Proceedings of the 9th WSEAS International Conference on Applied Mathematics, Istanbul, Turkey, May 27-29, 2006 (pp477-482)

Application of Wavelets to the Optimal Control of a Parallel System of Beams

Ibrahim Sadek, Taher Abualrub, and Marwan Abukhaled
Department of Mathematics and Statistics, American University of Sharjah, Sharjah-UAE

Abstract

The optimal control of a distributed system consisting of two Euler-Bernoulli beams coupled in parallel with pointwise controllers is considered. The optimal control problem is to minimize a given performance index over these forces and subject to the equation of motion governing the structural vibrations, the imposed initial condition as well as the boundary conditions. A computationally attractive method by using finite wavelets for evaluating the modal optimal control and trajectory of the lumped parameter system is suggested. A numerical example is provided to demonstrate the applicability and effectiveness of the proposed method.

1 Introduction

Orthogonal functions have received considerable attention in obtaining approximate solutions of dynamical systems [1]. The main idea of this technique is that it reduces these problems to those of solving a system of algebraic equations and thus it greatly simplifies the problem. The state and/or control involved in the equations are approximated by finite terms of orthogonal series and using an operational matrix of integration to eliminate the integral operations. The form of the operational matrix of integration depends on the particular choice of the orthogonal functions [1]. In this study, we use wavelet functions to approximate both the control and state functions. It offers a different approach from the standard variational method [3] and has the advantage of being attractive computationally. It avoids the difficult integral equations created from variational methods by reducing the problem to solving an algebraic system of equations, thus providing a computationally more efficient approach. In addition, solving a system of coupled initial-boundary-terminal-value problems, a requirement for the maximum principle [2], is avoided. In this study, the sine-cosine wavelets will be used as they are widely used in many fields of engineering.

The present study deals with the optimal control of two parallel simply supported beams coupled by pointwise springs with pointwise controllers applied along the beams. The basic optimal control problem is to minimize a given performance index in a given period of time with the minimum expenditure of force. Using modal expansion, the optimal control of a distributed parameter system is reduced to the optimal control of a lumped parameter system. The parameterization approach is used to approximate the state variable and each component of the control varying using finite-term wavelet with unknown coefficients. The quadratic problem is then transformed into a mathematical programming problem with the objective of minimizing the unknown coefficients to obtain suboptimal solution. A necessary condition for optimality of the unknown coefficients is derived as a system of linear algebraic equations, for which the solution is used to obtain the optimal control force and optimal state function.

To demonstrate the effectiveness of the suggested approach, a numerical example which simulates the application of a single actuator on each beam is presented. Results indicate that the proposed method significantly minimizes the energy of the system.

2 Optimal Control Problem Formulation

An elastically connected double-beam system consists of two parallel beams of the same length l through m-discrete springs with constant stiffness modulus k_j placed at $x_j^s, j = 1, 2, ..., m$, as shown in Figure 1. The transverse vibrations of a double beam system are governed by the following differential equations based on the Bernoulli-Euler theory [6] and the application of pointwise controllers $f_{ij}(t)$ at n_i discrete points $0 < x_{i1}^a < x_{i2}^a < \ldots < x_{in_i}^a < l, i = 1$ or 2 refers to beam 1 or 2, respectively.

$$\begin{aligned} & m_i \partial_t^2 u^i(x,t) + K_i \partial_x^4 u^i(x,t) \\ & +(-1)^{i+1} \sum_{j=1}^{m} k\left(u^1 - u^2\right)(x_j^s, t)]\delta(x - x_j^s) \\ = & \sum_{j=1}^{n_i} \gamma_{ij} f_{ij}(t) \delta(x - x_{ij}^a), \end{aligned} \quad (1)$$

