Proceedings of the 9th WSEAS International Conference on Applied Mathematics, Istanbul, Turkey, May 27-29, 2006 (pp367-371)

Nonsmooth Continuous-Time Multiobjective Optimization Problems with Invexity

S. K. MISHRA
Pant University
Department of Mathematics
Pantnagar
INDIA

S. Y. WANG
Chinese Academy of Sciences
Institute of Systems Sciences
Academy of Mathematics
CHINA

K. K. LAI
City University
Management Sciences
Hong Kong SAR

Abstract: A few Karush-Kuhn-Tucker type of sufficient optimality conditions are given in this paper for nonsmooth continuous-time nonlinear multi-objective optimization problems in the Banach space $L_\infty^n[0,\ T]$ of all n-dimensional vector-valued Lebesgue measurable functions which are essentially bounded, using Clarke regularity and generalized convexity. Further, we establish duality theorems for Wolfe and Mond-Weir types of dual problems under the assumptions of invexity, pseudo-invexity and quasi-invexity on the functions involved.

Key–Words: Multiobjective optimization; Nonsmooth Optimization; Generalized convexity; Continuous-time nonlinear optimization; Optimality; Duality

1 Introduction

The relationship between mathematical programming and classical calculus of variation was explored and extended by Hanson [2]. Optimality conditions and duality results are obtained for scalar valued variational problems in Mond and Hanson [4] under convexity. Mishra and Mukherjee [3] extended the work of Mond *et al.* [5] for multiobjective variational problems. For other works on variational problems, one can see [6, 7]. However, very few work has been done on the kind of variational problems is considered in this paper, see for example [6, 7]. In this paper, we develop sufficiency and duality results for nonsmooth continuous-time optimization problems under suitable invexity assumption.

2 Problem Formulation

Consider the following continuous-time nonlinear multi-objective programming problem for short (CNMP):

Min $\phi(x) = \left(\int_0^T f_1(t,\ x(t))\ dt, ..., \int_0^T f_p(t,\ x(t))\ dt\right)$

subject to

$g_j(t, x(t)) \leq 0$ a. e. in $[0,\ T]$,

$$j \in J = \{1, ..., m\},\ x \in X,$$

where X is an open, nonempty convex subset of the Banach space $L_\infty^n[0,\ T]$ of all n-dimensional vector-valued Lebesgue measurable functions, which are essentially bounded, defined on the compact interval $[0,\ T] \subset R$, with the norm $\|\cdot\|_\infty$ defined by

$$\|x\|_\infty = \max_{1\leq k\leq n}\ ess\ \sup\{\ |x_k(t)|\ ,\ 0 \leq t \leq T\}\ ,$$

where for each $t \in [0,\ T]$, $x_k(t)$ is the k^{th} component of $x(t) \in R^n$, ϕ is a real-valued function defined on X, $g(t, x(t)) = \gamma(t)\, x(t)$, and $f(t, x(t)) = \Gamma(x)(t)$, where γ is a map from X into the normed space $\Lambda_1^m[0,\ T]$ of all Lebesgue measurable essentially bounded m-dimensional vector functions defined on $[0,\ T]$, with the norm $\|\cdot\|_1$ defined by

$$\|x\|_1 = \max_{1\leq k\leq m} \int_0^T |y_k(t)|\, dt\ ,$$

and Γ is a map from X into the normed space $\Lambda_1^p[0,\ T]$.

Let Z be a Banach space and $\psi : Z \to R$ be a locally Lipschitz function; i.e., for each $x \in Z$, there exist $\varepsilon > 0$ and a constant $K > 0$, depending on ε, such that

$$|\psi(x_1) - \psi(x_2)| \leq K\,\|x_1 - x_2\|\ \forall x_1, x_2 \in x + \varepsilon B,$$

where B is the open unit ball of Z.

The Clarke generalized directional derivative of ψ at x in the direction of a given

$v \in Z$, denoted by $\psi^0(x; v)$, is defined by

Proceedings of the 9th WSEAS International Conference on Applied Mathematics, Istanbul, Turkey, May 27-29, 2006 (pp367-371)

$$\psi^0(x;v) = \limsup_{\substack{y \to x \\ s \to 0^+}} \frac{\psi(y+sv) - \psi(y)}{s}.$$

The Clarke generalized gradient of ψ at x, denoted by $\partial\psi(x)$, is defined by

$$\partial\psi(x) = \left\{\xi \in Z^* : \langle \xi, v\rangle \le \psi^0(x;v)\ \forall \mathrm{v} \in \mathrm{Z}\right\}.$$

Here, Z^*denotes the dual space of continuous linear functionals on Z,and $\langle\cdot,\cdot\rangle : Z^* \times Z \to R$ is the duality pairing. Please refer to [1] for more details.

