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High-order approximations to the Mie–series for electromagnetic scattering in three dimensions

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Abstract: Approximations to the Mie–series are fundamental in many applications. In particular, for validation of efficient electromagnetic scattering algorithms (to compute the radar cross section of three dimensional targets) or to understand the electromagnetic scattering by rain drops, it is important to evaluate the Mie–series with accuracy close to machine precision. In this paper, we propose a quadrature based algorithm to approximate the Mie–series to very high–order accuracy, and demonstrate that the approach leads to better approximations than obtained using a industrial standard Mie–series scattering code.

Key–Words: Mie–series, electromagnetics, scattering, radar cross section, quadrature

1 Introduction

Approximations to the Mie–series play an important and indirect role in understanding and developing physical processes in many applications ranging, for example, from the radar cross section simulations in stealth technology to the backscattering of sunlight by cloud water droplets. The concept of trying to understand light scattering through approximations started a few centuries ago, and analytical (Mie–series) solutions for scattering by spherical particles were developed about one century ago. Geometrical optics and computational approximations to the Mie–series have been developed over the last half a century, and industrial standard algorithms for efficient evaluation of the Mie–series were developed over the last three decades. In this work we develop a quadrature based approximation to the Mie–series with almost machine precision accuracy.

The geometrical optics or asymptotic expansions do not yield good accuracy when the light wavelength is similar to, or larger than, the particle diameter, as the light interaction area is larger than the geometric cross section of the particle. Efficient computation of the partial sums of the Mie–series is essential in such cases [4]. Computational approximations to the Mie–series have been developed by many researchers, and a celebrated industrial algorithm and code is due to Wiscombe and collaborators [5, 6]. Included in [5] is a widely accepted formula for the number of terms that should be used in the partial sum.

This paper is motivated by the validation of our spectrally accurate electromagnetic scattering algorithm (currently at the final stage of development). This algorithm is an electromagnetic counterpart of our acoustic scattering algorithm [2]. Spectral (and almost machine) accuracy for acoustic plane wave scattering by the one wavelength sphere is demonstrated in [2, Page 231], and the accuracy is shown to increase with increasing number of unknowns for low to medium frequency scattering by the sphere, and other obstacles [2, Page 232-234]. For the validation of our algorithm for electromagnetic scattering by the sphere, as it is standard in literature, we used the Wiscombe Mie–series approximations (and code) [5] as a benchmark. Despite choosing the recommended number terms (and even more terms) in the partial sum in the Wiscombe code [5], we observed stagnated error of the order $O(10^{-10})$ between our spectrally accurate solution and the Wiscombe Mie–series solution. This led to us investigate the approximations to the Mie–series considered in this paper.

In general, the Mie-series computation with accuracy of the order $O(10^{-10})$ is considered to be sufficient, as most electromagnetic scattering algorithms give only a few decimal places of accuracy due to limitations of computer power. However, as we moved from crude approximations to Wiscombe code type accuracy over a half century ago, it is useful to develop the important Mie-series computation to almost machine accuracy, for validation of high–order electromagnetic scattering algorithms and in further understanding of scattering by various targets and processes.

For validation of electromagnetic scattering algorithms for benchmark radar targets that are not spherical [7], a standard approach is to place an off–center point source inside a three dimensional target. In this case the exact far–field is known, it is the field created by the source itself. The point–source induced far–field requires a simple function evaluation without any series. Such an approach is useful for validation because the partial differential (Maxwell) equations and their discrete matrix approximations are independent of the incident waves.

The analytical (series) solution of the time-harmonic Maxwell equations for the electric and magnetic far field induced by any tangential (including the point source and plane wave) incident field on the sphere can be easily derived, provided an expansion of the incident field in spherical waves is available. We start our approximations with a general series representation, and the plane wave scattering Mie–series solution is a special case of the series. We use quadrature rules on the sphere that are exact for polynomials of degree equal to the chosen number of terms in the partial sums of the general series. In our numerical approximations, we show that our approach yields close to machine accuracy for the electric and magnetic dipole point-source problems, and also that the error between our approximations and the Wiscombe code approximations remains stagnated with order $O(10^{-10})$.