where $(x,t) \in \Omega = \Omega_x \times \Omega_t = [0,l] \times [0,t_f], i = 1,2$, ∂_t and ∂_x represent the partial derivatives with respect to the time t, and space variable x, respectively. $u^i(x,t), i = 1,2$ are the vertical displacement of the beams measured from the horizontal equilibrium positions. The system parameters are $K_i = E_i I_i,\ \ m_i = \rho_i A_i$, where K_i is the flexural rigidity of the beam, E_i is Young's modulus of elasticity, I_i is the moment of inertia of the beam cross-section, A_i is the cross-sectional area of the beam, ρ_i is the mass density, t_f is the terminal time, and $f_{ij}(t)$ is the amplitude (or the influence) of actuators located at discrete points $x_{ij}^a \in (0,l)$ for $j = 1, 2, \ldots n_i$.The term $(-1)^i \sum_{j=1}^{m} k_j \left[u^1(x_j^s,t) - u^2(x_j^s,t)\right] \delta(x - x_j^s)$ in equation [1] represents the coupling between the two beams, where $\delta(x - x_j^s)$ are Dirac distributions with discrete points $0 < x_j^s < l$. For simplicity, the elastic coupling constants, k_j, are assumed to be independent of the spatial parameter. Associated with the dynamic model of parallel beams, some appropriate boundary and initial conditions need to be prescribed.

For simplicity of the analysis, all four ends of beams are assumed to be simply supported. The boundary conditions are:

$$\begin{aligned} u^i(0,t) &= u^i(l,t) = 0 \quad (2) \\ \partial_x^2 u^i(0,t) &= \partial_x^2 u^i(l,t) = 0, i = 1,2,\ t \in \Omega_t \end{aligned}$$

The initial conditions are

$$u^i(x,0) = u_0^i(x),\ \partial_t u^i(x,0) = u_1^i(x) \quad (3)$$

Consider the set of admissible distributions:

$$U_{\text{ad}} = \left\{ \begin{array}{c} f_{ij}(t) : [0,t_f] \to \mathbb{R} | \int_0^{t_f} f_{ij}^2(t) dt < \infty, \\ i = 1,2,\ j = 1, 2, \ldots n_i \end{array} \right\}. \quad (4)$$

In order to measure the performance of the system under the influence of the applied control forces $f_{ij}(t) \in U_{\text{ad}}$,for all i and j, we introduce the following performing index:

$$J(\overrightarrow{f}(t)) = E(t_f) + F(t_f) \quad (5)$$

where

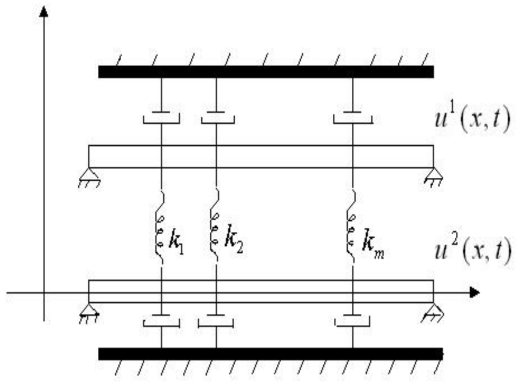

Figure 1: A geometric graph of two Euler-Bernoulli beams

$$E(t_f) = \frac{1}{2}\int_0^l \left\{ \begin{array}{c} \sum_{i=1}^{2} \mu_i \left[u^i(x,t_f)\right]^2 \\ +\sum_{i=1}^{2} \mu_{i+2} \left[\partial_t u^i(x,t_f)\right]^2 \end{array} \right\} dx$$

$$F(t_f) = \frac{1}{2}\int_0^{t_f} \sum_{i=1}^{2} \left(\sum_{j=1}^{n_i} \mu_{i+4} f_{ij}^2(t) \right) dt,$$

$$\overrightarrow{f}(t) = \left(f_{11}(t), \ldots f_{1n_1}(t), f_{21}(t), \ldots, f_{2n_2}(t)\right),$$

where $\mu_i \geq 0$, are the weighting constants, which reflect the relative weighting attached to each term of (5), and $\sum_{i=1}^{4} \mu_i \neq 0$. $E(t_f)$ is the contributions of the modified energies of the double beam, and $F(t_f)$ represents the contribution of the control energy that accumulates over the control duration Ω_t.

