Proceedings of the 9th WSEAS International Conference on Applied Mathematics, Istanbul, Turkey, May 27-29, 2006 (pp214-218)

Benjamin-Feir type instability of Sine-Gordon equation and spectrum of Lamé equation II

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Abstract: The Benjamin-Feir type instability of the nonhomoclinic solution of the Sine-Gordon equation against the spatially periodic small perturbation is observed by relating the Fourier coefficients of the first order approximate solution to the spectral structure of the second order ordinary differential equation. In particular, a numerical observation is also reported for the purterbation with the wave number belonging to the instability zone and the stability zone.
Keyword Sine-Gordon equation, Benjamin-Feir instability, Nonhomoclinic solution, Approximate solution of Fourier type

1. Introduction

In this paper, we study the Benjamin-Feir type instability of the Sine-Gordon equation

$$\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2} + \sin\phi = 0. \tag{1}$$

The present work is the sequal to the previous papers [12] and [13]. In this paper, we mainly report the numerical analysis of the Benjamin-Feir instability.

It is easy to see that the Sine-Gordon equation (1) has the following three types of the x-independent solutions $\phi_s(t,k)$, $k>0$;

$$\phi_s(t,k) = \begin{cases} 2\sin^{-1}(k\,\mathrm{sn}(t,k)), & 0<k<1, \\ 2\sin^{-1}(\mathrm{sn}(kt,k^{-1})), & 1<k, \\ 4\tan^{-1}(e^{-t})+\pi, & k=1, \end{cases} \tag{2}$$

where $\mathrm{sn}(t,k)$ is the Jacobi elliptic function with the modulus k. The Benjamin-Feir instability problem is related to the modulation instability. However, the last solution $\phi_s(t,1)$ is a homoclinic orbit, and there is no modulation wave number. So we omit it from our consideration. On the other hand, $\phi_s(t,k)$, $k\neq 1$ are the nonhomoclinic orbits, so, we call them the nonhomoclinic solutions.

The Benjamin-Feir instability zone $B(k)$ is the set of the wave numbers such that the spatially periodic small perturbation of the solution $\phi_s(t,k)$ becomes unstable for large t, if modulation wave number belongs to $B(k)$. In [12] and [13], it is shown that the Benjamin-Feir instability zone $B(k)$ are the open intervals

$$B(k) = \begin{cases} (0,k), & 0<k<1 \\ (\sqrt{k^2-1},k), & 1<k. \end{cases} \tag{3}$$

The purpose of the present work is to clarify numerically the asymptotic behavior of the purturbed solutions for large t. More precisely, the boundary of the Benjamin-Feir zone can be calculated exactly, since they coincide with the simple periodic spectrum of a kind of 2nd Lamé equation. Moreover, the first order approximate solution of Fourier type of modulated wave solution can be expressed explicitly in terms of the corresponding eigenfunctions which are expressed exactly by the Jacobi elliptic functions. Hence those asymptotic behavior can be exactly clarified.

Conversely, if the modulation wave number belongs to $B(k)$, then such exact treatment can not be carried out. Hence, in such a case, numerical treatment is necessary. In general, the finite diffrence scheme is not so reliable for such nonlinear unstable phenomena. Therefore we reduce the problem to the 2nd order linear ordinary differential equation, and consider the asymptotic behavior using the Runge-Kutta scheme.

2. The review of the paper [12]

We briefly explain the fundamental materials obtained mainly in [12].

Let $\phi_s(t,k)$ be the nonhomoclinic solution of the SG equation (1) defined by (2). Define the function $\phi_\varepsilon(x,t)$ by

$$\phi_\varepsilon(x,t) = \phi_s(t,k) + \varepsilon \sum_{n=0}^{\infty} \eta_n(t)\cos(\mu_n x), \tag{4}$$

where

$$\mu_n = \frac{2\pi n}{L}$$

for $L > 0$, where L is the spatial period of the perturbation, and fixed in what follows.

Put $\phi_\varepsilon(x,t)$ into the SG equation (1), then we have

$$\frac{\partial^2\phi_\varepsilon}{\partial t^2} - \frac{\partial^2\phi_\varepsilon}{\partial x^2} + \sin\phi_\varepsilon = \varepsilon\sum_{n=0}^{\infty}\left\{\frac{d^2\eta_n}{dt^2} + (\cos\phi_s)\eta_n + \mu_n^2\eta_n\right\} + O(\varepsilon^2).$$

Hence, if $\eta_n(t)$ solves the 2nd order ordinary differential equation

$$\frac{d^2\eta_n}{dt^2} + (\cos\phi_s)\eta_n + \mu_n^2\eta_n = 0. \tag{5}$$

for all $n = 0, 1, 2, \cdots$, then

$$\frac{\partial^2\phi_\varepsilon}{\partial t^2} - \frac{\partial^2\phi_\varepsilon}{\partial x^2} + \sin\phi_\varepsilon = O(\varepsilon^2)$$

follows. If the condition (5) are satisfied, we call $\phi_\varepsilon(x,t)$ defined by (4) the first order approximate solution of Fourier type.