2 Problem Formulation

The spatial components $\boldsymbol{\mathcal{E}}, \boldsymbol{\mathcal{H}}$ of the time–harmonic electromagnetic wave scattered by a sphere satisfy the harmonic Maxwell equations [1, Page 154]

$$\begin{aligned} \mathbf{curl}\, \boldsymbol{\mathcal{E}}(\boldsymbol{x}) - ik\boldsymbol{\mathcal{H}}(\boldsymbol{x}) &= \mathbf{0}, \\ \mathbf{curl}\, \boldsymbol{\mathcal{H}}(\boldsymbol{x}) + ik\boldsymbol{\mathcal{E}}(\boldsymbol{x}) &= \mathbf{0}, \quad \boldsymbol{x} \in \mathbb{R}^3 \setminus \overline{D}, \end{aligned}$$

where $k = 2\pi\omega/c$ is the wavenumber, w the frequency of the wave, c is the speed of light, $\overline{D} = D \cup \partial D$, and D is a ball with perfectly conducting spherical surface ∂D of diameter equal to some constant multiple of the incident wavelength $\lambda = 2\pi/k = c/\omega$.

In case of electromagnetic waves induced by an incident plane wave $\boldsymbol{E}^{\mathrm{i}}, \boldsymbol{H}^{\mathrm{i}}$, with direction $\widehat{\boldsymbol{d}}_0$ and polarization $\widehat{\boldsymbol{p}}_0$,

$$\begin{aligned} \boldsymbol{E}^{\mathrm{i}}(\boldsymbol{x}) &= \frac{i}{k}\mathbf{curl\ curl}\ \left\{\widehat{\boldsymbol{p}}_0 e^{ik\boldsymbol{x}\cdot\widehat{\boldsymbol{d}}_0}\right\} \\ &= ik\left[(\widehat{\boldsymbol{d}}_0 \times \widehat{\boldsymbol{p}}_0) \times \widehat{\boldsymbol{d}}_0\right] e^{ik\boldsymbol{x}\cdot\widehat{\boldsymbol{d}}_0}, \quad (1) \\ \boldsymbol{H}^{\mathrm{i}}(\boldsymbol{x}) &= \mathbf{curl}\ \left\{\widehat{\boldsymbol{p}}_0 e^{ik\boldsymbol{x}\cdot\widehat{\boldsymbol{d}}_0}\right\} \\ &= ik\left[\widehat{\boldsymbol{d}}_0 \times \widehat{\boldsymbol{p}}_0\right] e^{ik\boldsymbol{x}\cdot\widehat{\boldsymbol{d}}_0}, \quad (2) \end{aligned}$$

the total fields $\boldsymbol{\mathcal{E}}, \boldsymbol{\mathcal{H}}$ are given by [1, Page 3]

$$\begin{aligned} \boldsymbol{\mathcal{E}}(\boldsymbol{x}) &= \boldsymbol{E}^{\mathrm{i}}(\boldsymbol{x}) + \boldsymbol{E}^{s}(\boldsymbol{x}), \\ \boldsymbol{\mathcal{H}}(\boldsymbol{x}) &= \boldsymbol{H}^{\mathrm{i}}(\boldsymbol{x}) + \boldsymbol{H}^{s}(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathbb{R}^3 \setminus \overline{D}, \quad (3) \end{aligned}$$

where the unknown scattered fields $\boldsymbol{E}^{\mathrm{s}}, \boldsymbol{H}^{\mathrm{s}}$ comprise a radiating solution of the Maxwell equations (1) that satisfies the Silver-Müller radiation condition

$$\lim_{|\boldsymbol{x}|\to\infty} \left[\boldsymbol{H}^{\mathrm{s}}(\boldsymbol{x}) \times \boldsymbol{x} - |\boldsymbol{x}|\boldsymbol{E}^{\mathrm{s}}(\boldsymbol{x})\right] = \mathbf{0}.$$

The radiating solution $\boldsymbol{E}^{\mathrm{s}}, \boldsymbol{H}^{\mathrm{s}}$ has the asymptotic behavior of an outgoing spherical wave [1, Theorem 6.8, Page 164]

$$\begin{aligned} \boldsymbol{E}^{\mathrm{s}}(\boldsymbol{x}) &= \frac{e^{ik|\boldsymbol{x}|}}{|\boldsymbol{x}|}\left\{\boldsymbol{E}_\infty(\widehat{\boldsymbol{x}}) + O\left(\frac{1}{|\boldsymbol{x}|}\right)\right\}, \\ \boldsymbol{H}^{\mathrm{s}}(\boldsymbol{x}) &= \frac{e^{ik|\boldsymbol{x}|}}{|\boldsymbol{x}|}\left\{\boldsymbol{H}_\infty(\widehat{\boldsymbol{x}}) + O\left(\frac{1}{|\boldsymbol{x}|}\right)\right\}, \quad (4) \end{aligned}$$

as $|\boldsymbol{x}| \to \infty$ uniformly in all directions $\widehat{\boldsymbol{x}} = \boldsymbol{x}/|\boldsymbol{x}|$. The vector fields $\boldsymbol{E}_\infty, \boldsymbol{H}_\infty$, defined on the unit sphere (denoted throughout the paper by ∂B), are the far field patterns of $\boldsymbol{E}^{\mathrm{s}}, \boldsymbol{H}^{\mathrm{s}}$.

