Proceedings of the 9th WSEAS International Conference on Applied Mathematics, Istanbul, Turkey, May 27-29, 2006 (pp530-535)

Solving the Initial Value Problem of Ordinary Differential Equation of Higher Order by Integration Method

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Abstract: We will introduce a numerical method for finding numerical solution of the initial value problem of ordinary differential equation of higher order. This new method uses the combination of numerical integration formula, trapezoidal formula, Simpson's formula and the Gaussian elimination method. Some examples will be illustrated and the results will be compared with those obtained by the classical Runge-Kutta method.

Key-Words: Gaussian Runge-Kutta Integration Trapezoidal Simpson

1. Introduction

The initial value problem of the ordinary differential equation of higher order is of the form

(1) $$y' = F(x, y, y', ..., y^{(n-1)})$$

with the initial conditions

$$y(x_0) = c_1$$

$$y'(x_0) = c_2$$

(2) $$y''(x_0) = c_3$$

M

$$y^{(n-1)}(x_0) = c_n .$$

The numerical method which most frequently used to find the numerical solution of the above equations is the Runge-Kutta method by transforming the above equations into the system of n first order equations of initial value problem of ordinary differential equations then solve this system simultaneously.

2. Formulation

To find the numerical solution of the system mentioned above, we come up with the following idea of finding the numerical definite integral. We know that the first order initial value problem of ordinary differential equation

(3) $$y'(x) = f(x, y), \qquad x \in [a, b]$$

with the initial condition

(4) $$y(a) = c$$

is equivalent to the integral equation

(5) $$y(x) = c + \int_a^x f(t, y)dt .$$

By the method of successive approximation, the sequence $\{y_n(x)\}$ converges to the solution of the above equations (3)-(4) where

(6) $$y_{m+1}(x) = c + \int_a^x f(t, y_m(t))dt .$$

So we will use the fact that the sequence $\{y_n(x)\}$ converges to the solution of the above equations (3)-(4). To approximate the numerical solution at the point $x = a + h$ we will use the numerical approximation

(7) $$\int_a^{a+h} f(t, y(t))dt = \frac{h}{2}\left[f(a) + f(a+h)\right]$$

which is trapezoidal formula and

(8) $$\int_a^{a+h} f(t, y(t))dt = \frac{h}{3}\left[f(a) + 4f\left(\frac{2a+h}{2}\right) + f(a+h)\right]$$

which is Simpson's formula. By equation (7) and (8), we obtain eight formulas. They are;

For $y' = f(x, y), \quad y(a) = A$ we have

$$y_{m+1} = y_m + \frac{h}{2}(f(x_m, y_m) + f(x_{m+1}, y_{m+1}))$$

.

For $y'' = f(x, y, y'), y(a) = A$ and

$y'(a) = A'$ which equivalent to

$$y_1' = y_2$$
$$y_2' = f(x, y_1.y_2)$$

thus we have

$$y_{1m+1} = y_{1m} + \frac{h}{2}(y_{2m} + y_{2m+1})$$
$$y_{2m+1} = y_{2m} + \frac{h}{2}(f(x_m, y_m, y_{2m}) + f(x_{m+1}, y_{1m+1}, y_{2m+1})).$$

For $y''' = f(x, y, y', y'')$, $y(a) = A$, $y'(a) = A'$, $y''(a) = A''$ which equivalent to

$$y_1' = y_2$$
$$y_2' = y_3$$
$$y_3' = f(x, y_1.y_2, y_3)$$

thus we have

$$y_{1m+1} = y_{1m} + \frac{h}{2}(y_{2m} + y_{2m+1})$$
$$y_{2m+1} = y_{2m} + \frac{h}{2}(y_{3m} + y_{3m+1})$$
$$y_{3m+1} = y_{3m} + \frac{h}{2}(f(x_m, y_m, y_{2m}, y_{3m}) + f(x_{m+1}, y_{1m+1}, y_{2m+1}, y_{3m+1})).$$