The optimal control of the double-beam can now be expressed as

$$\min J(\overrightarrow{f}) \text{ with } u^i(x,t) \text{ is subjected to (2)-(3)}, \quad (6)$$

$i = 1, 2.$

3 Control Problem in Model-Space

The distributed parameter system (1) can be transformed into a modal lumped parameter problem by means of the eigenfunction technique [3]. The functions $u^i(x,t)$ can be expanded in terms of the finite orthonormal eigenfunctions $\varphi_n(x) = \sqrt{2/l}\sin(\lambda_n x)$, $\lambda_n = n\pi/l$.

$$u^i(x,t) = \sum_{n=1}^{N} u_n^i(t)\varphi_n(x) \text{ or } \overrightarrow{v}(x,t) = \overrightarrow{r}^T(t)\overrightarrow{\varphi}(x), \quad (7)$$

where

$$\begin{aligned} \overrightarrow{v}(x,t) &= [u^1(x,t), u^2(x,t)]^T, \\ \overrightarrow{r}(t) &= [u_1^1(t), \ldots, u_N^1(t), u_1^2(t), \ldots, u_N^2(t)]_{2N\times 1}^T, \\ \overrightarrow{\varphi}(x) &= [\varphi_1(x), \ldots, \varphi_N(x), \varphi_1(x), \ldots, \varphi_N(x)]_{2N\times 1}^T, \end{aligned}$$

and T stands for a vector transpose.

Substituting from (7) into equation (1), taking the dot product of the resulting expression by $\vec{\varphi}(x)$ and integrating with respect to x over the spatial domain Ω_x, we obtain

$$\frac{d^2}{dt^2}\overrightarrow{r}(t) + D\overrightarrow{r}(t) = \Phi^a \overrightarrow{f}(t), \quad t \in \Omega_1 = [0,1]. \quad (8)$$

where

$$\begin{aligned} D_{2N\times 2N} &= C + \Phi^s G \\ \Phi^a_{2N\times 2N} &= \text{diag}\left[(t_f^2 \gamma_{ij}/m_i)\varphi_j(x_{ij}^a)\right], \\ i &= 1,2,\ j = 1,2,\ldots,n_i. \\ \overrightarrow{f}_{\overline{m}\times 1}(t) &= [f_{11}(t), \ldots, f_{1n_1}(t), f_{21}(t), \ldots, f_{2n_2}(t)]^T, \end{aligned}$$

in which

$$\begin{aligned} C_{2N\times 2N} &= t_f^2 \left(\frac{\pi}{l}\right)^4 \left[\begin{array}{c} \text{diag}\left[\left(\frac{K_1 j}{m_1}\right)\right] \\ \text{diag}\left[\left(\frac{K_1 j}{m_1}\right)\right] \end{array} \right], \\ j &= 1,2,\cdots,N \\ \Phi^s_{2N\times 2m} &= (t_f^2)\text{diag}[(k_i/m_i)\,\varphi_j(x_i^s)], \\ i &= 1,2, \quad j = 1,2,\ldots,n_i. \\ G_{\overline{m}\times 2N} &= \left[\begin{array}{cc} [g] & -[g] \\ -[g] & [g] \end{array} \right],\ \overline{m} = n_1 + n_2. \end{aligned}$$

where

$$[g] = \left[\begin{array}{ccc} \varphi_1(x_1^s) & \cdots & \varphi_N(x_1^s) \\ \vdots & & \vdots \\ \varphi_1(x_m^s) & \cdots & \varphi_N(x_m^s) \end{array} \right]$$

Since $u_0^i(x)$ and $u_1^i(x)$ are elements of $H = L^2(\Omega_x)$, one can approximate them with the finite-dimensional basis $\{\varphi_i(x)\}_{i=1}^N$ of H and thus the modal initial condition of equation (3) become

$$\begin{aligned} u_{0_j}^i(0) &= \left\langle u_0^i(x), \varphi_j(x)\right\rangle_{\Omega_x}, \\ u_{1_j}^i(0) &= \left\langle u_1^i(x), \varphi_j(x)\right\rangle_{\Omega_x}, \\ i &= 1,2,\ j = 1,2,\cdots,N \end{aligned} \quad (9)$$

and the corresponding cost functional (5) in the first N controlled modes results into

$$J_N\left(\overrightarrow{f}(t)\right) = \frac{1}{2}\left[\begin{array}{c} \langle \overrightarrow{r}(t_f), R_1 \overrightarrow{r}(t_f)\rangle_{\Omega_t} \\ +\langle \frac{d}{dt}\overrightarrow{r}(t_f), R_2 \frac{d}{dt}\overrightarrow{r}(t_f)\rangle_{\Omega_t} \\ +t_f \left\langle \overrightarrow{F}(t), R_3 \overrightarrow{F}(t)\right\rangle_{\Omega_t} \end{array} \right] \quad (10)$$

where R_i =diag$[\mu_{ij}]$, $i = 1,2$, and $j = 0,1,\cdots,N$, and $\langle,\rangle$ denotes the inner product.