A boundary condition is required in order to solve the exterior Maxwell equations with the Silver-Müller radiation condition. The perfect conductor assumption on D [1, Page 155] leads to the Dirichlet boundary condition $\boldsymbol{n} \times \boldsymbol{E} = \mathbf{0}$ on ∂D, where $\boldsymbol{n}(\boldsymbol{y})$ denotes the unit outward normal at the point $\boldsymbol{y} \in \partial D$. Hence for the plane wave problem (with incident direction $\widehat{\boldsymbol{d}}_0$ and polarization $\widehat{\boldsymbol{p}}_0$), using (3) and (1), we get the boundary condition

$$\boldsymbol{n}(\boldsymbol{x}) \times \boldsymbol{E}^{\mathrm{s}}(\boldsymbol{x}) = \boldsymbol{f}(\boldsymbol{x}), \qquad \boldsymbol{x} \in \partial D, \qquad (5)$$

where

$$\boldsymbol{f}(\boldsymbol{x}) = -\boldsymbol{n}(\boldsymbol{x}) \times \boldsymbol{E}^{\mathrm{i}}(\boldsymbol{x}). \qquad (6)$$

The unique analytical solution of the spherical scattering problem (1)–(6) is given by the Mie-series [4]. As described in the introduction, to check the accuracy of approximations to the Mie–series, it is useful to consider the point–source radiation problem with simple exact solutions. To this end, we also consider the boundary conditions

$$\begin{aligned} \boldsymbol{n}(\boldsymbol{x}) \times \boldsymbol{E}^{\rm ED}(\boldsymbol{x}) &= \boldsymbol{f}(\boldsymbol{x}), \\ \boldsymbol{n}(\boldsymbol{x}) \times \boldsymbol{E}^{\rm MD}(\boldsymbol{x}) &= \boldsymbol{f}(\boldsymbol{x}), \qquad \boldsymbol{x} \in \partial D, \end{aligned} \quad (7)$$

where the boundary data $\boldsymbol{f}$ is induced by the following electric and magnetic dipole solutions $\boldsymbol{E}^{\rm ED}, \boldsymbol{H}^{\rm ED}$ and $\boldsymbol{E}^{\rm MD}, \boldsymbol{H}^{\rm MD}$ of the Maxwell equations with radiation from a point source with polarization $\widehat{\boldsymbol{p}}_{\rm src}$ located at $\boldsymbol{x}_{\rm src} \in D$ [1, (6.20), (6.21), Page 163]:

$$\begin{aligned} \boldsymbol{E}^{\rm ED}(\boldsymbol{x}) &= -\frac{1}{ik}\mathbf{curl}_{\boldsymbol{x}}\mathbf{curl}_{\boldsymbol{x}}\left\{\widehat{\boldsymbol{p}}_{\rm src}\Phi(\boldsymbol{x}, \boldsymbol{x}_{\rm src})\right\}, \\ \boldsymbol{H}^{\rm ED}(\boldsymbol{x}) &= \mathbf{curl}_{\boldsymbol{x}}\left\{\widehat{\boldsymbol{p}}_{\rm src}\Phi(\boldsymbol{x}, \boldsymbol{x}_{\rm src})\right\}, \end{aligned} \quad (8)$$

and

$$\begin{aligned} \boldsymbol{E}^{\rm MD}(\boldsymbol{x}) &= \mathbf{curl}_{\boldsymbol{x}}\left\{\widehat{\boldsymbol{p}}_{\rm src}\Phi(\boldsymbol{x}, \boldsymbol{x}_{\rm src})\right\}, \\ \boldsymbol{H}^{\rm MD}(\boldsymbol{x}) &= \frac{1}{ik}\mathbf{curl}_{\boldsymbol{x}}\mathbf{curl}_{\boldsymbol{x}}\left\{\widehat{\boldsymbol{p}}_{\rm src}\Phi(\boldsymbol{x}, \boldsymbol{x}_{\rm src})\right\} \end{aligned} \quad (9)$$

where

$$\Phi(\boldsymbol{x}, \boldsymbol{y}) = \frac{1}{4\pi}\frac{e^{ik|\boldsymbol{x}-\boldsymbol{y}|}}{|\boldsymbol{x}-\boldsymbol{y}|} \quad (10)$$

