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Integration Algorithms to Construct Semi-analytical Planetary Theories

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Abstract: The aim of this paper is to describe the construction of a set of algorithms that allow several types of anomalies to be used as integration variables in the Lagrange planetary equations. The method, based on the relation between the mean anomaly and the other anomalies taken as temporal variables, involves a set of algorithms that can be used to expand the inverse of the distance according to the anomalies that are used, the construction of an iterative algorithm of integration for each set of variables, and a Poisson processor to manage Fourier or Poisson series.

Key–Words: Celestial mechanics, Planetary theory, Algorithms, Computational algebra, Orbital mechanics, Perturbation theory

1 Introduction

One of main problems in celestial mechanics is the construction of the models (containing the Sun and a set of planets in motion around it),that serve to explain planetary motion in the solar system. Such models are called planetary theories and can be classified into three types: analytical theories, when the expansions are given using literal expansions in all variables; semi-analytical theories, when the developments are given using Fourier series whose coefficients are given by numerical values and their temporal variables are literal; and numerical theories, when the solutions are given directly from numerical quadrature formulas.
To construct a planetary theory, it is very common to use the well-known two-body problem as a zero-order approximation for each planet, and from this initial solution we can improve the model by usung the perturbation theory. The solution to the two-body problem is fully defined by means of the orbital elements $(a, e, i, \Omega, \omega, M)$ (set III of Brower& Clemence [3]).
The relative position of the secondary with respect to the primary on the orbital plane is given by:

$$\xi = r \cos V \,, \quad \eta = r \sin V \tag{1}$$

where r is the vector radius and V is the true anomaly. The true anomaly V is connected with the mean anomaly M by means of the center equation:

$$V - M = \sum_{k=1}^{\infty} C_k(e) \sin kM \tag{2}$$

The values of the eccentricity functions $C_k(e)$ can be obtained from Tisserand [14]. The vector radius can be given by the equation of the relative orbit:

$$r = \frac{a(1-e^2)}{1 + e \cos V} \tag{3}$$

The mean anomaly is connected with the time by $M = M_0 + n(t - t_0)$ where n is the mean motion.
The position of the secondary in the spatial system can be obtained as

$$(x, y, z)^t = R_3(-\Omega) R_1(-i) R_3(-\omega)(\xi, \eta, 0)^t \tag{4}$$

The position of the secondary in the orbital plane can be given, according to the eccentric anomaly E, as:

$$\xi = a(\cos E - e) \quad , \quad \eta = a\sqrt{1-e^2}\sin E \qquad (5)$$

the vector radius is given by:

$$r = a(1 - e\cos E) \qquad (6)$$

The eccentric anomaly satisfies the Kepler equation:

$$E - e\sin E = M \qquad (7)$$

and according to the elliptic argument u defined by $\mathrm{am}\, u = E$ (Brumberg [6]) the secondary position can be obtained as:

$$\xi = a(\mathrm{sn}\, u - e) \quad , \quad \eta = a\sqrt{1-e^2}\mathrm{cn}\, u \qquad (8)$$

where u is defined as:

$$u = F(\phi, e) = \int_0^{\phi} \frac{d\phi}{\sqrt{(1-e^2\sin^2\phi)}} \quad , \quad \phi = E + \frac{\pi}{2} \qquad (9)$$

the vector radius can be given by:

$$r = a(1 - e\,\mathrm{sn}\, u) \qquad (10)$$

The elliptic argument u satisfies the modified Kepler equation:

$$\mathrm{am}\, u - e\,\mathrm{cn}\, u = M + \frac{\pi}{2} \qquad (11)$$

where $\mathrm{am}\, u$ is the elliptic amplitude of u.
Nevertheless, it is more convenient to introduce a new variable ω called the elliptic anomaly, which is defined by:

$$\omega = \frac{\pi u}{2K} - \frac{\pi}{2} \qquad (12)$$

where $K(e) = \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1-e^2\sin^2\pi}}$ is the complete elliptic integral of the first kind.
The perturbed motion solution can be obtained by the variation constant method using the Lagrange planetary equations.

