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The series expansions of generalized hypergeometric functions

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Abstract: - Our aim is to give a method for the determination of the power series expansion of the solution of the initial value problem of nonlinear ordinary differential equations.

Key-Words: - Nonlinear ordinary differential equation, initial value problem, series expansion

1 Introduction

We consider the differential equation of the second order

$$y''|y'|^{p-1} \pm y|y|^{p-1} = 0, \tag{1}$$

where p is a positive real number. We examine the solutions y_1 and y_2 of the initial value problem of (1) with initial conditions

$$y(0) = 0, \ y'(0) = 1, \tag{2}$$

or

$$y(0) = 1, \ y'(0) = 0, \tag{3}$$

for both the + sign and – sign in (1).

Equation (1) is called half-linear one since its solution set is homogeneous but not additive. The existence of the solution of the initial value problem (1)-(2) and (1)-(3) was showed in [3].

For $p = 1$ equation (1) is the linear equation

$$y'' \pm y = 0, \tag{4}$$

and the solution of (4) for '+' sign with (2) is $y = \sin x$ and with (3) is $y = \cos x$. Furthermore, when $p = 1$, the solution of (4) for '-' sign with (2) is $y = \sinh x$ and with (3) is $y = \cosh x$.

If $p \neq 1$ then the solution of

$$y''|y'|^{p-1} + y|y|^{p-1} = 0 \tag{5}$$

with (2) is called generalized sine function and denoted by $y = S_p(x)$. The generalization of sine function was introduced by E. Lundberg [5] and Á. Elbert [3]. The properties of $S_p(x)$ was examined by V. I. Levin [4] and E. Schmidt [6]. The application of the generalized sine function is useful in the study of the boundary value problem of some nonlinear partial differential equations (see [1]).

If $p \neq 1$ then the solution of (5) with (3) is called generalized cosine function and we denote it by $y = C_p(x)$. We note that $C_1(x) = \cos x$.

If $p \neq 1$ then the solution of

$$y''|y'|^{p-1} - y|y|^{p-1} = 0 \tag{6}$$

with (2) is called generalized sine hyperbolic function and denoted by $y = Sh_p(x)$. In the special case $p = 1$, $Sh_1(x) = \sinh x$.

If $p \neq 1$ then the solution of (6) with (3) is called generalized cosine hyperbolic function, denoted by $y = Ch_p(x)$, and $Ch_1(x) = \cosh x$.

2 Problem Formulation

We show that the solution of the non linear problem (1) with initial condition (2) or (3) can be given of the form

$$y(x) = x^{\alpha} \sum_{n=0}^{\infty} \gamma_n x^{\beta n} \tag{7}$$

for both the + and – sign.

2.1 The generalized sine function

We define the generalized sine function as the solution of the nonlinear initial value problem

$$y''|y'|^{p-1} + y|y|^{p-1} = 0,$$

$$y(0) = 0, \ y'(0) = 1.$$

If $y > 0$, $y' > 0$ the solution S_p can be given by

$$x = \int_0^{S_p} \frac{dy}{\sqrt[p+1]{1 - y^{p+1}}}$$

in the interval $\left[0, \hat{\pi}/2\right]$, where

$$\frac{\hat{\pi}}{2} = \int_0^1 \frac{1}{\sqrt{1-y^{p+1}}} dy,$$

therefore

$$\frac{\hat{\pi}}{2} = \frac{\pi}{p+1} \frac{1}{\sin \frac{\pi}{p+1}}.$$

Function S_p can be extended as a $2\hat{\pi}$ periodic function to the whole real axis as follows:

$$S_p(x) = \begin{cases} S_p(\hat{\pi} - x) & \text{if} \quad \frac{\hat{\pi}}{2} \le x \le \hat{\pi} \\ -S_p(x - \hat{\pi}) & \text{if} \quad \hat{\pi} \le x \le 2\hat{\pi} \\ S_p(x + 2k\hat{\pi}) & \text{where} \quad k = \pm 1, \pm 2, \dots \end{cases}$$

moreover for the zeros of S_p we have that

$$S_p(x_0) = 0, \ if \ x_0 = 0, \pm\hat{\pi}, \pm 2\hat{\pi}, \dots .$$

For function S_p the generalization of the Pythagorean relation

$$\sin^2 x + \cos^2 x = 1$$

has the form

$$\left|S_p(x)\right|^{p+1} + \left|S'_p(x)\right|^{p+1} = 1.$$

Our aim is to give solution S_p of the form

$$\begin{aligned} S_p(x) &= x \sum_{i=0}^{\infty} a_i x^{i(p+1)} \\ &= a_0 x + a_1 x^{p+2} + a_2 x^{2p+3} + \dots \end{aligned} \tag{8}$$

and to give a method for the determination of coefficients $a_0, a_1, a_2, \dots$.