is the fundamental solution of the Helmholtz equation. The point source radiating fields $\boldsymbol{E}^{\rm ED}, \boldsymbol{H}^{\rm ED}$ and $\boldsymbol{E}^{\rm MD}, \boldsymbol{H}^{\rm MD}$ satisfy the asymptotic expansions (4), leading to four far field patterns. The far field patterns can be computed by applying the following asymptotics to the Mie–series and dipole solutions (8)–(9) and comparing with (4). For a continuous vector field $\boldsymbol{a}$ on ∂D, the fundamental solution in (10) satisfies the asymptotics

$$\begin{aligned} &\mathbf{curl}_{\boldsymbol{x}}\left\{\boldsymbol{a}(\boldsymbol{y})\Phi(\boldsymbol{x}, \boldsymbol{y})\right\} \\ &= \frac{ik}{4\pi}\frac{e^{ik|\boldsymbol{x}|}}{|\boldsymbol{x}|}\left\{e^{-ik\widehat{\boldsymbol{x}}\cdot\boldsymbol{y}}\widehat{\boldsymbol{x}} \times \boldsymbol{a}(\boldsymbol{y}) + O\left(\frac{|\boldsymbol{a}(\boldsymbol{y})|}{|\boldsymbol{x}|}\right)\right\}, \\ &\mathbf{curl}_{\boldsymbol{x}}\mathbf{curl}_{\boldsymbol{x}}\left\{\boldsymbol{a}(\boldsymbol{y})\Phi(\boldsymbol{x}, \boldsymbol{y})\right\} \\ &= \frac{k^2}{4\pi}\frac{e^{ik|\boldsymbol{x}|}}{|\boldsymbol{x}|}\left\{e^{-ik\widehat{\boldsymbol{x}}\cdot\boldsymbol{y}}\widehat{\boldsymbol{x}} \times (\boldsymbol{a}(\boldsymbol{y}) \times \widehat{\boldsymbol{x}}) + O\left(\frac{|\boldsymbol{a}(\boldsymbol{y})|}{|\boldsymbol{x}|}\right)\right\}, \end{aligned}$$

as $|\boldsymbol{x}| \to \infty$ uniformly for all $\boldsymbol{y} \in \partial D$ [1, Page 164], where $\widehat{\boldsymbol{x}} = \boldsymbol{x}/|\boldsymbol{x}| \in \partial B$. In the remaining sections of the paper, $\boldsymbol{E}$ denotes the radiating electric field $\boldsymbol{E}^{\rm s}$ or $\boldsymbol{E}^{\rm ED}$ or $\boldsymbol{E}^{\rm MD}$; $\boldsymbol{x}$ denotes a point on ∂D, and points on the unit sphere are denoted by $\widehat{\boldsymbol{x}}$.

3 Problem Solution

Using (6) for plane wave scattering and (7) for point–source radiation, it is clear that the smooth (infinitely continuously differentiable) boundary data $\boldsymbol{f}$ is tangential on the scattering sphere ∂D, and hence can be represented using the Fourier (Laplace) series

$$\begin{aligned} \boldsymbol{f}(\boldsymbol{x}) &= \sum_{l=1}^{\infty}\sum_{|j|\le l} A_{l,j}\boldsymbol{U}_{l,j}(\widehat{\boldsymbol{x}}) + \sum_{l=1}^{\infty}\sum_{|j|\le l} B_{l,j}\boldsymbol{V}_{l,j}(\widehat{\boldsymbol{x}}), \\ &\qquad \boldsymbol{x} \in \partial D,\ \widehat{\boldsymbol{x}} = \boldsymbol{x}/|\boldsymbol{x}| \in \partial B, \end{aligned} \quad (11)$$

where $\boldsymbol{U}, \boldsymbol{V}$ are the tangential vector harmonics on the unit sphere

$$\begin{aligned} \boldsymbol{U}_{l,j}(\widehat{\boldsymbol{x}}) &= \frac{1}{\sqrt{l(l+1)}}\text{Grad } Y_{l,j}(\widehat{\boldsymbol{x}}), \\ \boldsymbol{V}_{l,j}(\widehat{\boldsymbol{x}}) &= \boldsymbol{n}(\widehat{\boldsymbol{x}}) \times \boldsymbol{U}_{l,j}(\widehat{\boldsymbol{x}}),\ l = 1, 2, \ldots,\quad |j| \le l, \end{aligned}$$

Grad is the surface gradient [1] and for $\widehat{\boldsymbol{x}} = \boldsymbol{p}(\theta, \phi) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)^T$,

$$Y_{l,j}(\widehat{\boldsymbol{x}}) = (-1)^{\frac{(j+|j|)}{2}}\sqrt{\frac{2l+1}{4\pi}\frac{(l-|j|)!}{(l+|j|)!}}P_l^{|j|}(\cos\theta)e^{ij\phi} \quad (12)$$

are the orthonormal scalar spherical harmonics.

