, i.e., $Z = Z(\tau)$, then

$$\frac{d\,Z(\tau)}{dt} = \frac{d\,Z(\tau)}{d\tau}\frac{d\tau}{dt} = \varepsilon\,\frac{d\,Z(\tau)}{d\tau}. \tag{23}$$

Computing the LHS of equation (17) by using the equations (20-22), with respect to the differentiation given by equation (23) and expanding the result in powers of ε, gives the following

$$\begin{aligned} &\mathbf{LHS} = \ddot{s} + \Omega^2 s = \\ &= \varepsilon\left\{\left[\frac{\partial A_1}{\partial \Psi_1}(\Omega - \nu) - 2\bar{s}_1\Omega B_1\right]\cos\xi_1 + \right. \\ &+ \left[\frac{\partial A_2}{\partial \Psi_2}(\Omega - \frac{1}{2}\nu) - 2\bar{s}_2\Omega B_2\right]\cos\xi_2 + \\ &+ \left[-2\Omega A_1 - \bar{s}_1(\Omega - \nu)\frac{\partial B_1}{\partial \Psi_1}\right]\sin\xi_1 + \\ &\left. + \left[-2\Omega A_2 - \bar{s}_2(\Omega - \frac{1}{2}\nu)\frac{\partial B_2}{\partial \Psi_2}\right]\sin\xi_2\right\}. \end{aligned} \tag{24}$$

Computing the RHS of equation (17) by using the equations (20-22), with respect to the differentiation given by equation (23) and expanding the result in powers of ε, gives

$$\begin{aligned} &\mathbf{RHS} = \varepsilon\left[-\Omega^2\mu\cos\theta\, s - (\alpha + \alpha_0\cos\theta)\dot{s} - \right. \\ &\left. -(\Omega^2\bar{\delta} + \gamma)s^3 + \tilde{\Gamma}\cos\theta\right] = \\ &= \varepsilon\left\{\left[\tilde{\Gamma}\cos\Psi_1 - \frac{3}{4}(\gamma + \Omega^2\bar{\delta})\bar{s}_1^3\right]\cos\xi_1 + \right. \\ &+ \left[-\frac{1}{2}\mu\Omega^2\bar{s}_2\cos 2\Psi_2 + \frac{1}{2}\alpha_0\Omega\,\bar{s}_2\sin 2\Psi_2 - \right. \\ &\left. -(\gamma + \Omega^2\bar{\delta})\left(\frac{3}{4}\bar{s}_2^3 + \frac{3}{2}\bar{s}_1^2\bar{s}_2\right)\right]\cos\xi_2 + \\ &+ \left[\tilde{\Gamma}\sin\Psi_1 + \alpha\,\Omega\,\bar{s}_1\right]\sin\xi_1 + \\ &+ \left[-\frac{1}{2}\alpha_0\Omega\bar{s}_2\cos 2\Psi_2 - \frac{1}{2}\mu\Omega^2\bar{s}_2\sin 2\Psi_2 + \right. \\ &\left.\left. + \alpha\,\Omega\,\bar{s}_2\right]\sin\xi_2\right\}. \end{aligned} \tag{25}$$

In expressions (24) and (25) the terms in ε^2 and ε^3 are neglected. In order that the assumed solution (20) may formally satisfy equation (17), was performed a harmonic balance within the coefficients of $\varepsilon\cos\xi_1, \varepsilon\sin\xi_1, \varepsilon\cos\xi_2$ and $\varepsilon\sin\xi_2$ for RHS and LHS expressed by (24) and (25). This yields the two following coupled systems of first order differential equations for the unknown functions A_1, A_2, B_1, B_2

$$\begin{cases} \dfrac{\partial A_1}{\partial \Psi_1}(\Omega - \nu) - 2\bar{s}_1\Omega B_1 = \tilde{\Gamma}\cos\Psi_1 - \dfrac{3}{4}\left(\gamma + \Omega^2\bar{\delta}\right)\bar{s}_1^3, \\ 2\Omega A_1 + \bar{s}_1(\Omega - \nu)\dfrac{\partial B_1}{\partial \Psi_1} = -\tilde{\Gamma}\sin\Psi_1 - \alpha\Omega\bar{s}_1, \end{cases} \tag{26}$$