$$\begin{aligned}
\frac{da}{dt} &= \frac{2}{na}\frac{\partial R}{\partial \sigma} \\
\frac{de}{dt} &= -\frac{\sqrt{1-e^2}}{na^2e}\frac{\partial R}{\partial \omega} + \frac{1-e^2}{na^2e}\frac{\partial R}{\partial \sigma} \\
\frac{di}{dt} &= -\frac{1}{na^2\sqrt{1-e^2}\sin i}\frac{\partial R}{\partial \Omega} + \frac{\mathrm{ctg}\, i}{na^2\sqrt{1-e^2}}\frac{\partial R}{\partial \omega} \\
\frac{d\Omega}{dt} &= \frac{1}{na^2\sqrt{1-e^2}\sin i}\frac{\partial R}{\partial i} \\
\frac{d\omega}{dt} &= \frac{\sqrt{1-e^2}}{na^2e}\frac{\partial R}{\partial e} - \frac{\cos i}{na^2\sqrt{1-e^2}\sin i}\frac{\partial R}{\partial i} \\
\frac{d\sigma}{dt} &= -\frac{2}{na}\frac{\partial R}{\partial a} - \frac{1-e^2}{na^2e}\frac{\partial R}{\partial e}
\end{aligned} \qquad (13)$$

where R is the disturbing potential defined by Levallois [11]:

$$R = \sum_{k=1}^{N} Gm_k \left[\left(\frac{1}{\Delta_k}\right) + \frac{xx_k + yy_k + zz_k}{r_k^3}\right] \qquad (14)$$

where Δ_k is the distance between the secondary and the disturbing body and r_k is the distance between the primary and the disturbing body.

2 Expansion of the Partial Derivatives of the Disturbing Potential

To evaluate the second member of the Lagrange planetary equations by semi-analytical expansions we can see Chapront [7], Hagihara [8], López [12], Simon [13] from which the inverse of the distance developments are obtained as a Fourier series of appropriate temporal variables.
To expand the inverse of the distance as a Fourier series of appropriate temporal variables:

$$\begin{aligned}
\Delta^2 &= r^2 + r'^2 - 2rr'\cos S = \\
&= \sum_{j_1,j_2} A_{j_1,j_2}\cos(j_1\Psi_1 + j_2\Psi_2 + \varphi_{j_1,j_2})
\end{aligned} \qquad (15)$$

where Ψ_1, Ψ_2 represent appropriate temporal variables (mean anomaly, eccentric anomaly, true anomaly, or elliptic anomaly), we use the Kovalewsky [10] iteration formula:

$$\left(\frac{1}{\triangle}\right)_{k+1} = \frac{3}{2}\left(\frac{1}{\triangle}\right)_k - \frac{1}{2}\left(\frac{1}{\triangle}\right)_k^3 \triangle^2 \qquad (16)$$

where k represents the number of iterations. An appropriate first approximation (Tisserand [14]) can be taken as:

$$\left(\frac{1}{\triangle}\right)_0 = \frac{1}{a'}\left[b_{1/2}^{(0)} + \sum_{j=1}^{\infty} b_{1/2}^{(j)}(\alpha)\cos jS\right] \qquad (17)$$

where $b_s^{(j)}$ are the Laplace coefficients defined as (Tisserand [14]):

$$b_s^{(j)} = \frac{(s)_j}{(1)_j}F\left(s, s+j, j+1; \alpha^2\right) \qquad (18)$$

where F is the Gauss hypergeometric function and $(s)_j$ are the Pochhammer symbols.
The cosine of multiples of angle S can be obtained by the recursive formula

$$\cos nS = 2\cos((n-1)S)\cos S - \cos((n-2)S) \qquad (19)$$

where

$$\cos S = \frac{xx' + yy' + zz'}{rr'} \qquad (20)$$

Equations (15-19) can be used to given the development of the inverse of the distance according to true or elliptic anomalies. To obtain this expansion according to eccentric anomalies we can use the Hansen functions $X^k_{i,j}(e)$ (see Tisserand [14]) or elliptic Hansen functions $B^k_{i,j}(e)$ (see Brumberg [5][6]).