Let Ω be the set of all feasible solutions to (CNP), i.e.,

$$\Omega = \{x \in X : g_j(t, x(t)) \le 0 \text{ a.e. in } [0,T],\ j \in J\}.$$

Assume that Ω is non-empty. Let Vbe an open convex subset of R^ncontaining the set

$$\{x(t) \in R^n : x \in \Omega, t \in [0,T]\}.$$

Let f_i and $g_j, i \in I, j \in J$ are real functions defined on $[0,\ T] \times V$. Function $t \to f_i(t, x(t))$ is assumed to be Lebesgue measurable and integrable for all $x \in X$. Assume that, given $a \in V$,there exist an $\varepsilon > 0$ and a positive number k such that $\forall t \in [0,\ T]$, and $\forall x_1, x_2 \in a + \varepsilon B$($B$ denotes the unit ball of R^n) we have

$$|f_i(t,x_1) - f_i(t,x_2)| \le k\,\|x_1 - x_2\|\ , \forall i \in I.$$

Similar hypothesis are assumed for $g_j,\ j \in J$. Hence, $f_i(t,\ \cdot)$ and $g_j(t,\ \cdot),\ i \in I, j \in J$are locally Lipschitz on Vthroughout $[0,\ T]$.

Assume $\bar{x} \in X$and $h \in L^n_\infty[0,\ T]$ are given. The continuous Clarke generalized directional derivatives of f_i and g_j's are given by

$$\mathrm{f}_i^0(t,\bar{x}(t);h(t)) = \Gamma_i^0(\bar{x};h)(t) = \limsup_{\substack{y \to \bar{x} \\ s \to 0^+}} \frac{\Gamma_i^0(y+sh)(t) - \Gamma_i^0(y)(t)}{s}$$

and

$$\mathrm{g}_j^0(t,\bar{x}(t);h(t)) = \gamma_j^0(\bar{x};h)(t) = \limsup_{\substack{y \to \bar{x} \\ s \to 0^+}} \frac{\gamma_j(\mathrm{y}+\mathrm{sh})(t) - \gamma_j(\mathrm{y})(t)}{\mathrm{s}}$$

a.e. in $[0,\ T]$.

It follows easily from the above assumptions that $t \to f_i^0(t, \bar{x}(t); h(t)), t \to g_j^0(t, \bar{x}(t); h(t)),\ i \in I, j \in J$ are Lebesgue measurable and integrable for all $\bar{x} \in X$and $h \in L^n_\infty[0,\ T]$. Let Ube a nonempty subset of $Z$and $\psi : U \to R$be a locally Lipschitz function on U. We introduce the following two duals to the problem (CNMP).

3 Wolfe Dual (WCMD)

$$Max \quad \varphi(\mathrm{u}) = \left(\int_0^T [f_1(t,u(t)) + \lambda(t)\,g(t,u(t))]dt, ..., \int_0^T [f_p(t,u(t)) + \lambda(t)\,g(t,u(t))]dt\right)$$

subject to

$$0 \le \int_0^T \left[\sum_{i=1}^p \tau_i(t) f_i^0(t,u(t);h(t)) + \sum_{j=1}^m \lambda_j(t)\,g_j^0(t,u(t);h(t))\right] dt\ \forall h \in L^n_\infty[0,\ T],$$

$$\lambda(t) \ge 0, \text{ a.e. in } [0,\ T],$$

$$\tau_i(t) \ge 1,\ 1 \le i \le p,\ \sum_{i=1}^p \tau_i = 1,$$

$$u \in X.$$

Let W_1 denote the set of all feasible solutions of (WCMD).