The optimal control problem in modal space can now be expressed as

$$\min J_N\left(\overrightarrow{f}(t)\right) \text{ with } \overrightarrow{r}(t) \text{ subject to (8)-(9)} \quad (11)$$

where $J_N\left(\overrightarrow{f}(t)\right)$ is defined in (10).

4 Wavelet-base Approach

In this section, a direct method for solving the modal control problem (11) by parametrizing the generalized state $\overrightarrow{r}(t)$ and control $\overrightarrow{f}(t)$ vectors is developed. The modal cost functional $J_N\left(\overrightarrow{f}(t)\right)$ of equation (10) reduces to a function of unknown parameters and thus yields an ordinary minimization problem over $\mathbb{R}^m$.

4.1 State and control parameterization

Let the orthogonal function series approximations of the generalized state variables $r_j(t)$ and the control variables $f_{ij}(t),\ t \in \Omega_1 = [0,1]$ be given by

$$r_j(t) = \sum_{m=0}^{2L}\sum_{n=0}^{2^k-1} a_{nm}^j s_{nm}(t), \quad (12)$$

$$f_{ij}(t) = \sum_{m=0}^{2L}\sum_{n=0}^{2^k-1} b_{nm}^{ij} s_{nm}(t), \quad (13)$$

where $i = 1,2$, and $j = 1,2,\cdots 2N$ and

$$a_{nm}^j = \langle r_j(t), s_{nm}(t)\rangle_{\Omega_1},\ b_{nm}^{ij} = \langle f_{ij}(t), s_{nm}(t)\rangle_{\Omega_1}$$

and s_{nm} are orthogonal wavelet functions defined on Ω_1 such as sine-cosine [5] or Legendre wavelets [4]. In the present study we use sine-cosine wavelets [5].

Equations (12) and (13) can be written in vector notation as follows

$$\overrightarrow{r}(t) = A\overrightarrow{s}(t),\ \overrightarrow{f}(t) = B\overrightarrow{s}(t) \quad (14)$$

where

$$A_{2N\times p}=\begin{bmatrix} a_{0,0}^1 & \cdots & a_{0,r}^1 & \cdots & a_{v,0}^1 & \cdots & a_{v,r}^1 \\ \vdots & & & & & & \vdots \\ a_{0,0}^N & \cdots & a_{0,r}^N & \cdots & a_{v,0}^N & \cdots & a_{v,r}^N \\ \vdots & & & & & & \vdots \\ a_{0,0}^{2N} & \cdots & a_{0,r}^{2N} & \cdots & a_{v,0}^{2N} & \cdots & a_{v,r}^{2N} \end{bmatrix}$$

$$\overrightarrow{s}_{p\times 1}(t) = \begin{bmatrix} s_{0,0}(t), s_{0,1}(t),\cdots,s_{0,r}(t), \\ \cdots, s_{v,0}(t),\cdots s_{v,r}(t) \end{bmatrix}^T \quad (15)$$

where $p = 2^k(r+1), v = 2^k - 1, r = 2L$.

Upon integrating (8) twice over the finite domain Ω_1, we obtain

$$\begin{aligned} \overrightarrow{r}(t) &= \overrightarrow{r}(0) + t\frac{d}{dt}\overrightarrow{r}(0) \\ &\quad -D\int_0^t\int_0^\eta \overrightarrow{r}(\tau)d\tau d\eta \\ &\quad +\int_0^t\int_0^\eta \Phi^a\overrightarrow{f}(\eta)d\tau d\eta \end{aligned} \quad (16)$$