For $y^{iv} = f(x, y, y', y'', y''')$, $y(a) = A$, $y'(a) = A'$, $y''(a) = A''$, $y'''(a) = A'''$ which equivalent to

$$y_1' = y_2$$
$$y_2' = y_3$$
$$y_3' = y_4$$
$$y_4' = f(x, y_1.y_2, y_3, y_4)$$

thus we have

$$y_{1m+1} = y_{1m} + \frac{h}{2}(y_{2m} + y_{2m+1})$$
$$y_{2m+1} = y_{2m} + \frac{h}{2}(y_{3m} + y_{3m+1})$$
$$y_{3m+1} = y_{3m} + \frac{h}{2}(y_{4m} + y_{4m+1})$$
$$y_{4m+1} = y_{4m} + \frac{h}{2}(f(x_m, y_m, y_{2m}, y_{3m}, y_{4m}) + (x_{m+1}, y_{1m+1}, y_{2m+1}, y_{3m+1}, y_{4m+1}))$$

In the first order, since y_m is given and y_{m+1} from trapezoidal, we solve for y_{m+2} as follows;

$$y_{m+2} = y_{1m} + \frac{h}{3}(y_m + 4y_{m+1} + y_{m+2}).$$

In the second order, since y_{1m}, y_{2m} are given and y_{1m+1}, y_{2m+1} from trapezoidal, we solve for y_{1m+2}, y_{2m+2} as follows;

$$y_{1m+2} = y_{1m} + \frac{h}{3}(y_{2m} + 4y_{2m+1} + y_{2m+2})$$
$$y_{2m+2} = y_{2m} + \frac{h}{3}(f(x_m, y_{1m}, y_{2m}) + 4f(x_{m+1}, y_{1m+1}, y_{2m+1}) + f(x_{m+2}, y_{1m+2}, y_{2m+2})).$$

In the third order, since y_{1m}, y_{2m}, y_{3m} are given and $y_{1m+1}, y_{2m+1}, y_{3m+1}$ can be obtained from trapezoidal, we solve for $y_{1m+2}, y_{2m+2}, y_{3m+2}$ as follows;.

$$y_{1m+2} = y_{1m} + \frac{h}{3}(y_{2m} + 4y_{2m+1} + y_{2m+2})$$
$$y_{2m+2} = y_{2m} + \frac{h}{3}(y_{3m} + 4y_{3m+1} + y_{3m+2})$$
$$y_{3m+2} = y_{3m} + \frac{h}{3}(f(x_m, y_{1m}, y_{2m}, y_{3m}) + 4f(x_{m+1}, y_{1m+1}, y_{2m+1}, y_{3m+1}) + f(x_{m+2}, y_{1m+2}, y_{2m+2}, y_{3m+2})).$$

In the fourth order, since $y_{1m}, y_{2m}, y_{3m}, y_{4m}$ are given and $y_{1m+1}, y_{2m+1}, y_{3m+1}, y_{4m+1}$ can be obtained from trapezoidal, we solve for $y_{1m+2}, y_{2m+2}, y_{3m+2}, y_{4m+2}$ as follows;.

$$y_{1m+2} = y_{1m} + \frac{h}{3}(y_{2m} + 4y_{2m+1} + y_{2m+2})$$
$$y_{2m+2} = y_{2m} + \frac{h}{3}(y_{3m} + 4y_{3m+1} + y_{3m+2})$$
$$y_{3m+2} = y_{3m} + \frac{h}{3}(y_{4m} + 4y_{4m+1} + y_{4m+2})$$
$$y_{4m+2} = y_{4m} + \frac{h}{3}(f(x_m, y_{1m}, y_{2m}, y_{3m}, y_{4m}) + 4f(x_{m+1}, y_{1m+1}, y_{2m+1}, y_{3m+1}, y_{4m+1}) + f(x_{m+2}, y_{1m+2}, y_{2m+2}, y_{3m+2}, y_{4m+2})).$$

3. Examples

There are four examples in this section. We will use the above eight systems to find the numerical solutions of these four examples.

Example 1

Find the numerical solution of the equation $y' = \frac{\sin(x)}{x^2} - \frac{2y}{x}$, $x \in [2,3]$ with the initial condition $y(2) = 1$.