4 Mond-Weir Dual (MWCMD)

$$Max\,\psi(u) = \left(\int_0^T f_1(t,u(t))\,dt, ..., \int_0^T f_p(t,u(t))\,dt\right)$$

subject to

$$0 \le \int_0^T \left[\sum_{i=1}^p \tau_i f_i^0(t,u(t);h(t)) +\right.$$

Proceedings of the 9th WSEAS International Conference on Applied Mathematics, Istanbul, Turkey, May 27-29, 2006 (pp367-371)

$$\sum_{j=1}^{m} \lambda_j(t)\, g_j^0(t, u(t); h(t))\Bigg] dt\ \forall h \in L_\infty^n[0,\ T],$$

$$\lambda(t)\, g(t, u(t)) \geq 0, \text{ a.e. in } [0,\ T]$$

$$\lambda(t) \geq 0, \text{ a.e. in } [0,\ T],$$

$$\tau_i(t) \geq 1,\ 1 \leq i \leq p,\ \sum_{i=1}^{p} \tau_i = 1,$$

$$u \in X.$$

Let W_2 denote the set of all feasible solutions of (MWCMD). Problems (WCMD) and (MWCMD) are the Wolfe type and Mond-Weir type of dual problems of (CNMP), respectively.

5 Problem Solution

In this section, we present a few sufficient conditions for a feasible solution to be an efficient solution (or a weakly efficient solution) to (CNMP).

THEOREM 1. Let $\bar{x} \in \Omega$.Suppose that $f_i(t,\ \cdot)$ and $g_j(t,\ \cdot)$ are strictly invex at $\bar{x}(t)$(with respect to V) throughout $[0,\ T]$ for each $i \in I$ and $j \in J$with the same $\eta(x(t),\ \bar{x}(t))$. Suppose further that there exist $0 \leq \int_0^T \Big[\sum_{i=1}^{p} \tau_i(t) f_i^0(t, u(t); h(t)) + \sum_{j=1}^{m} \lambda_j(t)\, g_j^0(t, u(t); h(t))\Big] dt\ \forall h \in L_\infty^n[0,\ T]$, and $\bar{\lambda} \in L_\infty^m[0,\ T]$ such that

$$0 \leq \int_0^T \Bigg[\sum_{i=1}^{p} \bar{\tau}_i f_i^0(t, \bar{x}(t); h(t))$$

$$+ \sum_{j=1}^{m} \bar{\lambda}_j(t)\, g_j^0(t, \bar{x}(t); h(t))\Bigg] dt\ \forall h \in L_\infty^n[0,\ T],$$

$$\bar{\tau}(t) \geq 0,\ \bar{\lambda}(t) \geq 0\ a.e. \text{ in } [0,\ T],$$

$$(\bar{\tau}(t),\ \bar{\lambda}(t)) \neq 0\ a.e. \text{ in } [0,\ T],$$

$$\bar{\lambda}_j g_j(t, \bar{x}(t)) = 0\ a.e. \text{ in } [0,\ T],\ j \in J.$$

Then $\bar{x}$ is a weakly efficient solution for (CNMP).

THEOREM 2. Let $\bar{x} \in \Omega$.Suppose that $f_i(t,\ \cdot)$ are pseudo-invex and $g_j(t,\ \cdot)$ are quasi-invex at $\bar{x}(t)$(with respect to V) throughout $[0,\ T]$ for each $i \in I$ and $j \in J$with the same $\eta(x(t),\ \bar{x}(t))$. Furthermore, suppose that there exist $0 \leq \int_0^T \Big[\sum_{i=1}^{p} \tau_i(t) f_i^0(t, u(t); h(t)) + \sum_{j=1}^{m} \lambda_j(t)\, g_j^0(t, u(t); h(t))\Big] dt\ \forall h \in L_\infty^n[0,\ T]$, and $\bar{\lambda} \in L_\infty^m[0,\ T]$ such that

$$0 \leq \int_0^T \Bigg[\sum_{i=1}^{p} \bar{\tau}_i f_i^0(t, \bar{x}(t); h(t)) +$$

$$\sum_{j=1}^{m} \bar{\lambda}_j(t)\, g_j^0(t, \bar{x}(t); h(t))\Bigg] dt\ \forall h \in L_\infty^n[0,\ T],$$

$$\bar{\tau}(t) \geq 0,\ \bar{\lambda}(t) \geq 0\ a.e. \text{ in } [0,\ T],$$

$$\bar{\lambda}_j g_j(t, \bar{x}(t)) = 0\ a.e. \text{ in } [0,\ T],\ j \in J.$$

Then $\bar{x}$ is an efficient solution for (CNMP).