$$\begin{cases} \dfrac{\partial A_2}{\partial \Psi_2}\left(\Omega - \dfrac{1}{2}\nu\right) - 2\Omega\,\bar{s}_2 B_2 = \\ = -\dfrac{1}{2}\mu\Omega^2\bar{s}_2\cos 2\Psi_2 + \dfrac{1}{2}\alpha_0\Omega\,\bar{s}_2\sin 2\Psi_2 - \\ \quad -(\gamma + \Omega^2\bar{\delta})\left(\dfrac{3}{4}\bar{s}_2^3 + \dfrac{3}{2}\bar{s}_1^2\bar{s}_2\right), \\ 2\Omega A_2 + \left(\Omega - \dfrac{1}{2}\nu\right)\bar{s}_2\dfrac{\partial B_2}{\partial \Psi_2} = \\ = \dfrac{1}{2}\alpha_0\Omega\bar{s}_2\cos 2\Psi_2 + \dfrac{1}{2}\mu\Omega^2\bar{s}_2\sin 2\Psi_2 - \alpha\,\Omega\,\bar{s}_2. \end{cases} \tag{27}$$

After solving the systems (26) and (27) and transforming all systems parameters back to their real time values, the systems of differential equations (21) and (22) become

$$\begin{cases} \dfrac{d\bar{s}_1}{dt} = -\dfrac{\tilde{\Gamma}}{\Omega+\nu}\sin\Psi_1 - \dfrac{1}{2}\alpha\bar{s}_1, \\ \dfrac{d\Psi_1}{dt} = \Omega - \nu - \dfrac{\tilde{\Gamma}}{\Omega+\nu}\dfrac{1}{\bar{s}_1}\cos\Psi_1 + \dfrac{\gamma+\Omega^2\bar{\delta}}{\Omega}\dfrac{3}{8}\bar{s}_1^2, \end{cases} \tag{28}$$

$$\begin{cases} \dfrac{d\bar{s}_2}{dt} = \dfrac{1}{2}\dfrac{\alpha_0\Omega}{\nu}\bar{s}_2\cos 2\Psi_2 + \dfrac{1}{2}\dfrac{\mu\Omega^2}{\nu}\bar{s}_2\sin 2\Psi_2 - \dfrac{1}{2}\alpha\bar{s}_2, \\ \dfrac{d\Psi_2}{dt} = \Omega - \dfrac{1}{2}\nu + \dfrac{1}{2}\dfrac{\mu\Omega^2}{\nu}\cos 2\Psi_2 - \dfrac{1}{2}\dfrac{\alpha_0\Omega}{\nu}\sin 2\Psi_2 + \\ \quad + \dfrac{\gamma+\Omega^2\bar{\delta}}{\Omega}\left(\dfrac{3}{4}\bar{s}_1^2 + \dfrac{3}{8}\bar{s}_2^2\right). \end{cases} \tag{29}$$

The systems (28) and (29) define the rate of the change of amplitudes and phase angles as functions the slowly varying frequency of external periodic excitation ν. By integrating this coupled systems of equations, the amplitudes $\bar{s}_1$, $\bar{s}_2$ and the phase angles Ψ_1, Ψ_2 can be obtained as functions of time and the relative displacement of the gears in the pitch plane s, that completely describes the non-stationary vibrational process in both the fundamental resonance region and the principal parametric resonance region, can then be computed from the equation (20) in the first approximation.

3. Stationary Response

To obtain stationary values of the phases angles Ψ_1, Ψ_2 **,** it is necessary to equate the right-hand sides of the systems (28) and (29) to zero. In the previous paper (part 1) it were computed the amplitudes $\bar{s}_1$, $\bar{s}_2$. The stationary response has the phase angle in the region of fundamental resonance, computed from the system (28), after the calculation of the amplitude $\bar{s}_1$,

$$\Psi_1 = \arcsin - \frac{\alpha\, \bar{s}_1(\Omega+\nu)}{2\tilde{\Gamma}}, \tag{30}$$

while the phase angle in the region of principal parametric resonance, computed from the system (29), after the determination of the amplitudes $\bar{s}_1$ and $\bar{s}_2$, has the expression

$$\Psi_2 = \frac{1}{2}\operatorname{arctg}\frac{e_3 + e_4\left(2\bar{s}_1^2 + \bar{s}_2^2\right)}{e_1 + e_2\left(2\bar{s}_1^2 + \bar{s}_2^2\right)}. \tag{31}$$

As can be seen from the relations (30) and (31) the amplitude in the region of fundamental resonance interacts with both the phases angles in the regions of fundamental resonance and of principal parametric resonance.