The Fourier coefficients in (11) are required to compute the Mie–series. Using orthonormality of $\boldsymbol{U}_{l,j}$ and $\boldsymbol{V}_{l,j}$ in (11), the Fourier coefficients are given by

$$\begin{aligned} A_{l,j} &= \langle \boldsymbol{f} \circ \boldsymbol{q}^{-1}, \boldsymbol{U}_{l,j}\rangle, \\ B_{l,j} &= \langle \boldsymbol{f} \circ \boldsymbol{q}^{-1}, \boldsymbol{V}_{l,j}\rangle, \quad 1 \le l,\ |j| \le l, \end{aligned} \quad (13)$$

where $\boldsymbol{q} : \boldsymbol{x} \mapsto \widehat{\boldsymbol{x}}$, and the orthonormality of the tangential vector harmonics is with respect to the inner product, defined for two 3-vector valued continuous function $\boldsymbol{G}, \boldsymbol{H}$ on the unit sphere by

$$\langle \boldsymbol{G}, \boldsymbol{H}\rangle = \mathcal{I}(\overline{\boldsymbol{H}}^T\boldsymbol{G}) = \int_{\partial B}\overline{\boldsymbol{H}}^T(\widehat{\boldsymbol{x}})\boldsymbol{G}(\widehat{\boldsymbol{x}})\ ds(\widehat{\boldsymbol{x}}).$$

Next we derive the far–field Mie–series representation, using the Fourier series representation (11) of the boundary data $\boldsymbol{f}$. Using [1, Theorem 6.25, Page 181], there exist constants $a_{l,j}, b_{l,j}$, for $l = 1, 2, \cdots, |j| \le l$, such that the electric radiating field $\boldsymbol{E}$ can be written as

$$\begin{aligned} \boldsymbol{E}(\boldsymbol{x}) &= \sum_{l=1}^{\infty}\sum_{|j|\le l}\left\{a_{l,j}\mathbf{curl}\left\{\boldsymbol{e}(\boldsymbol{x})\right\}\right. \\ &\qquad \left.+ b_{l,j}\mathbf{curl}\ \mathbf{curl}\left\{\boldsymbol{e}(\boldsymbol{x})\right\}\right\}, \end{aligned} \quad (14)$$

where

$$\boldsymbol{e}(x) = \boldsymbol{x}h_l^{(1)}(k|\boldsymbol{x}|)Y_{l,j}(\widehat{\boldsymbol{x}}). \quad (15)$$

Here $h_l^{(1)}$ denotes the lth degree spherical Hankel function of the first kind. The corresponding electric far field is given by [1, Theorem 6.26, Page 182]

$$\boldsymbol{E}_\infty(\widehat{\boldsymbol{x}}) = \frac{1}{k}\sum_{l=1}^{\infty}\frac{1}{i^{l+1}}\sum_{|j|\le l}\Big\{ikb_{l,j}\sqrt{l(l+1)}\boldsymbol{U}_{l,j}(\widehat{\boldsymbol{x}}) - a_{l,j}\sqrt{l(l+1)}\boldsymbol{V}_{l,j}(\widehat{\boldsymbol{x}})\Big\}. \quad (16)$$

Using [1, Equations (6.64) and (6.65), Page 180] and $\widehat{\boldsymbol{x}}\times(\boldsymbol{U}(\widehat{\boldsymbol{x}})\times\widehat{\boldsymbol{x}}) = \boldsymbol{U}(\widehat{\boldsymbol{x}})$, we get

$$\widehat{\boldsymbol{x}}\times\mathbf{curl}\,\{\boldsymbol{e}(\boldsymbol{x})\} = h_l^{(1)}(k|\boldsymbol{x}|)\sqrt{l(l+1)}\boldsymbol{U}_{l,j}(\widehat{\boldsymbol{x}}),$$

and

$$\widehat{\boldsymbol{x}}\times\mathbf{curl}\,\mathbf{curl}\,\{\boldsymbol{e}(\boldsymbol{x})\} = \frac{1}{|\boldsymbol{x}|}\{h_l^{(1)}(k|\boldsymbol{x}|) + k|\boldsymbol{x}|h_l^{(1)'}(k|\boldsymbol{x}|)\}\sqrt{l(l+1)}\boldsymbol{V}_{l,j}(\widehat{\boldsymbol{x}}),$$

where $h_l^{(1)'}$ is the derivative of the lth degree spherical Hankel function of the first kind.