On the other hand, the single component perturbation $\phi_\varepsilon^{(1)}(x,t,\mu)$ is defined by

$$\phi_\varepsilon^{(1)}(x,t,\mu) = \phi_s(t,k) + \varepsilon\eta(t)\cos\mu x, \tag{6}$$

where

$$\frac{d^2\eta}{dt^2} + (\cos\phi_s)\eta + \mu^2\eta = 0. \tag{7}$$

Definition 1. Let $\phi_s(t,k)$ be the nonhomoclinic solution and $\phi_\varepsilon^{(1)}(x,t,\mu)$ be the single component perturbation. The subset $B(k)$ of the positive axis $\boldsymbol{R}_+$ is called the Benjamin-Feir zone if the following condition is fulfilled; Assume $(\eta(0),\eta'(0)) \neq (0,0)$, i.e. $\eta(t) \not\equiv 0$, then

$$\lim_{t\to\infty}\sup_{-\infty<x<\infty}|\phi_\varepsilon^{(1)}(x,t,\mu)| = \infty$$

holds if and only if $\mu \in B(k)$.

By direct calculation, we have the following potential $U_k(t) = -\cos\phi_k(t)$;

$$U_k(t) = \begin{cases} 2k^2\mathrm{sn}^2(t,k) - 1, & 0 < k < 1 \\ 2\mathrm{sn}^2(kt, 1/k) - 1, & k > 1 \end{cases} \tag{8}$$

Define the 2nd order ordinary differential operator H_k by

$$H_k = -\frac{d^2}{dt^2} + U_k(t).$$

Then the function $\eta_n(t)$, $n = 0, 1, \cdots$ are the solutions of the eigenvalue problem

$$H_k\eta_n(t) = \mu_n^2\eta_n(t). \tag{9}$$

Hence, one can clarify the asymptotic behavior of the single component perturbation $\phi_\varepsilon(x,t)$ for large t by analyzing the behavior of the solution of the eigenvalue problem (9). The potential $U_k(t)$ is periodic. Hence H_k is a simple example of Hill's operator. The spectrum of Hill's operator has the so called band structure. That is, let

$$-\infty = \lambda_{-1} < \lambda_0 < \lambda_1 \leq \lambda_2 < \lambda_3 \leq \lambda_4 < \cdots < \lambda_{4j-1} \leq \lambda_{4j} < \lambda_{4j+1} \leq \lambda_{4j+2} < \cdots$$

be the periodic spectrum of Hill's operator. Let $I_i = (\lambda_{2i-1}, \lambda_{2i}), i = 0, 1, 2, \cdots$, which are called the instability zones, and if the spectral parameter is in I_i, then all nontrivial solutions of the eigenvalue problem are unbounded. On the one hand, let $J_i = (\lambda_{2i}, \lambda_{2i+1})$, $i = 0, 1, 2, \cdots$, which is called the stability zone, and if the spectral parameter is in J_i, then the all solutions of the eigenvalue problem are bounded, and are not periodic. The points λ_j,

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$j = 0, 1, 2, \cdots$ are periodic spectrum. We refer the reader [7] for more precise description about the band structure of the spectrum of Hill's operator.

Concerned with the spectrum of H_k with the modulus $0 < k < 1$, we have the following.

Theorem 1. *Assume* $0 < k < 1$. *Then,* $U_k(t)$ *is the 1-st algebro-geometric elliptic potential with the real period* $2K$, *where* K *is defined by*

$$K = \int_0^1 \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}. \tag{10}$$

Moreover, the simple periodic eigenvalues of the operator H_k *are* $k^2 - 1$, 0, *and* k^2, *where* $k^2 - 1 < 0 < k^2$, *and the instability zones are the intervals* $(-\infty, k^2 - 1)$ *and* $(0, k^2)$.