Let

$$\overrightarrow{r}(0) = U^0\ \overrightarrow{s}(t),\ t\frac{d}{dt}\overrightarrow{r}(0) = U^1\overrightarrow{s}(t) \quad (17)$$

where

$$U_{2N\times s}^0 = \begin{bmatrix} u_1^1(0) & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ u_N^1(0) & 0 & \cdots & 0 \\ u_1^2(0) & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ u_N^2(0) & 0 & \cdots & 0 \end{bmatrix}$$

and

$$U_{2N\times s}^1 = \begin{bmatrix} \overrightarrow{g}^T\frac{d}{dt}u_1^1(0) \\ \vdots \\ \overrightarrow{g}^T\frac{d}{dt}u_N^1(0) \\ \overrightarrow{g}^T\frac{d}{dt}u_1^2(0) \\ \vdots \\ \overrightarrow{g}^T\frac{d}{dt}u_N^2(0) \end{bmatrix}$$

in which g is given by [4]

$$\overrightarrow{g} = \begin{bmatrix} 1, 0, \cdots, 0, \frac{-\sqrt{2}}{\pi}, \frac{-\sqrt{2}}{2\pi}, \cdots, \frac{-\sqrt{2}}{\pi L} | \\ 3, 0, \cdots, 0, \frac{-\sqrt{2}}{\pi}, \frac{-\sqrt{2}}{2\pi}, \cdots, \frac{-\sqrt{2}}{\pi L} | \\ \cdots | 2^{k+1} - 1, 0, \cdots, 0, \frac{-\sqrt{2}}{\pi}, \frac{-\sqrt{2}}{2\pi}, \cdots, \frac{-\sqrt{2}}{\pi L} \end{bmatrix}^T.$$

Substitute from (14) into (16) and drop $\overrightarrow{s}(t)$ from it to get the matrix equation

$$A + Q + DAP^2 = \Phi^a BP^2 \tag{18}$$

where $Q = -(U^0 + U^1)$ and P is the sine-cosine wavelets matrix, A and B are unknown matrices to be determined.

Equation (18) can then be written in the form of vec [7]

$$\text{vec}(A) + \text{vec}(Q) + \text{vec}(DAP^2) = \text{vec}(\Phi^a BP^2) \tag{19}$$

Between the vec and Kronecker product $\otimes$ [7], equation (**??**) can be easily reduced to

$$[I + B \otimes \Gamma]\overrightarrow{a} + \overrightarrow{q} = (\Phi^a \otimes \Gamma)\overrightarrow{b} \tag{20}$$

where $\overrightarrow{a} = \text{vec}(A)$ and $\overrightarrow{b} = \text{vec}(B)$ are the $r \times 2Np$-dimensional state and control unknown parameters. Moreover $\overrightarrow{q}_{r\times 1} = \text{vec}(Q)$ and $\Gamma = P^2$.

Let

$$X = I + B \otimes \Gamma, \qquad Y = \Phi^a \otimes \Gamma \tag{21}$$

and use equation (20) to solve the state unknown parameters $\overrightarrow{b}$, that is

$$\overrightarrow{a} = W\overrightarrow{b} + \overrightarrow{w} \tag{22}$$

provided that X^{-1}exists, and where $W = Y^{-1}X$ and $\overrightarrow{w} = -X^{-1}\overrightarrow{q}$.

4.2 Approximate of modal performance index

Refine equations (14) as

$$\overrightarrow{r}(t) = S(t)\ \overrightarrow{a},\ \overrightarrow{f}(t) = S(t)\ \overrightarrow{b} \tag{23}$$

where the $2N \times r$ matrix is given by $S_{2N\times r}(t) = \text{diag}\left[\overrightarrow{s}^T(t)\right]$.

The relations (23) can be used to approximate the modal performance index $J_N(\overrightarrow{f})$ given in equation (10). The resulted approximation, which we shall denote by $J_{N,r}(\overrightarrow{f})$, entirely depends on the state of parameter $\overrightarrow{a}$. Upon the use of the relation (22), $J_{N,r}(\overrightarrow{f})$ may be expressed, after some algebraic manipulations, as

$$J_{N,r}(\overrightarrow{f}) = \frac{1}{2}\left[\overrightarrow{b}^T Y\overrightarrow{b} + \overrightarrow{b}^T\overrightarrow{y}_1 + \overrightarrow{y}_2\overrightarrow{b} + \alpha\right] \tag{24}$$