In this case, in (7) for the values of α and β we have $\alpha = 1$ and $\beta = p+1$.

According to (8), we want to find S'_p and S''_p in the form

$$\begin{aligned} S'_p(x) &= \sum_{i=0}^{\infty} b_i x^{i(p+1)} \\ &= b_0 + b_1 x^{p+1} + b_2 x^{2p+2} + \dots \end{aligned}$$

and

$$\begin{aligned} S''_p(x) &= x^p \sum_{i=0}^{\infty} c_i x^{i(p+1)} \\ &= c_0 x^p + c_1 x^{2p+1} + c_2 x^{3p+2} + \dots . \end{aligned}$$

From the initial conditions (2) we get

$$S_p(0) = 0,$$
$$S'_p(0) = b_0,$$

hence

$$b_0 = 1.$$

The relations between coefficients a_i, b_i and c_i are the following

$$\begin{cases} b_i = a_i\left(i(p+1)+1\right) \\ c_i = b_{i+1}(i+1)(p+1), \quad i = 0,1,2,\dots . \end{cases} \tag{9}$$

From this for $i = 0$ we get that

$$a_0 = b_0 = 1$$

and from differential equation (5) we get that

$$c_0 = -1.$$

The series expansion of form (8) is considered for $\left[0, \hat{\pi}/2\right]$, where $S_p(x) \ge 0$, $S'_p(x) \ge 0$, therefore (5) has the form

$$\left[S'_p(x)\right]^{p-1} \cdot S''_p(x) + \left[S_p(x)\right]^p = 0.$$

We introduce the notion of elasticity of function f by

$$E(f(x)) = \frac{x}{f(x)} \cdot f'(x).$$

The elasticity has the properties:

$$E(f(x) \cdot g(x)) = E(f(x)) + E(g(x)),$$
$$E(x^\alpha) = \alpha,$$
$$E(f^\alpha(x)) = \alpha \cdot E(f(x))$$
$$E(-f(x)) = E(f(x)).$$

Let us evaluate the elasticity of $S_p(x)$ by

$$E(S_p(x)) = 1 + (p+1) \cdot \frac{\sum_{i=0}^{\infty} a_i \cdot i \cdot x^{i(p+1)}}{\sum_{i=0}^{\infty} a_i \cdot x^{i(p+1)}}.$$

Taking into consideration condition $a_0 = 1$ and making the division it has the form

$$E(S_p(x)) = 1 + (p+1)\sum_{k=1}^{\infty} A_k x^{k(p+1)},$$

where $A_k = k \cdot a_k - \sum_{i=1}^{k-1} A_i a_{k-i}$.

Similarly,

$$E(S'_p(x)) = (p+1)\cdot \frac{\sum_{i=0}^{\infty} b_i \cdot i \cdot x^{i(p+1)}}{\sum_{i=0}^{\infty} b_i \cdot x^{i(p+1)}}$$

with $b_0 = 1$ it can be written that

$$E(S'_p(x)) = (p+1)\cdot \sum_{k=1}^{\infty} B_k \cdot x^{k(p+1)},$$

where $B_k = kb_k - \sum_{i=1}^{k-1} B_i b_{k-i}$, moreover

$$E(S''_p(x)) = p + (p+1)\cdot \frac{\sum_{i=0}^{\infty} c_i \cdot i \cdot x^{i(p+1)}}{\sum_{i=0}^{\infty} c_i \cdot x^{i(p+1)}},$$

$$E(S''_p(x)) = p + (p+1)\cdot \sum_{k=1}^{\infty} C_k \cdot x^{k(p+1)},$$

where $C_k = -kc_k + \sum_{i=1}^{k-1} C_i c_{k-i}$.