3 Integration Algorithms

To integrate the Lagrange Planetary equations, we proceed by means of the perturbation method. For a generic element σ in the first order we have the differential equation:

$$\dot{\sigma} = \sum_{j_1,j_2} C_{j_1,j_2} \cos(j_1 M_1 + j_2 M_2 + \varphi_{j_1,j_2}) \qquad (21)$$

The integration of this equation is immediate, but the series can be very longs and very expensive as regards computational time.
To use shorter expansions, it is interesting to use the other anomalies as temporal variables, but in this case the integration may not be immediate and it thus becomes necessary to obtain a representation of the mean anomaly according to the anomaly that is used.

3.1 Algorithms using eccentric anomalies as temporal variable

Let $C_{j_1,j_2} \cos(j_1 E_1 + j_2 E_2 + \varphi_{j_1,j_2})$ be a term of the development of $\dot{\sigma}$.
To evaluate the integral

$$\int_{t_0}^{t} \cos(j_1 E_1 + j_2 E_2 + \varphi_{j_1,j_2}) dt$$

we define a new integration variable $\xi = j_1 E_1 + j_2 E_2$. To obtain a relation between this variable and the time we derive the Kepler equation.

$$n dt = dM = (1 - e \sin E) dE \qquad (22)$$

from this relation we obtain:

$$j_1 dM_1 = j_1 dE_1 - e_1 \cos E_1 dE_1 \qquad (23)$$

$$j_2 dM_2 = j_2 dE_2 - e_e \cos E_2 dE_2 \qquad (24)$$

and so

$$dt = \frac{1}{j_1 n_1 + j_2 n_2} d\xi - - \frac{j_1 e_1 \cos E_1}{j_1 n_1 + j_2 n_2} dE_1 - \frac{j_2 e_2 \cos E_2}{j_1 n_1 + j_2 n_2} dE_2 \qquad (25)$$

To represent dE according to dt and $\cos kE$ we use the expansion:

$$\frac{1}{1 - e\cos E} = \frac{a}{r} = \frac{P_0}{2} + \sum_{k=0}^{\infty} P_k \cos kE \qquad (26)$$

where P_k are defined according to the Cauchy numbers $N_{i,j,q}$ (Tisserand [14]) as:

$$P_k = 2 \sum_{s=0}^{\infty} \sum_{q=0}^{\infty} \frac{N_{-i,s,q}}{q!} \left(\frac{e}{2}\right)^{s+q} \qquad (27)$$

Replacing () in () we get

$$dt = \frac{d\xi}{j_1 n_1 + j_2 n_2} + (f_1 + f_2) dt \qquad (28)$$

where

$$f_1 = -\frac{j_1 n_1 e_1 \cos E_1}{j_1 n_1 + j_2 n_2} \left[\frac{P_0(e_1)}{2} + \sum_{k=0}^{\infty} P_k(e_1) \cos kE_1 \right]$$

$$f_2 = -\frac{j_2 n_2 e_2 \cos E_2}{j_1 n_1 + j_2 n_2} \left[\frac{P_0(e_2)}{2} + \sum_{k=0}^{\infty} P_k(e_2) \cos kE_2 \right] \qquad (29)$$

The functions f_1, f_2 are of first order in e_1, e_2.
This method can be applied as an iterative method where each iteration increases the order in the eccentricities for the residual integrals by one.

3.2 Algorithms using true anomalies as temporal variable

Let $C_{j_1,j_2} \cos(j_1 V_1 + j_2 V_2 + \varphi_{j_1,j_2})$ be a term of the development of $\dot{\sigma}$. To integrate this term in the time span $[t_0, t]$ it is necessary to obtain a representation of the mean anomaly according to the true anomaly. Let $C = \frac{2\pi ab}{T}$ be the integral of the areas, and $n = \frac{2\pi}{T}$ the mean motion. The constant of the areas satisfies $r^2 dV = C$, and from equation (3) we have:

$$\frac{\left(1 - e^2\right)^{\frac{3}{2}}}{\left(1 + e\cos V\right)^2} dV = n dt = dM \qquad (30)$$