This fact is classical one. However, an alternative elementary proof is given in [12]

Next we consider the case $k > 1$. In this case, by (8), we have

$$\begin{aligned} &H_k\eta_n(t) = \\ &\quad -\frac{d^2}{dt^2}\eta_n(t) + (2\mathrm{sn}^2(kt, \frac{1}{k}) - 1)\eta_n(t) \\ &\quad = \mu_n^2\eta_n(t). \end{aligned}$$

Let $\tau = kt$, then, by direct calculation, we have

$$\begin{aligned} &-\frac{d^2}{d\tau^2}\tilde{\eta}_n(\tau) + \\ &\quad \left(2\left(\frac{1}{k}\right)^2 \mathrm{sn}^2(\tau, \frac{1}{k}) - 1\right)\tilde{\eta}_n(\tau) \\ &\quad = \tilde{\mu}_n^2\tilde{\eta}_n(\tau), \end{aligned} \tag{11}$$

where

$$\tilde{\eta}_n(\tau) = \eta_n(\frac{\tau}{k}), \quad \tilde{\mu}_n^2 = \frac{1}{k^2} - 1 + \frac{\mu_n^2}{k^2}. \tag{12}$$

Let

$$\widetilde{H}_k = -\frac{d^2}{d\tau^2} + 2\left(\frac{1}{k}\right)^2 \mathrm{sn}^2(\tau, \frac{1}{k}) - 1,$$

then, by Theorem 1, the simple periodic eigenvalues of the operator $\widetilde{H}_k$ are $\frac{1}{k^2} - 1$, 0 and $\frac{1}{k^2}$, and the instability zones are the intervals $(-\infty, \frac{1}{k^2} - 1)$ and $(0, \frac{1}{k^2})$. If $\tilde{\mu}_n^2 = \frac{1}{k^2} - 1$ then $\mu_n^2 = 0$ follows. Similarly, if $\tilde{\mu}_n^2 = 0$, and $\tilde{\mu}_n^2 = \frac{1}{k^2}$ then $\mu_n^2 = k^2 - 1$ and $\mu_n^2 = k^2$ follow respectively. Hence, the instability zones of the operator H_k itself are $(-\infty, 0)$ and $(k^2 - 1, k^2)$.

Now we consider the Benjamin-Feir instability zone $B(k)$ applying the above results.

First suppose $0 < k < 1$. If μ_n^2 is in the one of the instability zones of Hill's operator H_k, then, arbitrary nontrivial solution $\eta_n(t)$ of the eigenvalue problem (9) is unbounded for $t \to \infty$. Since μ_n are real numbers, $\mu_n^2 > 0$ holds. Hence, by Theorem 1, the instability zone which includes μ_n^2 is only $(0, k^2)$. Therefore, the Benjamin-Feir zone $B(k)$ coincides with the interval $(0, k)$.

Similarly, if $1 < k$, then one can verify that the Benjamin-Feir zone $B(k)$ coincides with the interval $(\sqrt{k^2-1}, k)$. These are the fact(3) mentioned in the introduction.

Thus we have the following Theorem 2.

Theorem 2. *The Benjamin-Feir zones* $B(k)$ *for the modulus* $0 < k$ *and* $k \neq 1$ *are given as follows.*

1. *If* $0 < k < 1$, $B(k) = (0, k)$ *holds.*
2. *If* $1 < k$, $B(k) = (\sqrt{k^2-1}, k)$ *holds.*

A part of this result itself seems to be already known. Actually, Ercolani et al. refer to this fact in [3] in the case $0 < k < 1$. But, the method of the proof is different from ours.

Now let us consider the behavior of the single component perturbation $\phi_\varepsilon^{(1)}(x, t, \mu)$ for $\mu \in \partial B(k)$.

Let $\theta_j(t, \lambda)$, $j = 1, 2$ be the eigenfunction of the eigenvalue problem

$$H_k\theta = \lambda\theta$$

such that

$$\begin{pmatrix} \theta_1(0,\lambda) & \theta_2(0,\lambda) \\ \theta_1'(0,\lambda) & \theta_2'(0,\lambda) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

First, we construct the real valued eigenfunction $\theta_1(t, \lambda)$ for the eigenvalue $\lambda = \mu^2$ such that $\mu \in \partial B(k)$ for $0 < k < 1$. By the Λ-operator method [9, p. 420, Theorem 11], we have

$$\theta_1(t, 0) = \mathrm{cn}(t, k),$$

which satisfies the initial condition

$$\theta_1(0, 0) = 1,\ \theta_1'(0, 0) = 0.$$

The solution $\theta_2(t, 0)$ such that

$$\theta_2(0, 0) = 0,\ \theta_2'(0, 0) = 1$$

is given by

$$\theta_2(t,0) = \mathrm{cn}(t,k)\int_0^t \frac{ds}{\mathrm{cn}^2(s,k)}.$$

For $\mu = k \in \partial B(k)$, similarly to the above, we have

$$\theta_2(t,k^2) = \mathrm{sn}(t,k),$$

$$\theta_1(t,k^2) = \mathrm{sn}(t,k)\int_t^K \frac{ds}{\mathrm{sn}^2(s,k)}.$$

It is easy to see that $\theta_2(t,0)$, and $\theta_1(t,k^2)$ increase linearly as $t \to \infty$.