From differential equation (5) we obtain

$$(p-1)E(S'_p(x)) + E(S''_p(x)) = pE(S_p(x)) \qquad (10)$$

Substituting the elasticity of S_p, S'_p, S''_p into equation (10) we get, by comparison of coefficients, the following recurrence relations

$$C_k = p\cdot A_k - (p-1)B_k \quad k = 1,2,..., \qquad (11)$$

from where

$$c_k = -\frac{1}{k}\sum_{i=1}^{k-1} C_i c_{k-i} - pa_k + \frac{p}{k}\cdot \sum_{i=1}^{k-1} A_i a_{k-i}$$

$$+(p-1)b_k - \frac{p-1}{k}\sum_{i=1}^{k-1} B_i b_{k-i}$$

$$= (p-1)b_k - pa_k - \frac{1}{k}\cdot$$

$$\sum_{i=1}^{k-1}\left[C_i c_{k-i} - pA_i a_{k-i} + (p-1)B_i b_{k-i}\right]$$

Coefficients a_i, b_i, c_i $(i = 0,1,2,...)$ can be determined from (9) and (11). The convergence of the power series is examined in [2].

Example Solve problem

$$y''|y'|^{p-1} + y|y|^{p-1} = 0,$$

$$y(0) = 0, \quad y'(0) = 1$$

for $p = 3$.

After the determination of coefficients a_i, $i = 0,1,2,...$ calculations we find the series expansion of the solution of the form

$$S_3(x) = x - 0.05000x^5 - 0.00486x^9$$
$$- 0.00124x^{13} - ...$$

in the interval $[0, 1.11072]$.

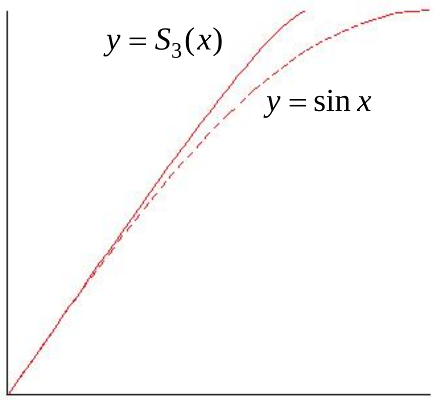

Fig.1.

2.2 The generalized cosine function

We define the generalized sine function as the solution of the nonlinear initial value problem

$$y''|y'|^{p-1} + y|y|^{p-1} = 0,$$

$$y(0) = 1, \quad y'(0) = 0.$$

The solution C_p of this problem in the interval $[0, \hat{\pi}/2]$ can be given by

$$x = -\int_0^{C_p} \frac{dy}{\sqrt[p+1]{1 - y^{p+1}}}$$

when $y > 0$, $y' < 0$, and its series expansion has the form

$$C_p(x) = \sum_{i=0}^{\infty} l_i \, x^{i\left(\frac{1}{p}+1\right)} =$$

$$= l_0 + l_1 x^{\frac{1}{p}+1} + l_2 x^{\frac{2}{p}+2} + \dots.$$

We note that now $\alpha = 0,\ \beta = \frac{1}{p} + 1$ in (7). Moreover, the exponent β in case of generalized sine function is $p+1$, in case of generalized cosine function is $\frac{1}{p}+1$, and the two exponents are dual pair since equation $\frac{1}{p+1} + \frac{1}{\frac{1}{p}+1} = 1$ is fulfilled.

Now C'_p and C''_p can be written as follows

$$C'_p(x) = x^{\frac{1}{p}} \sum_{i=0}^{\infty} m_i\, x^{i\left(\frac{1}{p}+1\right)},$$

and

$$C''_p(x) = x^{\frac{1}{p}-1} \sum_{i=0}^{\infty} n_i\, x^{i\left(\frac{1}{p}+1\right)}.$$

From the initial conditions (3) we have that

$$l_0 = 1,\ m_0 = \sqrt[p]{-p}, \tag{12}$$

and from the differential equation (5)

$$n_0 = \sqrt[p]{-p}\,/\,p. \tag{13}$$

We note that the series expansion of C_p can be achieved for such values of p when $\sqrt[p]{-p}$ is defined. In this case we have the following connections between the coefficients

$$m_i = (i+1)\left(\frac{1}{p}+1\right) l_{i+1},$$
$$n_i = m_i\left[i\left(\frac{1}{p}+1\right) + \frac{1}{p}\right] \tag{14}$$

for $i = 0,1,2,...$.