Let $z = \exp(\sqrt{-1}V)$, and let $G(e, z)$ be the function defined as:

$$G(e, z) = \frac{\left(1 - e^2\right)^{\frac{3}{2}}}{\left[1 + e\frac{z + z^{-1}}{2}\right]^2} \qquad (31)$$

Let z_1, z_2 be the roots of the denominator, for each $e \in]0, 1[$ we have $z_1 \neq z_2$ and $z_1 z_2 = 1$ then $0 <$

$|z_1| < 1 < |z_2|$, so $k_1\,k_2 \in \mathbb{R}$, $0 < |z_1| < k_1 < 1 < k_2 < |z_2|$ where $G(e,z)$ is a holomorphic function in the ring $\mathcal{H} = \{z \in \mathbb{C} | k_1 \le |z| \le k_2\}$. This ring contains the circumference of radius one.
Expanding the function $G(e,z)$ in the Laurent series in the ring we give the development as

$$G(e,z) = \sum_{k=-\infty}^{\infty} Q_k(e) z^k \tag{32}$$

Expanding the generatrix function $G(e,z)$ in the Laurent series, and identifying the coefficients, we get:

$$Q_k(e) = (-1)^k \frac{e^k}{2^k} \sum_{s=0}^{\infty} (k+2s+1) \binom{s+2k}{s} \frac{e^{2s}}{2^{2s}} \tag{33}$$

The functions $Q_k(e)$ satisfies the relations:

$$\forall k \ge 0; \qquad Q_0(e) = 1; \qquad Q_{-k}(e) = Q_k(e) \tag{34}$$

$$\frac{ke}{2} Q_{k+2}(e) + (k+1) Q_{k+1}(e) + \left(\frac{k}{2} + 1\right) e Q_k = 0 \tag{35}$$

To obtain the functions $Q_k(e)$ in a closed form we can compute $Q_2(e)$ directly as:

$$Q_2(e) = \frac{1}{2\pi} \int_0^{2\pi} G(e, \exp(\sqrt{-1}V)) \cos 2V dV = = \frac{e^2}{1+\sqrt{1-e^2}} (1 + 2\sqrt{1-e^2}) \tag{36}$$

and by using the recurrence relations we get, in closed form:

$$Q_k(e) = (-1)^k \frac{e^k}{1+\sqrt{1-e^2}} (1 + k\sqrt{1-e^2}) \tag{37}$$

To obtain the mean anomaly according to the true anomaly we can integrate equation (), we use:

$$M = V - 2e \sin V + \sum_{k=2}^{\infty} \frac{2Q_k(e)}{k} \sin kV \tag{38}$$

To evaluate the integral

$$\int_{t_0}^{t} \cos(j_1 V_1 + j_2 V_2 + \varphi_{j_1,j_2}) dt$$

we define a new integration variable $\xi = j_1 V_1 + j_2 V_2$, and from the relations

$$j_1 dM_1 = j_1 dV_1 + 2 j_1 \sum_{k=1}^{\infty} Q_k(e_1) \cos(kV_1) dV_1$$

$$j_2 dM_2 = j_2 dV_1 + 2 j_2 \sum_{k=1}^{\infty} Q_k(e_2) \cos(kV_2) dV_2 \tag{39}$$

so

$$dt = \frac{d\xi}{j_1 n_1 + j_2 n_2} + (g_1 + g_2) dt \tag{40}$$

where:

$$g_1 = \frac{j_1}{j_1 n_1 + j_2 n_2} \sum_{k=1}^{\infty} 2Q_k(e_1) \cos(kV_1) \cdot \cdot \left[(1-e_1^2)^{-\frac{3}{2}} \left(1 + \frac{e_1^2}{2} + 2e_1 \cos V_1 + \frac{e_1^2}{2} \cos 2V_1 \right) \right]$$

$$g_2 = \frac{j_2}{j_1 n_1 + j_2 n_2} \sum_{k=1}^{\infty} 2Q_k(e_2) \cos(kV_2) \cdot \cdot \left[(1-e_2^2)^{-\frac{3}{2}} \left(1 + \frac{e_2^2}{2} + 2e_2 \cos V_2 + \frac{e_2^2}{2} \cos 2V_2 \right) \right]$$

The functions g_1, and g_2 are of first order in e_1, e_2.
This method can be applied as an iterative method where each iteration increases the order in the eccentricities for the residual integrals by one.
The method can be extended to Poisson terms $C_{j_1,j_2}(t) \cos(j_1 V_1 + j_2 V_2)$ by using the integral by parts formula.