where

$$\begin{aligned} Y &= W^T G_1(t_f)W + G_2(t_f),\ \overrightarrow{y}_1 = W^T G_1(t_f)\overrightarrow{w}, \\ \overrightarrow{y}_2 &= \overrightarrow{w}^T G_1(t_f)W,\ \alpha = \overrightarrow{w}^T G_1(t_f)\overrightarrow{w} \end{aligned}$$

in which

$$\begin{aligned} G_1(t_f) &= S^T(t_f)R_1 S(t_f) + \frac{d}{dt}S^T(t_f)R_2\frac{d}{dt}S(t_f), \\ G_2(t_f) &= t_f\int_0^1 S^T(t)R_1 S(t)dt \end{aligned}$$

4.3 Optimality Condition

The necessary condition of optimality can be now obtained by differentiating the performance index (24) with respect to the unknown vector $\overrightarrow{b}$ and using the differentiation properties of [7], that is

$$\frac{\partial}{\partial\overrightarrow{b}}J_{N,r}(\overrightarrow{f}) = 0 \Rightarrow \overrightarrow{b^*} = -Q^{-1}\overrightarrow{y} \tag{25}$$

provided that $Q^{-1} = (Y + Y^T)$ exists and vector $\overrightarrow{y}$ is defined as $\overrightarrow{y} = \overrightarrow{y}_1 + \overrightarrow{y}_2^T$. Note that equation (25) provides necessary conditions, where

$$\frac{\partial^2}{\partial\overrightarrow{b}^2}J_{N,r}(\overrightarrow{f}) = \text{vec}\left(Y + Y^T\right) \tag{26}$$

is a non negative vector which provides conditions for $\overrightarrow{b^*}$, and $\overrightarrow{a^*}$ to be the optimal control and state parameter, respectively. Finally the optical control $\overrightarrow{f^*}(t)$ and $\overrightarrow{v^*}(x,t)$ are obtained in terms of sine-cosine wavelets series expansion from equations (7).

5 Numerical Example

In this section, we consider the behavior of the controlled beams and compare it with beams on which no control force is applied. Moreover, the effect of various problem parameters on the control and motion of beams are investigated. For simplicity of the analysis, it is assumed that the double-beam system is subjected to the initial conditions (3) of the form

$$u^i(x,0) = \varphi_1(x), \ \partial_t u^i(x,0) = 0 \qquad (27)$$

where $\varphi_1(x)$ is the fundamental mode of the system.

In numerical simulations, it is assumed both beams are geometrically and physically identical. The values of the parameters used in the numerical calculations [6]

$m_i = 1 \times 10^3 \text{kgm}^{-1}$, $E_i = 1 \times 10^{10} \text{Nm}^{-2}$, $I_i = 4 \times 10^{-4} \text{m}^4$, $i = 1, 2$, $k_1 = 2 \times 10^5 \text{Nm}^{-2}$ and $l = 10m$.

Table 1 shows the effect of forces acting on both beams at the points $x_1^s = x_2^s = 3.0$, $x_1^a = 4.0$, $x_2^a = 8.0$, $\mu_i = 1$, $i = 1, 2, 3, 4$, $\mu_5 = \mu_6 = 0.001$, and $t_f = 1$.

Actuators	$E(t_f)$	$F(t_f)$
$f_{11} = 0, \ f_{21} = 0$	0.5285	0.0
$f_{11} \neq 0, \ f_{21} \neq 0$	0.0007	0.0161
$f_{11} = 0, \ f_{21} \neq 0$	0.004	0.0388
$f_{11} \neq 0, \ f_{21} = 0$	0.0016	0.0244

Table 1: Effect of forces on parallel beams.

It is observed from Table 1 that the system achieves a substantial reduction in energy when both actuators are applied to both beams.

6 Conclusions

Based on the modal space and the finite terms of orthogonal wavelets, an attractive computational formulation for evaluating the optimal control and state functions of a distributed system consisting of two Euler-Bernoulli beams in parallel is established. Compared to the other methods, the formulation is straightforward and convenient to digital computation. The main aspect in the nature of this approach is that it converts the quadratic programming problem into a mathematical programming problem where the necessary conditions of optimality are derived as a system of algebraic equations. A numerical example is provided to substantiate the theoretical results.

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