The analytical solution of the above equation is

$$y(x) = \frac{4 + \cos(2) - \cos(x)}{x^2}.$$

For Trapezoidal formula, we obtain the equation

$$y_{m+1} = \frac{y_m(1 - \frac{h}{x_m}) + \frac{h}{2}(\frac{\sin x_m}{x_m^2} + \frac{\sin x_{m+1}}{x_{m+1}^2})}{1 + \frac{h}{x_m}}.$$

.For Simpson's formula, we obtain the equation $y_{m+2} = M/(1 + \frac{2h}{3x_{m+2}})$ where

$$M = y_m(1 - \frac{2h}{3x_m}) - \frac{8hy_{m+1}}{3x_{m+1}}$$

$$+ \frac{h}{3}(\frac{\sin x_m}{x_m^2} + 4\frac{\sin x_{m+1}}{x_{m+1}^2} + \frac{\sin x_{m+2}}{x_{m+2}^2}).$$

The numerical results are as follow;

At $x = 2.1$, h=0.1, y=0.92714269117
Error of Trapezoidal 0.00013284506713
Error of Simpson 0.00000000062664
Error of RK2 0.00007084114259
Error of RK3 0.00000173431545

At $x = 2.1$, h=0.01, y=0.92714269117
Error of Trapezoidal 0.00000132545483
Error of Simpson 0.00000000007640
Error of RK2 0.00000066328266
Error of RK3 0.00000000163800

At $x = 2.1$, h=0.001, y=0.92714269117
Error of Trapezoidal 0.00000001324588
Error of Simpson 0.00000000001364
Error of RK2 0.00000000660930
Error of RK3 0.00000000001273

At $x = 2.1$, h=0.0001, y=0.92714269117
Error of Trapezoidal 0.00000000061664
Error of Simpson 0.00000000076852
Error of RK2 0.00000000082400
Error of RK3 0.00000000076670

At $x = 3.0$, h=0.1, y=0.50820507334
Error of Trapezoidal 0.00037354729557
Error of Simpson 0.00000171579177
Error of RK2 0.00023297424923
Error of RK3 0.00000442352120

At $x = 3.0$, h=0.01, y=0.50820728672
Error of Trapezoidal 0.00000373132752
Error of Simpson 0.00000000019281
Error of RK2 0.00000221333175
Error of RK3 0.00000000424370

At $x = 3.0$, h=0.001, y=0.50820728672
Error of Trapezoidal 0.00000003727109
Error of Simpson 0.00000000008640
Error of RK2 0.00000002209526
Error of RK3 0.00000000007094

At $x = 3.0$, h=0.0001, y=0.50820728672
Error of Trapezoidal 0.00000000328600
Error of Simpson 0.00000000373075
Error of RK2 0.00000000383534
Error of RK3 0.00000000361979

Example 2

Find the numerical solution of $y'' = \frac{1}{x^2 e^{2x}} - 4y' - 4y$, $x \in [1, 2]$ with the initial condition $y(1) = 0$ and $y'(1) = -\frac{1}{e^2}$. The analytical solution of the above equation is $y(x) = -\frac{\ln x}{e^{2x}}$.

For Trapezoidal formula, we obtain

$$y_{1,m+1} = y_{1,m} + \frac{h}{2}(y_{2,m} + y_{2,m+1})$$

$$y_{2,m+1} = \frac{M}{1 + 2h + h^2} \text{ where}$$

$$M = y_{2,m}(1 - 2h - h^2) - 4hy_{1,m} + \frac{h}{2}\left(\frac{1}{x_m^2 e^{2x_m}} + \frac{1}{x_{m+1}^2 e^{2x_{m+1}}}\right).$$

