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Sensitivity of Closed Loop System Eigenvalues to Structured Perturbations in System Parameters

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Abstract:- The dynamic behavior of any control system is totally dictated by the eigenvalues of the closed loop system matrix (A-BF). Any small changes in system parameters lead to real problems of unknown results. In this paper a methodology has been introduced to predict and calculate the sensitivities of each closed loop eigenvalue due to any perturbation in the elements of the system matrix A.

Key-words:- Control Systems, Eigenvalue assignment, sensitivity ,Structured perturbation

1 Introduction

Eigenvalue assignment techniques to modify the dynamic response of linear systems are among the most studied problems in modern control theory. Consider a linear constant coefficient dynamical system in state space form,

$$\dot{x}(t) = Ax(t) + Bu(t) \quad (1)$$

$$y(t) = Cx(t) \quad (2)$$

Where $A \in R^{n*n}$ is the system matrix, $B \in R^{n*m}$ is the input matrix ,$C \in R^{l*n}$ is the output matrix, $x(t) \in R^{n*1}$ is the state vector at the time t, $u \in R^{m*1}$, is the control input of the system $y \in R^{l*1}$ is the output vector, n, m, and l are numbers of state variables, inputs and outputs respectively.

The classical eigenvalue assignment problem is to find a state feedback control law

$$u = -Fx(t) \quad (3)$$

such that the closed loop system

$$\dot{x}(t) = (A - BF)x(t) + Bv \quad (4)$$

has prescribed desired eigenvalues.

Equation 4 indicates that the system stability will no longer be determined by the system matrix A, but by the modified system matrix A-BF.

There is a large literature on this problem, in particular on numerical methods for its solution [see 1,2,3,4 for example]. It is well known from the literature of control theory, that when a parameter in the system matrix is changed, the locations of the eigenvalues on the complex plane will change consequently. This leads to the problem of sensitivity. By sensitivity of eigenvalue assignment problem we mean the prediction of eigenvalue displacements owing to perturbations in system parameters. The variations are the consequence of component aging, environmental conditions, properties of the machine arithmetic and the computational algorithm.

Several authors [5,6,7,8] gave their consideration about the sensitivity of the closed-loop eigenvalues due to changes in the system parameters. Most of their approaches are based on the idea of the eigndecopmpsition discussed by Wilkinson [9] or numerical accuracy and problems [10].

In this paper, the sensitivity analysis is introduced for the time-invariant linear control systems using the structural concepts for the perturbation in system matrix.

2 Sensitivity Analysis for Structural Perturbation

The basic problem in the structural perturbation is to determine the sensitivity of the eigenvalues , owing to changes in each element of the system matrix separately. The causes of the perturbation which are considered in this paper are due to the system components and the computed feedback matrix only. In this case an analytical analysis is necessary to avoid noise and distortion of the computation method applied.

The structural study is necessary for the selection of system parameters, so that the eigenvalues are as insensitive as possible to changes in those parameters.

Consider the sensitivity of an einvalue λ_r to a small change in the ij-th element of A, δa_{ij}, is denoted by $\delta\Delta(\lambda_r)$. The closed loop characteristic equation is described by the polynomial,

$$\Delta(\lambda) = |\lambda I - (A - BF))| \quad (5)$$

$$= \lambda^n - q_i \lambda^{n-1} - q_2 \lambda^{n-2} - ....q_n \quad (6)$$

or,

$$= (\lambda - a_{jj} + b_j f_j) A_{jj} - \sum_{i=1, j=i}^{n} (a_{ij} - b_j f_j) A_{ij} \quad (7)$$

Where A_{jj} and A_{ij} are the corresponding cofactors of the specific element in the matrix (A-BF). Taking the derivation of the characteristic polynomial(6) with respect to the eigenvalue λ_r, we can get,

$$\Delta'(\lambda_r) = \frac{\partial \Delta(\lambda_r)}{\partial \lambda_r} = n\lambda_r^{n-1} - \sum_{i=1}^{n-1} (n-i) q_i \lambda_r^{n-i-1} \quad (8)$$

The sensitivity coefficient S_{ij} for the eigenvalue λ_r to a small change δa_{ij} is defined by Stojic[11] as,

$$S_{ij}(\lambda_r) = \frac{\partial \lambda_r}{\partial a_{ij}}, i, j = 1,2,3,.., n \quad (9)$$

To Compute S_{ij}, it is necessary to substitute $\lambda=\lambda_r$ into equation (5) and then to differentiate the characteristic equation $\Delta(\lambda_r)=0$ with respect to the element a_{ij}, hence:

$$\frac{\partial \Delta(\lambda_r)}{\partial \lambda_r} \cdot \frac{\delta \lambda_r}{\delta a_{ij}} + \frac{\delta \Delta(\lambda_r)}{\delta a_{ij}} = 0 \quad (10)$$

From equation (7), it can be found that,

$$\frac{\delta \Delta(\lambda_r)}{\delta a_{ij}} = -A_{ij} \quad (12)$$

Substituting equations (8) and (10) in the definition (9), the sensitivity coefficient is found to be:

$$S_{ij}(\lambda_r) = \frac{1}{\Delta'(\lambda_r)} A_{ij}, i, j = 1,2,3,.., n \quad (13)$$

Equation (13) is a n*n matrix referred to as the sensitivity matrix. Each element of $S_{ij}(\lambda_r)_{ij}$ determines the sensitivity weight in the correspondent eigenvalue λ_r due to a small change in the correspondent element in the system matrix A.