Assume that the Clarke regularity holds in the sequel of this section. We define the Lagrangian function $L : X \times L_\infty^p[0,\ T] \times L_\infty^m[0,\ T] \to R$ by

$$L(x, \tau; \lambda) = \int_0^T \Bigg[\sum_{i=1}^{p} \tau_i(t) f_i(t, x(t)) +$$

$$\sum_{j=1}^{m} \lambda_j(t)\, g_j(t, x(t))\Bigg] dt.$$

Let $L_x'(\bar{x}, \tau, \lambda; h)$ denote the usual directional derivative of $L(\cdot, \tau, \lambda)$ at $\bar{x}$ in the direction $h \in L_\infty^n[0,\ T]$,and let $\partial_x L(x, \tau, \lambda)$ denote the Clarke generalized gradient of $L(\cdot, \tau, \lambda)$.

THEOREM 3. Let $\bar{x} \in \Omega$. Suppose that $f_i(t,\ \cdot)$ and $g_j(t,\ \cdot)$ are strictly invex at $\bar{x}(t)$(with respect to V) throughout $[0,\ T]$ for each $i \in I$ and $j \in J$with the same $\eta(x(t),\ \bar{x}(t))$ for all functions. Suppose further that there exist $\bar{\tau} \in L_\infty^p[0,\ T]\ \bar{\lambda} \in L_\infty^m[0,\ T]$ such that

$$0 \in \partial_x L(\bar{x}, \bar{\lambda}_0, \bar{\lambda}),$$

$$\bar{\tau}(t) \geq 0,\ \bar{\lambda}(t) \geq 0\ a.e. \text{ in } [0,\ T],$$

$$(\bar{\tau}(t),\ \bar{\lambda}(t)) \neq 0\ a.e. \text{ in } [0,\ T],$$

Proceedings of the 9th WSEAS International Conference on Applied Mathematics, Istanbul, Turkey, May 27-29, 2006 (pp367-371)

$$\bar{\lambda}_j g_j\left(t, \bar{x}\left(t\right)\right) = 0 \; a.e. \text{ in } \left[0,\ T\right], j \in J.$$

Then $\bar{x}$ is a weakly efficient solution for (CNMP).

THEOREM 4. Let $\bar{x} \in \Omega$.Suppose that $f_i\left(t,\ \cdot\right)$ are pseudo-invex and $g_j\left(t,\ \cdot\right)$ are quasi-invex at $\bar{x}\left(t\right)$(with respect to V) throughout $\left[0,\ T\right]$ for each $i \in I$ and $j \in J$with the same $\eta\left(x\left(t\right),\ \bar{x}\left(t\right)\right)$. Suppose further that there exist $\bar{\tau} \in L^p_\infty\left[0,\ T\right]$ and $\bar{\lambda} \in L^m_\infty\left[0,\ T\right]$such that

$$0 \in \partial_x L\left(\bar{x}, \bar{\tau}, \bar{\lambda}\right),$$

$$\bar{\tau}\left(t\right) \geq 0,\ \bar{\lambda}\left(t\right) \geq 0 \; a.e. \text{ in } \left[0,\ T\right],$$

$$\bar{\lambda}_j g_j\left(t, \bar{x}\left(t\right)\right) = 0 \; a.e. \text{ in } \left[0,\ T\right], j \in J.$$

Then $\bar{x}$ is an efficient solution for (CNMP).

The next two results extend the Propositions 4.3 and 4.4 of Rojas-Medar et al. [6].

THEOREM 5. Let $\bar{x} \in \Omega$. Suppose that $f_i\left(t,\ \cdot\right)$ are pseudo-invex and $g_j\left(t,\ \cdot\right)$ are quasi-invex at $\bar{x}\left(t\right)$(with respect to V) throughout $\left[0,\ T\right]$ for each $i \in I$ and $j \in J$with the same $\eta\left(x\left(t\right),\ \bar{x}\left(t\right)\right)$. If there exist $\bar{\tau} \in L^p_\infty\left[0,\ T\right]$ and $\bar{\lambda} \in L^m_\infty\left[0,\ T\right]$such that

$$0 \in \partial_x L\left(\bar{x}, \bar{\tau}, \bar{\lambda}\right)$$

$$\bar{\tau}\left(t\right) \geq 0,\ \bar{\lambda}\left(t\right) \geq 0 \; a.e. \text{ in } \left[0,\ T\right],$$

$$\bar{\lambda}_j g_j\left(t, \bar{x}\left(t\right)\right) = 0 \; a.e. \text{ in } \left[0,\ T\right], j \in J.$$

Then $\bar{x}$ is an efficient solution for (CNMP).