Let r be the radius of the scattering sphere. Writing $\boldsymbol{x} = r\widehat{\boldsymbol{x}}$ and taking the vector product of $\widehat{\boldsymbol{x}}$ with (14) gives

$$\widehat{\boldsymbol{x}}\times\boldsymbol{E}(r\widehat{\boldsymbol{x}}) = \sum_{l=1}^{\infty}\sum_{|j|\le l}a_{l,j}h_l^{(1)}(kr)\sqrt{l(l+1)}\boldsymbol{U}_{l,j}(\widehat{\boldsymbol{x}}) + \sum_{l=1}^{\infty}\sum_{|j|\le l}b_{l,j}\frac{1}{r}\{h_l^{(1)}(kr) + krh_l^{(1)'}(kr)\}\sqrt{l(l+1)}\boldsymbol{V}_{l,j}(\widehat{\boldsymbol{x}}).$$

Hence, using the boundary condition $\boldsymbol{n}\times\boldsymbol{E} = \boldsymbol{f}$, and equating coefficients with (11), we get

$$\boldsymbol{E}_\infty(\widehat{\boldsymbol{x}}) = \frac{1}{k}\sum_{l=1}^{\infty}\frac{1}{i^{l+1}}\sum_{|j|\le l}\left\{\frac{ikrB_{l,j}}{h_l^{(1)}(kr)+krh_l^{(1)'}(kr)}\boldsymbol{U}_{l,j}(\widehat{\boldsymbol{x}}) - \frac{A_{l,j}}{h_l^{(1)}(kr)}\boldsymbol{V}_{l,j}(\widehat{\boldsymbol{x}})\right\}. \quad (17)$$

In practical Mie–series computations, the series (17) must be truncated. An empirical study in [5, Section V] recommended truncation after the first $N_{\max}$ terms for plane wave problems, where

$$N_{\max} = \begin{cases} x + 4x^{1/3} + 1, & 0.02 \le x \le 8, \\ x + 4.05x^{1/3} + 2, & 8 < x < 4200, \\ x + 4x^{1/3} + 2, & 4200 \le x \le 20000, \end{cases} \quad (18)$$

and $x = 2\pi r/\lambda$, where r is the radius of the scattering sphere and λ is the incident wavelength.

For some special cases (for example, an incident plane wave [3, Page 419]) the Fourier coefficients of the boundary data are known analytically. In general we require a quadrature rule on the sphere to compute the Fourier coefficients in (13). To this end, we use an m-point quadrature rule with points $\widehat{\boldsymbol{x}}_q^m \in \partial B$ and weights $\zeta_q^m \in \mathbb{R}$, $q = 1,\cdots,m$ of the form

$$(\boldsymbol{G},\boldsymbol{H})_m = Q_m(\overline{\boldsymbol{H}}^T\boldsymbol{G}) = \sum_{q=1}^{m}\zeta_q^m\overline{\boldsymbol{H}(\widehat{\boldsymbol{x}}_q^m)}^T\boldsymbol{G}(\widehat{\boldsymbol{x}}_q^m). \quad (19)$$

In our algorithm, for approximations to the Mie–series by its partial sum containing at most the first n terms, we choose the number of quadrature points m such that $(\boldsymbol{G},\boldsymbol{H})_m = \langle\boldsymbol{G},\boldsymbol{H}\rangle$, if $\boldsymbol{G}$ and $\boldsymbol{H}$ are tangential vector harmonics of degree at most n. Hence in the rest of the paper we use the notation $m(n)$ to identify the connection between the number of quadrature points and the number of terms in the partial sums.

One such quadrature rule that we use in our computations is the $2(n+1)\times(n+1)$-point rectangle-Gauss rule with $m(n) = 2(n+1)^2$:

$$Q_m\boldsymbol{G} = \sum_{r=0}^{2n+1}\sum_{s=1}^{n+1}\mu_r^n\nu_s^n\boldsymbol{G}(\boldsymbol{p}(\theta_s^n,\phi_r^n)), \quad (20)$$

where $\theta_s^n = \cos^{-1}z_s$, where z_s are the zeros of the Legendre polynomial of degree $n+1$, ν_s^n are the corresponding Gauss-Legendre weights ($s = 1,\cdots,n+1$), and

$$\mu_r^n = \frac{\pi}{n+1}, \quad \phi_r^n = \frac{r\pi}{n+1}, \qquad r = 0,\ldots,2n+1. \quad (21)$$