Substituting C_p, C'_p, C''_p into the differential equation and taking the elasticity, we get after simplifications the following recurrence relations

$$N_k = p \cdot L_k - (p-1) \cdot M_k,$$

where

$$L_k = k \cdot l_k - \sum_{i=1}^{k-1} L_i \cdot l_{k-i},$$

$$M_k = k \cdot \frac{m_k}{m_0} - \sum_{i=1}^{k-1} M_i \cdot \frac{m_{k-i}}{m_0},$$

$$N_k = k \cdot \frac{n_k}{n_0} - \sum_{i=1}^{k-1} N_i \cdot \frac{n_{k-i}}{n_0}.$$

Example Solve problem

$$y''|y'|^{p-1} + y|y|^{p-1} = 0,$$
$$y(0) = 1,\ y'(0) = 0$$

for $p = 3$.

The calculations shows that

$$C_3(x) = 1 - 1.081x^{4/3} + 0.2106x^{8/3} - 0.039x^4 + ...$$

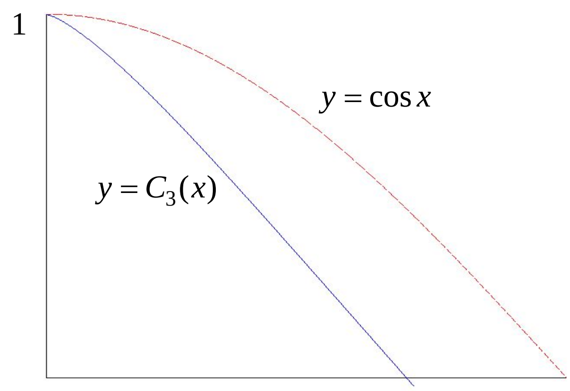

Fig.2.

2.3 The generalized hyperbolic sine function

We define the generalized hyperbolic sine function as the solution of the nonlinear initial value problem

$$y''|y'|^{p-1} - y|y|^{p-1} = 0,$$
$$y(0) = 0,\ y'(0) = 1.$$

The solution Sh_p satisfies

$$x = \int_0^{Sh_p} \frac{dy}{\sqrt[p+1]{y^{p+1}+1}},\quad x \in R.$$

We want to find the series expansion in the form

$$Sh_p(x) = x \sum_{i=0}^{\infty} t_i x^{i(p+1)}$$

and we use the notation for its derivatives

$$Sh'_p(x) = \sum_{i=0}^{\infty} u_i x^{i(p+1)},$$

$$Sh''_p(x) = x^p \sum_{i=0}^{\infty} v_i x^{i(p+1)} .$$

Here $\alpha = 1$, $\beta = p+1$ as in Section 2.1. From the initial conditions we get

$$t_0 = 1,\ u_0 = 1,\ v_0 = 1$$

and for the coefficients using the elasticity of Sh_p, Sh'_p, Sh''_p we have the relation

$$V_k = pT_k - (p-1)U_k ,$$

where

$$V_k = -kv_k + \sum_{i=1}^{k-1} V_i v_{k-i},$$

$$T_k = kt_k - \sum_{i=1}^{k-1} T_i t_{k-i},$$

$$U_k = ku_k - \sum_{i=1}^{k-1} U_i u_{k-i}.$$

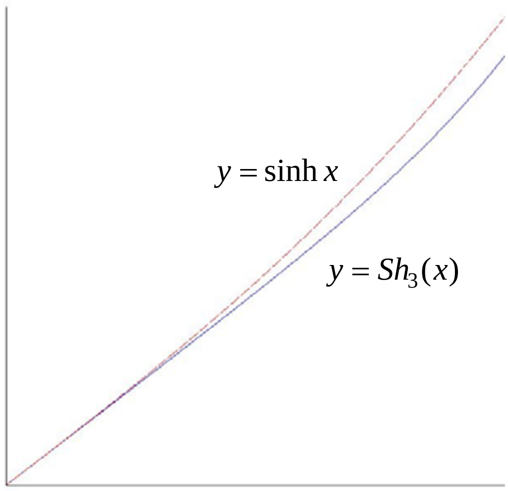

Fig.3.

Example Solve problem

$$y''|y'|^{p-1} - y|y|^{p-1} = 0 ,$$

$$y(0) = 0,\ \ y'(0) = 1$$

for $p = 3$.