THEOREM 6. Let $\bar{x} \in \Omega$.Suppose that $\phi\left(\ \cdot\right)$are pseudo-invex and $\sum\limits_{j=1}^{m} \bar{\lambda}_j\left(t\right) g_j(t, \bar{x}(t))dt$ are quasi-invex at $\bar{x}\left(t\right)$(with respect to V) throughout $\left[0,\ T\right]$ for each $i \in I$ and $j \in J$with the same $\eta\left(x\left(t\right),\ \bar{x}\left(t\right)\right)$. If there exist $\bar{\tau} \in L^p_\infty\left[0,\ T\right]$ and $\bar{\lambda} \in L^m_\infty\left[0,\ T\right]$, such that $\left(\bar{x}, \bar{\tau}, \bar{\lambda}\right)$ satisfies (21)-(23). Then $\bar{x}$ is an efficient solution for (CNMP).

THEOREM 7 (Weak Duality). Assume that for all $x \in \Omega$ and for all $\left(u, \tau, \lambda\right) \in W_1$, and $f_i\left(\cdot\right)$ and $\lambda\left(t\right) g\left(\cdot\right)$are invex with respect to the same η. Then, $\phi\left(x\right) \not\leq \varphi\left(u\right)$.

THEOREM 8 (Weak Duality). Assume that for all $x \in \Omega$ and for all $\left(u, \tau, \lambda\right) \in W_2$, $\tau_i f_i\left(\cdot\right)$ are pseudo-invex and $\lambda_j\left(t\right) g_j\left(\cdot\right)$ are quasi-invex with respect to the same η. Then, $\phi\left(x\right) \not\leq \psi\left(u\right)$.

THEOREM 9 (Strong Duality). Letx^*be an efficient solution for (WCMD) and $f\left(t, \cdot\right)$ and $g\left(t, \cdot\right)$be uniformly Lipschitz. If the constraint qualification holds at x^*, then there exist λ such that $\left(x^*, \lambda\right)$ is feasible for (WCMD). Moreover, if the weak duality Theorem 1 holds, then $\left(x^*, \lambda\right)$ is efficient to (WCMD).

THEOREM 10 (Strong Duality). Letx^*be an efficient solution for (MWCMD) and $f\left(t, \cdot\right)$ and $g\left(t, \cdot\right)$be uniformly Lipschitz. If the constraint qualification holds at x^*, then there exist λ such that $\left(x^*, \lambda\right)$ is feasible for (MWCMD). Moreover, if the weak duality Theorem 2 holds, then $\left(x^*, \lambda\right)$ is efficient to (MWCMD).

6 Conclusion

We considered a nonsmooth continuous-time problem similar to the one considered in [6] and establish Kuhn-Tucker type sufficint optimality conditions and duality theorems for Wolfe as well as Mond-Weir type of dual models for the problem under suitable invexity assumption.

References:

[1] F. H. Clarke, *Optimization and Nonsmooth Analysis,* Wiley, New York, 1983.

[2] M. A. Hanson, On sufficiency of the Kuhn-Tucker conditions, *Journal of Mathematical Analysis and Applications* 80 (1981), 545-550.

[3] S. K. Mishra and R. N. Mukherjee, Generalized continuous nondifferentiable fractional programming problems with invexity, *Journal of Mathematical Analysis and Applications* 195 (1995), 191-213.

[4] B. Mond and M. A. Hanson, Duality for variational problems, *Journal of Mathematical Analysis and Applications* 18 (1967) 355-364.

[5] B. Mond, S. Chandra and I. Husain, Duality for variational problems with invexity, *Journal of Mathematical Analysis and Applications* 134 (1988) 322-328.

[6] M. A. Rojas-Medar and A. J. V. Brandao, Nonsmooth Continuous-time optimization prob-

lems: sufficient conditions, *Journal of Mathematical Analysis and Applications* 227 (1998), 305-318.

[7] G. J. Zalmai, Sufficient optimality conditions in continuous-time nonlinear programming, *Journal of Mathematical Analysis and Applications* 111 (1985), 130-147.