For Simpson's formula, we obtain

$$y_{1,m+2} = y_{1,m} + \frac{h}{3}(y_{2,m} + 4y_{2,m+1} + y_{2,m+2})$$

$$y_{2,m+2} = \frac{M}{1 + \frac{4}{3}h + \frac{4}{9}h^2} \text{ where}$$

$$M=(1-\frac{4}{3}h-\frac{4}{9}h^2)y_{2,m}-(\frac{16}{3}h+\frac{16}{9}h^2)y_{2,m+1}$$
$$-\frac{8}{3}hy_{1,m}-\frac{16}{3}hy_{1,m+1}$$
$$+\frac{h}{3}\left(\frac{2^{-2x_m}}{x_m^2}+\frac{e^{-2x_{m+1}}}{x_{m+1}^2}+\frac{e^{-2x_{m+2}}}{x_{m+2}^2}\right).$$

The numerical solutions are as follow;

At $x=1.1$, h=0.1, y= -0.010150146243
Error of Trapezoidal 0.00015524673117
Error of Simpson 0.00000000020027
Error of RK2 0.00041052270392
Error of RK3 0.00003125036133

At $x=1.1$, h=0.01,
y = -0.010150146243
Error of Trapezoidal 0.00000152200182
Error of Simpson 0.00000000015881
Error of RK2 0.00000334368453
Error of RK3 0.00000002504973

At $x=1.1$, h=0.001,
y = -0.010150146243
Error of Trapezoidal 0.00000001521452
Error of Simpson 0.00000000000251
Error of RK2 0.00000003276871
Error of RK3 0.00000000002274

At $x=2.0$, h=0.1,
y = - 0.012695419162
Error of Trapezoidal 0.00003106919285
Error of Simpson 0.00000279127178
Error of RK2 0.00019009434600
Error of RK3 0.00001609703141

At $x=2.0$, h=0.01,
y = - 0.012695419162
Error of Trapezoidal 0.00000031214103
Error of Simpson 0.00000000000355
Error of RK2 0.00000146320170
Error of RK3 0.00000001249283

At $x=2.0$, h=0.001,
y = - 0.012695419162
Error of Trapezoidal 0.00000000312001
Error of Simpson 0.00000000000091
Error of RK2 0.00000001429055
Error of RK3 0.00000000001643

Example 3

Find the numerical solution of the equation $y'''+y''-3y'-\frac{6e^x}{x^4}$,
$x\in[1,2],\ \ y(1)=e,\ \ y'(1)=0,\ \ y''(1)=e.$

The analytical solution of the equation is
$$y(x)=\frac{e^x}{x}.$$

For the Trapezoidal formula, we obtain the equations
$$y_{1,m+1}=y_{1,m}+\frac{h}{2}(y_{2,m}+y_{2,m+1})$$
$$y_{2,m+1}=y_{2,m}+\frac{h}{2}(y_{3,m}+y_{3,m+1})$$
$$y_{3,m+1}=\frac{M}{1+\frac{3}{2}h+\frac{3}{4}h^2-\frac{h^3}{8}}\ \text{ where}$$
$$M=(1-\frac{3}{2}h-\frac{3}{4}h^2+\frac{h^3}{8})y_{3,m}+(-3h+\frac{h^2}{2})y_{2,m}$$
$$+hy_{1,m}-3h\left(\frac{e^{x_m}}{x_m^4}+\frac{e^{x_{m+1}}}{x_{m+1}^4}\right).$$

For Simpson's formula, we obtain the equations
$$y_{1,m+2}=y_{1,m}+\frac{h}{3}\left(y_{2,m}+4y_{2,m+1}+y_{2,m+2}\right)$$
$$y_{2,m+2}=y_{2,m}+\frac{h}{3}\left(y_{3,m}+4y_{3,m+1}+y_{3,m+2}\right)$$
$$y_{3,m+2}=\frac{M}{1-h+\frac{h^2}{3}-\frac{h^3}{27}}\ \text{ where}$$
$$M=\left(1+h-\frac{h^2}{3}+\frac{h^3}{27}\right)y_{3,m}+\left(4h-\frac{4}{3}h^2+\frac{4}{27}h^3\right)y_{3,m+1}$$
$$+\left(-2h+\frac{2}{9}h^2\right)y_{2,m}+\left(-4h+\frac{4}{9}h^2\right)y_{2,m+1}$$
$$+\frac{2}{3}hy_{1,m}+\frac{4}{3}hy_{1,m+1}$$
$$-2h\left(\frac{e^{x_m}}{x_m^4}+\frac{4e^{x_{m+1}}}{x_{m+1}^4}+\frac{e^{x_{m+2}}}{x_{m+2}^4}\right).$$