After calculations we obtain

$$Sh_3(x) = x + 0.05000x^5 + 0.00486x^9 - 0.00030x^{13} - \ldots$$

(see the graph in Fig.3).

2.4 The generalized hyperbolic cosine function

We define the generalized hyperbolic sine function as the solution of the nonlinear initial value problem

$$y''|y'|^{p-1} - y|y|^{p-1} = 0 ,$$

$$y(0) = 1,\ \ y'(0) = 0 .$$

The solution of this problem is

$$x = \int_0^{Ch_p} \frac{dy}{\sqrt[p+1]{y^{p+1} - 1}} , \quad x \ge 0$$

and we search the series expansion of Ch_p in the form

$$Ch_p(x) = \sum_{i=0}^{\infty} q_i\, x^{i\left(\frac{1}{p}+1\right)}$$

and

$$Ch'_p(x) = x^{\frac{1}{p}} \sum_{i=0}^{\infty} r_i\, x^{i\left(\frac{1}{p}+1\right)} ,$$

$$Ch''_p(x) = x^{\frac{1}{p}-1} \sum_{i=0}^{\infty} s_i\, x^{i\left(\frac{1}{p}+1\right)} ,$$

and

$$r_i = q_{i+1}(i+1)\left(\frac{1}{p}+1\right),$$

$$s_i = r_i\left(\frac{1}{p} + i\left(\frac{1}{p}+1\right)\right),\ \ i = 0,1,2,\ldots .$$

Here $\alpha = 0$, $\beta = \dfrac{1}{p} + 1$ as in Section 2.2. From the initial conditions

$$q_0 = 1,\ \ r_0 = \sqrt[p]{p} ,$$

and from the differential equation (5)

$$s_0 = \sqrt[p]{p} / p .$$

Moreover from the application of the elasticity we obtain the relations

$$S_k = p \cdot Q_k - (p-1) \cdot R_k ,\ k = 1,2,\ldots,$$

where

$$Q_k = k \cdot q_k - \sum_{i=1}^{k-1} Q_i \cdot q_{k-i} ,$$

$$R_k = k \cdot \frac{r_k}{r_0} - \sum_{i=1}^{k-1} R_i \cdot \frac{r_{k-i}}{r_0}$$

$$S_k = k \cdot \frac{s_k}{s_0} - \sum_{i=1}^{k-1} S_i \cdot \frac{s_{k-i}}{s_0}$$

Example Solve problem

$$y''|y'|^{p-1} - y|y|^{p-1} = 0,$$

$$y(0) = 1,\ y'(0) = 0$$

for $p = 3$.

The calculations shows that

$$Ch_3(x) = 1 + 1.081x^{4/3} + 0.4387x^{8/3} + 0.1260x^4 + \ldots$$

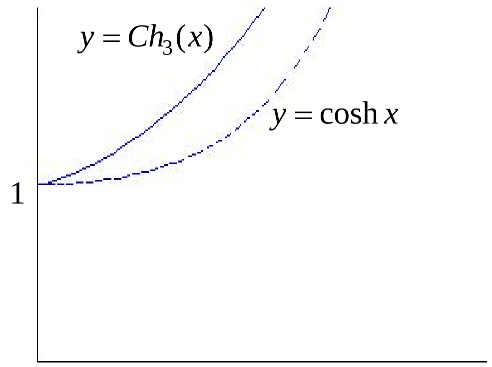

Fig.4.

3 Conclusion

In this paper we intend to give the solution in power series form for nonlinear initial value problems. We gave a method for the determination of the coefficients with the application of the elasticity of functions in power series form. This method enables us to compare the coefficients even so the equation is strongly nonlinear; it is nonlinear in the derivatives too. In the four cases above we dealt with the generalization of hyper geometric functions such as sine, cosine, hyperbolic sine and hyperbolic cosine. In the examples, for a special value $p = 3$, the numerical calculations are presented. Accordingly, in the figures the graphs of the generalized functions are compared with the usual hyper geometric functions.

We point out that the powers α, β are different in the cases of the two different initial conditions (2) and (3), but values of β are the dual pairs.

We note that this method can be applied for more general form of nonlinear initial value problems:

$$y''|y'|^{p-1} + p(x)|y|^{p-1} = 0$$

with initial conditions (2) or (3) if the coefficient function $p(x)$ has the proper power series form.

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