The deviation of $\delta\lambda_r$ in a single eigenvalue of the system with state feedback to a small change δa_{ij} in the element a_{ij} is determined from,

$$\delta \lambda_r = \delta a_{ij} * S(\lambda_r)_{ij} \quad (14)$$

3 Numerical Example

Now we shall apply the proposed sensitivity analysis of section 2 to a randomly generated system. Using MATLAB 6.R12, The system and input matrices for the generated system are respectively,

$$A = \begin{matrix} 0.2113 & 0.6654 & 0.8782 & 0.7264 \\ 0.7560 & 0.6284 & 0.0684 & 0.1985 \\ 0.0002 & 0.8497 & 0.5608 & 0.5443 \\ 0.3303 & 0.6857 & 0.6624 & 0.2321 \end{matrix}$$

$$B = \begin{matrix} 0.2312 & 0.3076 \\ 0.2165 & 0.9330 \\ 0.8834 & 0.2146 \\ 0.6524 & 0.3126 \end{matrix}$$

Let us assign the closed-loop system eigenvalues at locations: -0.1, -0.1, -0.1±j on the complex plane. The feedback matrix F using the command 'place' in MATLAB.6 R12 is computed and equals to:

$$F = \begin{matrix} -0.0874 & 0.7274 & 0.5422 & 0.4106 \\ 0.8903 & 0.6817 & 0.7499 & 0.2483 \end{matrix}$$

Using equation 13, the sensitivity matrix for the eigenvalues (-0.1+j0.1, -0.1) are :

$$S_1=\begin{matrix} 0.0005 & 0.0002 & 0.001 & 0.0015 \\ 0.0002 & 0.0005 & 0.001 & 0.0001 \\ 0.0001 & 0.0004 & 0.0004 & 0.0001 \\ 0.0004 & 0.0002 & 0.0001 & 0.0015 \end{matrix}$$

The sensitivity weights for a specified eigenvalue due to all individual system matrix elements are shown in Fig. 1. The same figure has to be drowning for each eigenvalue. The hight at each row-column intersection point represents the sensitivity weight for that eigenvalue with that element.

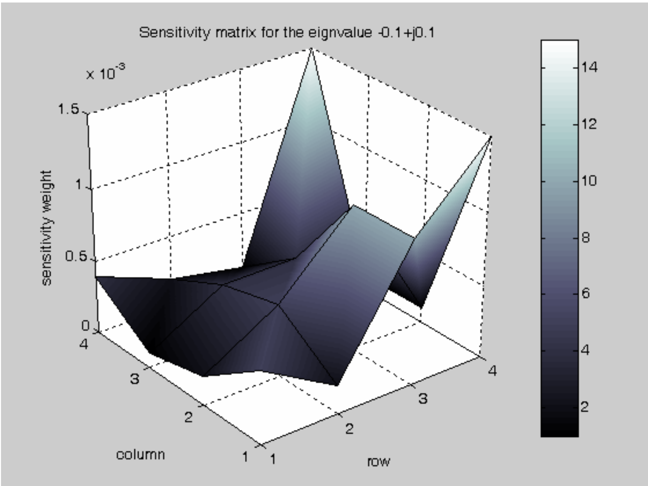

Fig.1 Sensitivity weights for λ_1 =-0.1+ j0.1

To test the accuracy of the proposed method, let us test a small change in the element A_{13} by 0.001, new A_{13}=0.8792, if we fix the feedback matrix F, which is the case of most control systems, using equation 14, the expected change δ in the closed-loop eignvalue λ_1 is $\delta\lambda$=0.0001 , or the eigenvalue λ_1 =-0.1001+ j0.1006.

The exact two complex-conjugate eigenvalues for the system matrix (A-BF) using MATLAB are:

λ_1= -0.1001 + 0.1006i
λ_2= -0.1001 - 0.1006i

Which gives an error of amount zero in the real part and 0.0005 in the complex part of the eigenvalues.

For evaluation of the sensitivity of the four eigenvalues, four sensitivity matrices have to be calculated. Each element in the S_i matrix represents the sensitivity of the correspondent element in A matrix associated with the i-th eigenvalue.

4 Conclusion

A method for sensitivity analysis of eigenvalue assignment problem for structural perturbation in system matrix has been suggested and tested in this paper. The method introduces a number of matrices called the sensitivity matrices. Each matrix related to one eigenvalue. These matrices represent weights for measuring sensitivity of each eigenvalue of the closed-loop system matrix to each element in the system matrix A.

Results show that a large element in the sensitivity matrix is an exact measure for the high sensitivity of the correspondent element to the eigenvalue. If a zero element appears in the sensitivity matrices, then the associated eigenvalue λ_r will be insensitive to any change in the corresponding element of A.

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