Our computable approximations to the Fourier coefficients $A_{l,j}, B_{l,j}$ in (13) are denoted by $\widetilde{A}_{l,j}^{m(n)}, \widetilde{B}_{l,j}^{m(n)}$, and these are defined using (19) as

$$\widetilde{A}_{l,j}^{m(n)} = \left(\boldsymbol{f}\circ\boldsymbol{q}^{-1},\boldsymbol{U}_{l,j}\right)_{m(n)}, \quad 1\le l\le n,$$
$$\widetilde{B}_{l,j}^{m(n)} = \left(\boldsymbol{f}\circ\boldsymbol{q}^{-1},\boldsymbol{V}_{l,j}\right)_{m(n)}, \quad |j|\le l. \quad (22)$$

Using the derivation above, and quadrature approximation of the Fourier coefficients of the boundary data induced by the incident field, a natural n-term computable quadrature approximation to the far–field $\boldsymbol{E}_\infty$ in (17) is denoted by $\boldsymbol{E}_{n,\infty}^{m(n)}$ and is defined for every observed direction as

$$\begin{aligned}
&\boldsymbol{E}_{n,\infty}^{m(n)}(\widehat{\boldsymbol{x}}) \\
&= \frac{1}{k}\sum_{l=1}^{n}\frac{1}{i^{l+1}}\sum_{|j|\le l}\left\{\frac{ikr\widetilde{B}_{l,j}^{m(n)}}{h_l^{(1)}(kr)+krh_l^{(1)'}(kr)}\boldsymbol{U}_{l,j}(\widehat{\boldsymbol{x}})\right. \\
&\qquad\left. -\frac{\widetilde{A}_{l,j}^{m(n)}}{h_l^{(1)}(kr)}\boldsymbol{V}_{l,j}(\widehat{\boldsymbol{x}})\right\}. \quad (23)
\end{aligned}$$

4 Numerical Experiments

We verify the accuracy of our approximations by comparison with benchmark solutions for the electric and magnetic dipole point source radiation problems (denoted ED and MD respectively), and for the plane wave scattering problem (denoted PW). The scattering object is a perfectly conducting sphere of radius 0.5 centered at the origin.

For the point source radiation problems our benchmarks are the exact solutions, computed using (7)–(10). In the radiation problems considered, the point source is located at $\boldsymbol{x}_{\mathrm{src}}$, which is at a distance 0.1 away from the origin in the direction $\theta = 30°$, $\phi = 90°$. The point source has polarization vector $\widehat{\boldsymbol{p}}_{\mathrm{src}} = (1,1,0)^T/\sqrt{2}$.

For the plane wave scattering problem our benchmark is the (truncated) Mie–series solution computed using Wiscombe's code. In the scattering problem considered, the incident wave has direction $\boldsymbol{e}_3 = (0,0,1)^T$ and polarization $\boldsymbol{e}_1 = (1,0,0)^T$.

We compare our computed far field $\boldsymbol{E}_{n,\infty}^{m(n)}$ with the benchmark far field $\boldsymbol{E}_\infty$ using the relative maximum error

$$\frac{\max_{\widehat{\boldsymbol{x}}\in\partial B}\left\{\sum_{k=1}^{3}\left|\boldsymbol{e}_k^T\left[\boldsymbol{E}_\infty(\widehat{\boldsymbol{x}})-\boldsymbol{E}_{n,\infty}^{m(n)}(\widehat{\boldsymbol{x}})\right]\right|\right\}}{\max_{\widehat{\boldsymbol{x}}\in\partial B}\left\{\sum_{k=1}^{3}\left|\boldsymbol{e}_k^T\boldsymbol{E}_\infty(\widehat{\boldsymbol{x}})\right|\right\}},$$

approximated using more than 1300 observed directions.

In Tables 1–5 our solution is compared against the benchmark solution for spheres of electromagnetic size $\lambda/2$ to 24λ. In the plane wave case, the benchmark solution is computed using Wiscombe's code with highest order term $N_{\max}$, where $N_{\max}$ is given by (18). Our solution is computed using (23) for a range of n bracketing $N_{\max}$.

The results show that our approximations are very accurate for the radiation problem. Choosing $n \ge N_{\max}$ is sufficient to obtain $O(10^{-13})$ accuracy in the far field approximations.