3.3 Algorithms using elliptic anomalies as temporal variable

Let $C_{j_1,j_2} \cos(j_1 \mathrm{w}_1 + j_2 \mathrm{w}_2 + \varphi_{j_1,j_2})$ be a term of the development of $\dot{\sigma}$.
To evaluate the integral

$$\int_{t_0}^{t} \cos(j_1 \mathrm{w}_1 + j_2 \mathrm{w}_2 + \varphi_{j_1,j_2}) dt$$

we define a new integration variable $\xi = j_1 \mathrm{w}_1 + j_2 \mathrm{w}_2$.
To obtain a relation between this variable and the time, in the Kepler equation we replace the elliptic functions with the following developments [2]:

$$\mathrm{am}\, u = \mathrm{w} + \frac{\pi}{2} + 2 \sum_{m=1}^{\infty} a_m \sin 2m\mathrm{w} \tag{41}$$

$$\mathrm{sn}\, u = \frac{2\pi}{eK(e)} \sum_{m=1}^{\infty} b_m \cos(2m-1)\mathrm{w} \tag{42}$$

$$\mathrm{cn}\, u = \frac{2\pi}{eK(e)} \sum_{m=1}^{\infty} c_m \sin(2m-1)\mathrm{w} \tag{43}$$

where

$$a_m = \frac{(-1)^m}{m}\frac{N(e)^m}{1+N(e)^{2m}}$$
$$b_m = \frac{(-1)^{m+1}N(e)^{m-\frac{1}{2}}}{1-N(e)^{2m-1}}$$
$$c_m = \frac{(-1)^m N(e)^{m-\frac{1}{2}}}{1+N(e)^{2m-1}}$$

and $N(e) = e^{-\frac{\pi K(\sqrt{1-e^2})}{K(e)}}$ is the Jacobi number (Brumberg [6]), and $K(e)$ is the complete elliptic integral of the first kind.

$$M = \mathrm{w} + \sum_{s=1}^{\infty} T_s \sin \mathrm{sw} \qquad (44)$$

where

$$T_s = \begin{cases} (-1)^{\left[\frac{s+1}{2}\right]}\frac{2N(e)^{\frac{s}{2}}}{1+N(e)^s}\frac{\pi}{K(e)} & \text{if s= 2m-1} \\ (-1)^{\left[\frac{s+1}{2}\right]}\frac{2N(e)^{\frac{s}{2}}}{1+N(e)^s}\frac{2}{s} & \text{if s=2m} \end{cases} \qquad (45)$$

where $[x]$ is the entire part of x.
The coefficients T_s are of order s in eccentricity.
Deriving the equation (44) we have:

$$j_1 n_1 dt = j_1 d\mathrm{w}_1 + j_1 \sum_{s=0}^{\infty} sT_s(e_1)\cos s\mathrm{w}_1 d\mathrm{w}_1$$
$$j_2 n_2 dt = j_2 d\mathrm{w}_2 + j_2 \sum_{s=0}^{\infty} sT_s(e_2)\cos s\mathrm{w}_2 d\mathrm{w}_2 \qquad (46)$$

and so:

$$\begin{aligned} dt = & \frac{1}{j_1 n_1 + j_2 n_2} d\xi + \\ & + \frac{j_1}{j_1 n_1 + j_2 n_2}\sum_{s=0}^{\infty} sT_s(e_1)\sin s\mathrm{w}_1 d\mathrm{w}_1 + \\ & + \frac{j_2}{j_1 n_1 + j_2 n_2}\sum_{s=0}^{\infty} sT_s(e_2)\sin s\mathrm{w}_2 d\mathrm{w}_2 \end{aligned} \qquad (47)$$