4. Results and Conclusions

Numerical results of the stationary responses for fundamental resonance and principal parametric resonance of helical gear pair system are shown in figures 2 through 4 for a wide variety of cases. All numerical results are based on equation (30) for Ψ_1 and equation (31) for Ψ_2. Incrementing the value of logarithmic decrement of the damping Δ has a neglecting effect on the phase angle in the region of fundamental resonance but in the region of principal parametric resonance induces sensitive changes of the phase angle as can be seen from figures 2, 3 and 4. As can be seen in figure 4, for the value Δ=0.03 the phase angle branches, in the region of principal parametric resonance, disappear, their values suddenly decrease and the non-linear effect doesn't manifest itself on the shape of the phase angle. This phenomenon can be explain by the fact that the damping has a mixed effect of stabilization for the vibrating motion, being in the mean time a factor of excitation, if it is taken into account the bi-parametric equation of the vibrating system (13). By this way the paper highlights all the qualitative connections between the values of a wide range of parameters that occur at the non-linear dynamic behavior of a helical gear-pair system parametrically excited.

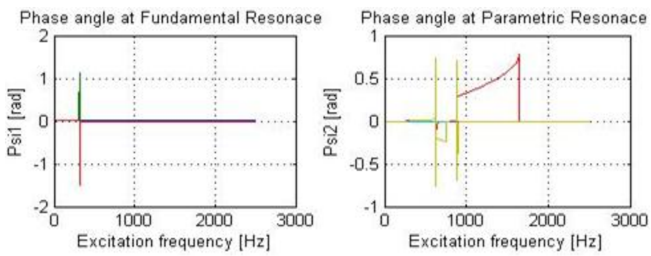

Figure 2. The phases angles Ψ_1, Ψ_2 for the stationary dynamic response in the region of fundamental resonance $\nu \cong \Omega$ and in the region of principal parametric resonance $\nu \cong 2\Omega$ for the parameters Δ=0.01; μ = 0.01; $\tilde{\Gamma}$ = 0.5; $\gamma > 0$; $\bar{\delta} > 0$.

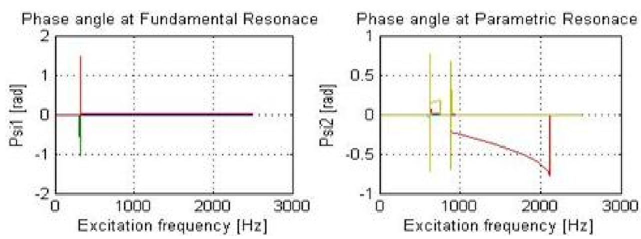

Figure 3. The phases angles Ψ_1, Ψ_2 for the stationary dynamic response in the region of fundamental resonance $\nu \cong \Omega$ and in the region of principal parametric resonance $\nu \cong 2\Omega$ for the parameters Δ=0.02; μ = 0.01; $\tilde{\Gamma}$ = 0.5; $\gamma > 0$; $\bar{\delta} > 0$.

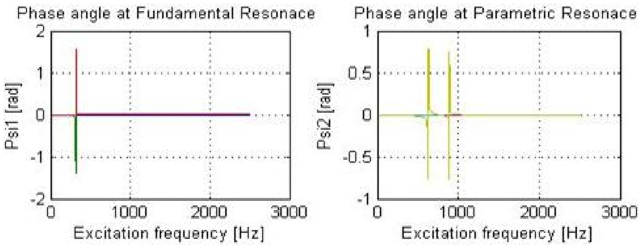

Figure 4. The phases angles Ψ_1, Ψ_2 for the stationary dynamic response in the region of fundamental resonance $\nu \cong \Omega$ and in the region of principal parametric resonance $\nu \cong 2\Omega$ for the parameters $\Delta=0.03$; $\mu = 0.01$; $\widetilde{\Gamma} = 0.5$; $\gamma > 0$; $\overline{\delta} > 0$.

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