The numerical solutions are as follow;

At $x=1.1$, h=0.1, y=2.7318732376
Error of Trapezoidal 0.00030599165595
Error of Simpson 0.00000002952947
Error of RK2 0.00081321582911
Error of RK3 0.00009287811190

At $x=1.1$, h=0.01, y=2.7318732376
Error of Trapezoidal 0.00000293977791
Error of Simpson 0.00000000063665
Error of RK2 0.00000533787170
Error of RK3 0.00000009884752

Proceedings of the 9th WSEAS International Conference on Applied Mathematics, Istanbul, Turkey, May 27-29, 2006 (pp530-535)

At $x = 1.1$, h=0.001, y=2.7318732376
Error of Trapezoidal 0.00000002942397
Error of Simpson 0.00000000002183
Error of RK2 0.00000005065158
Error of RK3 0.00000000012005

At $x = 1.1$, h=0.0001, y=2.7318732376
Error of Trapezoidal 0.00000000001091
Error of Simpson 0.00000000025466
Error of RK2 0.00000000065484
Error of RK3 0.00000079450183

At $x = 2.0$, h=0.1, y=3.6443723151
Error of Trapezoidal 0.01494241224600
Error of Simpson 0.00021100712911
Error of RK2 0.05015573438500
Error of RK3 0.00216866279520

At $x = 2.0$, h=0.01, y=3.6443723151
Error of Trapezoidal 0.00049050454640
Error of Simpson 0.00000010041913
Error of RK2 0.00057297762396
Error of RK3 0.00000207840276

At $x = 2.0$, h=0.001, y=3.6443723151
Error of Trapezoidal 0.00000149049447
Error of Simpson 0.00000000010114
Error of RK2 0.00000578748222
Error of RK3 0.00000000219370

At $x = 2.0$, h=0.0001, y=3.6443723151
Error of Trapezoidal 0.00000001015724
Error of Simpson 0.00000000105865
Error of RK2 0.00000005369657
Error of RK3 0.00000000468935

Example 4

Find the numerical solution of the equation

$$y^{iv} = x + y - xy' + y'' + xy''' + \frac{2}{x^3},$$

$x \in [1,2]$ with the initial condition $y(1) = 0,\ y'(1) = 1, y''(1) = 1,\ \ y'''(1) = -1.$

The analytical solution of the above equation is $y(x) = x \ln x$.

For trapezoidal formula, we obtain the equation

$$y_{1m+2} = y_{1m} + \frac{h}{3}(y_{2m} + 4y_{2m+1} + y_{2m+2})$$

$$y_{2m+2} = y_{2m} + \frac{h}{3}(y_{3m} + 4y_{3m+1} + y_{3m+2})$$

$$y_{3m+1} = y_{3m} + \frac{h}{2}(y_{4m} + y_{4m+1})$$

$$y_{4,m+1} = \frac{M}{1 - \frac{h}{2}x_{m+1} - \frac{h^2}{4} + \frac{h^3}{8}x_{m+1} - \frac{h^4}{16}} \quad \text{where}$$

$$M = \left(1 + \frac{h}{2}x_m + \frac{h^2}{4} - \frac{h^3}{8}x_{m+1} + \frac{h^4}{16}\right)y_{3,m} + \left(h - \frac{h^2}{2}x_{m+1} + \frac{h^3}{4}\right)y_{3,m} + \frac{h}{2}\left(h - x_m - x_{m+1}\right)y_{2,m} + hy_{1,m} + h\left(\frac{x_m}{2} + \frac{x_{m+1}}{2} + \frac{1}{x_m^3} + \frac{1}{x_{m+1}^3}\right).$$