Deriving the Kepler equation (11) we get:

$$dM = (1 - e\,\mathrm{sn}\,u)\mathrm{dn}u\,du$$

replacing (10),(12) in this equation we get:

$$d\mathrm{w} = \frac{\pi}{2K(e)}\frac{a}{r}\mathrm{nd}\,u\,dM \qquad (48)$$

where $\mathrm{nd}\,u = \frac{1}{\mathrm{dn}\,u}$.
The function $\frac{a}{r}$ can be developed according to elliptic anomaly by means the Hansen formulas (Brumberg [5]) as:

$$\frac{a}{r} = B^0_{(-1,0)}(e) + \sum_{s=1}^{\infty} 2B^s_{(-1,0)}(e)\cos s\mathrm{w} \qquad (49)$$

where

$$B^s_{(-1,0)}(e) = \begin{cases} \frac{E(e)}{\sqrt{1-e^2}K(e)} & s=0 \\ \frac{\pi^2}{2\sqrt{1-e^2}K(2)^2}|s|\frac{N(e)^{\frac{|s|}{2}}}{1-N(e)^{|s|}} & s\neq 0 \end{cases} \qquad (50)$$

where $E(e)$ is the complete elliptic integral of the second kind.
The elliptic function $\mathrm{nd}\,u$ can be developed as [2]:

$$\mathrm{nd}\,u = \frac{\pi}{2q(e)K(e)}\left[1 + 4\sum_{s=1}^{\infty}\frac{N(e)^s}{1+N(e)^{2s}}\cos 2s\mathrm{w}\right] \qquad (51)$$

where $q(e) = \sqrt{1-e^2}$.
Replacing (49), (51) in (48) we get:

$$dt = \frac{1}{j_1 n_1 + j_2 n_2} d\xi + [h_1 + h_2]\,dt \qquad (52)$$

where $h_i \quad i = 1, 2$ are defined as:

$$\begin{aligned} h_i = & \frac{j_i n_i}{j_1 n_1 + j_2 n_2}\frac{\pi^2}{4q(e_i)K(e_i)^2}\cdot \\ & \cdot\left[\sum_{s=0}^{\infty} sT_s(e_i)\cos s\mathrm{w_i}\right]\cdot \\ & \cdot\left[B^0_{(-1,0)}(e_i) + \sum_{s=1}^{\infty} 2B^s_{(-1,0)}(e_i)\cos s\mathrm{w_i}\right]\cdot \\ & \cdot\left[1 + 4\sum_{s=1}^{\infty}\frac{N(e_i)^s}{1+N(e_i)^{2s}}\cos 2s\mathrm{w_i}\right] \end{aligned} \qquad (53)$$

The functions h_1, and h_2 are of first order in e_1, e_2.
This method can be applied as an iterative method where each iteration increases the order in the eccentricities for the residual integrals by one.
To extend the method to Poisson terms we can use integration by parts.

4 Poisson Series Processor

The integration of the Lagrange planetary equation in the first order of perturbation produces periodic and secular terms. To obtain the differential equation in second and higher orders of perturbation, it is necessary to manage objects of a special type called Poisson series. For a generic element σ we have

$$\sigma(t) = \sigma_0 + \sum_{j_1,j_2} C_{j_1,j_2}(t)\cos(j_1\Psi_1 + j_2\Psi_2 + \varphi_{j_1,j_2}) \tag{54}$$

where $C_{j_1,j_2}(t)$ are polynomials in t.
Poisson series processors in C and Fortran has been developed by Abad [1], Brumberg[4], Ivanova [9].

5 Concluding Remarks

The expansion of the disturbing potential according to eccentric, true or elliptic anomalies is shorter than if we use mean anomalies as temporal variables.
The set of equations (28), (40), (52) are appropriate to integrate these developments by an iterative method. This method can be implemented in the Mathematica or C or Fortran 95 languages.

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