The results also show very good agreement with the Wiscombe code for the plane wave scattering problem. However, the approximately

Scattering by sphere of diameter 0.5λ			
$N_{\max}(0.5\lambda) = 7$			
	Error		
n	ED	MD	PW
2	1.37e-02	1.28e-02	4.10e-02
7	7.22e-10	6.36e-10	2.13e-12
12	4.43e-14	5.15e-14	2.06e-10
17	6.21e-14	7.21e-14	2.06e-10
22	8.28e-14	9.72e-14	2.06e-10

Table 1: Scattering from perfectly conducting sphere of size 0.5λ, with $k = 3.1416$.

$O(10^{-10})$ error obtained in this case is several orders of magnitude larger than the error obtained for the radiation problems.

The errors in the plane wave problem stagnate for $n \ge N_{\max} + 5$ and cannot be reduced by increasing n. We remark that these errors cannot be reduced by using more than $N_{\max}$ terms in the Wiscombe code.

The high accuracy of our approximation has been demonstrated for the radiation problems. We therefore believe that the larger error observed for the plane wave scattering problem is due to a restriction of the accuracy of the Wiscombe code to about $O(10^{-10})$. Comparisons with our spectrally accurate scheme for plane wave scattering have shown that we can obtain errors of $O(10^{-14})$ between the approximations described in this paper and our spectrally accurate solution, compared with errors of about $O(10^{-10})$ between the benchmark and our spectrally accurate solution.

In Table 6 we give AMD Opteron (2 GHz) CPU timings for computing three approximate far fields (for the two radiation and the plane wave scattering problems) using our algorithm and code, and demonstrate that only a fraction of a minute of CPU time is required to compute the quadrature based approximations to the Mie–series at 1352 observed directions for Mie scattering by spheres of various electromagnetic size.

5 Conclusion

The approximate Mie–series using numerically computed expansion of the incident field, as described in this paper, has been shown to produce solutions accurate almost to machine precision. We conclude that the quadrature based approximation to the Mie–series is a suitable benchmark, and note that it has wider applicability and is not restricted to plane waves and certain forms of polarizations.

Scattering by sphere of diameter λ			
$N_{\max}(\lambda) = 10$			
	Error		
n	ED	MD	PW
5	3.36e-05	3.14e-05	1.99e-03
10	4.21e-12	3.85e-12	5.60e-09
15	7.70e-14	8.12e-14	5.48e-09
20	1.05e-13	1.10e-13	5.48e-09
25	1.28e-13	1.37e-13	5.48e-09

Table 2: Scattering from perfectly conducting sphere of size λ, with $k = 6.2832$.

Scattering by sphere of diameter 8λ			
$N_{\max}(8\lambda) = 39$			
	Error		
n	ED	MD	PW
34	2.73e-13	2.78e-13	7.79e-07
39	2.81e-13	2.85e-13	3.01e-10
44	2.96e-13	3.26e-13	3.45e-10
49	8.18e-13	7.55e-13	3.45e-10
54	6.88e-13	6.45e-13	3.45e-10

Table 3: Scattering from perfectly conducting sphere of size 8λ, with $k = 50.2655$.

Scattering by sphere of diameter 16λ			
$N_{\max}(16\lambda) = 67$			
	Error		
n	ED	MD	PW
62	1.12e-12	9.96e-13	5.75e-07
67	9.87e-13	8.95e-13	1.26e-10
72	8.46e-13	8.23e-13	2.88e-10
77	7.09e-13	7.99e-13	2.88e-10
82	6.59e-13	7.54e-13	2.88e-10

Table 4: Scattering from perfectly conducting sphere of size 16λ, with $k = 100.5310$.

Scattering by sphere of diameter 24λ			
$N_{\max}(24\lambda) = 95$			
	Error		
n	ED	MD	PW
90	7.93e-13	8.34e-13	2.09e-07
95	7.59e-13	8.29e-13	6.28e-10
100	7.97e-13	8.62e-13	8.14e-10
105	8.36e-13	8.97e-13	8.14e-10
110	8.43e-13	9.10e-13	8.14e-10

Table 5: Scattering from perfectly conducting sphere of size 24λ, with $k = 150.7964$.

CPU time to compute three far-field approximations at 1352 directions			
n	sphere size	x	time
4	0.1λ	0.31	0.06 secs
7	0.5λ	1.6	0.11 secs
10	λ	3.1	0.18 secs
39	8λ	25	2.8 secs
67	16λ	50	12 secs
95	24λ	75	31 secs

Table 6: Time taken to compute three far-fields (MD, ED and PW) at 1352 directions for perfectly conducting spheres of size λ to 24λ with $n = N_{\max}$.

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