For Simpson's formula, we obtain the equation

$$y_{1m+2} = y_{1m} + \frac{h}{3}(y_{2m} + 4y_{2m+1} + y_{2m+2})$$

$$y_{2m+2} = y_{2m} + \frac{h}{3}(y_{3m} + 4y_{3m+1} + y_{3m+2})$$

$$y_{3m+2} = y_{3m} + \frac{h}{3}(y_{4m} + 4y_{4m+1} + y_{4m+2})$$

$$y_{4m+2} = \{\frac{h}{3}(x_m + 4x_{m+1+x_{m+2}} + \frac{2}{x_m^3} + \frac{8}{x_{n+1}^3} + \frac{2}{x_{m+2}^3}) + \frac{2h}{3}y_{1m} + \frac{4h}{3}y_{1m+1} + \frac{h}{3}y_{2m}(\frac{2h}{3} - x_m - x_{m+2}) + \frac{4h}{3}y_{2m+1}(\frac{h}{3} - x_{m+1}) + \frac{2h}{3}y_{3m}(1 - \frac{hx_{m+2}}{3} + \frac{h^2}{9}) + \frac{4h}{3}y_{3m+1}(1 - \frac{hx_{m+2}}{3} + \frac{h^2}{9}) + y_{4m}(1 + \frac{hx_m}{3} + \frac{h^2}{9} - \frac{h^3x_{m+2}}{27} + \frac{h^4}{81}) + \frac{4h}{3}y_{4m+1}(x_{m+1} + \frac{h}{3} - \frac{h^2x_{m+2}}{9} + \frac{h^3}{27})\}/M \text{ where}$$

$$M = \frac{1}{1 - \frac{h}{2}x_{m+1}} - \frac{h^2}{4} + \frac{h^3}{8}x_{m+1} - \frac{h^4}{16}.$$

The numerical results are in as follow;

At $x = 1.1$, h=0.1, y=1.05
Error of Trapezoidal 0.00006929845259
Error of Simpson 0.00000000070384
Error of RK2 0.00015880221258
Error of RK3 0.00000786445423

At $x = 1.1$, h=0.01, y=1.05

Error of Trapezoidal 0.00000068867234
Error of Simpson 0.00000064145761
Error of RK2 0.00000139831525
Error of RK3 0.00000000627642

At $x = 1.1$, h=0.001, y=1.05
Error of Trapezoidal 0.00000000690989
Error of Simpson 0.00000645738442
Error of RK2 0.00000001379260
Error of RK3 0.00000000002979

At $x = 1.1$, h=0.0001, y=1.05
Error of Trapezoidal 0.00000000070372
Error of Simpson 0.00000000084503
Error of RK2 0.00000000091177
Error of RK3 0.00000000076966

At $x = 2.0$, h=0.1, y=1.3877975821
Error of Trapezoidal 0.00013412624503
Error of Simpson 0.00077794139906
Error of RK2 0.00150322096490
Error of RK3 0.00003451793236

At $x = 2.0$, h=0.01, y=3. 1.3877975821
Error of Trapezoidal 0.00000126594568
Error of Simpson 0.00001004191290
Error of RK2 0.00001637324749
Error of RK3 0.00000003301830

At $x = 2.0$, h=0.001, y=1.3877975821
Error of Trapezoidal 0.00000001227090
Error of Simpson 0.00005593667083
Error of RK2 0.00000016431841
Error of RK3 0.00000000032742

At $x = 2.0$, h=0.0001, y=1.3877975821
Error of Trapezoidal 0.00000001243461
Error of Simpson 0.00000113504575
Error of RK2 0.00000001390799
Error of RK3 0.00000001216722

4. Conclusion

The new method, integration method, is the good numerical method for solving the initial value problem of ordinary differential equations. The results from the examples are satisfactory. This new integration method will give mathematician more freedom to select the way to find the numerical solution of the initial value problem of ordinary differential equations.

We strongly recommend the Simpson's formula for this new integration method since its error is of the form $O(h^5)$. We may use the Midpoint formula in this integration method. However we recommend the integration formula with four and five points which we strongly believe that will give us the very best result.

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