For $1 < k$, we consider the eigenfunction $\theta_1(t,k^2-1)$. Since $\mu^2 = k^2-1$, by (12), we have $\tilde{\mu}^2 = 0$. Hence, by the elementary transformation formulas of Jacobi elliptic function, we have

$$\theta_1(t,k^2-1) = \mathrm{cn}(\tau,\frac{1}{k}) = \mathrm{cn}(kt,\frac{1}{k}) = \mathrm{dn}(t,k),$$

$$\theta_2(t,k^2-1) = \mathrm{dn}(t,k)\int_t^K \frac{ds}{\mathrm{dn}^2(s,k)},$$

$$\theta_2(t,k^2) = \mathrm{sn}(t,k^2),$$

$$\theta_2(t,k^2) = \mathrm{sn}(t,k)\int_t^K \frac{ds}{\mathrm{sn}^2(s,k)}.$$

Therefore we have the following Theorem.

Theorem 3. *Suppose $\eta(0) \neq 0$, then*

$$\lim_{t\to\infty}\sup_{-\infty<x<\infty}|\phi_\varepsilon^{(1)}(x,t,k)| = \infty$$

holds for all $k \neq 1$. Moreover, suppose $\eta'(0) \neq 0$, then

$$\lim_{t\to\infty}\sup_{-\infty<x<\infty}|\phi_\varepsilon^{(1)}(x,t,\mu)| = \infty$$

holds for $\mu = 0$ if $0 < k < 1$, and for $\mu = \sqrt{k^2-1}$ if $k > 1$.

Fig.1 are the graphs of $\phi_\varepsilon^{(1)}(x,t,\mu)$ for $k = 2$ such that $\mu^2 = k^2 - 1 = 3$ and $\eta(0) = 1$, $\eta'(0) = 0$.

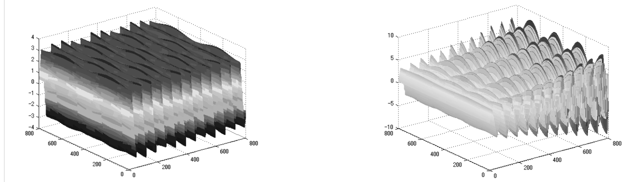

FIG. 1. Behavior of the approximate solutions at the boundary

3. Numerical consideration

Generally speaking, if μ^2 is the eigenvalue such that $\mu \in B(k)$, then one can not obtain any exact solutions of the eigenvalue problem

$$H_k\theta = \mu^2\theta. \tag{13}$$

Of course, since the set

$$\widehat{B}(k) = \{\mu^2 | \mu \in B(k)\}$$

is included in the resolvent set of the operator H_k considered in the space $L^2(-\infty,\infty)$, it is well known that any nontrivial solution $\theta(t)$ of (13) are unbouded, or increase exponetially as $|t| \to \infty$. However, it is essential to clarify the asymptotic behavior of the single component perturbation $\phi_\varepsilon^{(1)}(x,t,\mu)$ for large t. Therefore we carry out the numerical consideration in the rest of the present paper.

Any numerical scheme for the nonlinear evolution equation are not so reliable to clarify the unstable phenomena. In another occasion [14], we tried Hirota's scheme for the Sine-Gordon equation with the initial condition

$$\phi(x,0) = \phi_\varepsilon^{(1)}(x,0,\mu) = \varepsilon\eta(0)\cos\mu^2 x,$$

$$\phi_t(x,0) = \phi_{\varepsilon,t}^{(1)}(x,0,\mu) = \varepsilon\eta_n'(0)\mu^2 x.$$

However, while some interesting results were obtained, those ware not so trustworthy. The root of this fact is that the Hirota's scheme was developed to construct the soliton solutions, that is, for the stable phenomena. For Hirota's scheme of the Sine-Gordon equation, we refer the reader to the article [1]. Hence, some alternative scheme is required. That is our method of the 1st order approximate solution of Fourier type. In this case, the problem is reduced to the linear ordinary differential equation. Therefore, there are many trustworthy numerical schemes. In the present work, we adopt a standard one, the Runge-Kutta scheme to solve numerically the eigenvalue problem (13). Here the numerical results only for $1 < k$ are exhibited. The results for $0 < k < 1$ are completely parallel.

(1) Stable cases

There are two stable zone $I_1 = (0, k^2-1)$ and $I_2 = (k^2, \infty)$. As numerical experiments, we carried out the computation for $k = 2$.

Fig.2 are the graphs of $\phi_\varepsilon^{(1)}(x,t,\mu)$ such that $\eta(0)=1,\eta'(0)=0$ and $\eta(0)=0$ and $\eta'(0)=1$ for $\mu^2=1\in I_1$ respectively. Therefore, for arbitrary initial condition, $\phi_\varepsilon^{(1)}(x,t,\mu)$ is stable. The graphs of $\phi_\varepsilon^{(1)}(x,t,\mu)$ for $\mu^2=5\in I_2$ are similar to them.

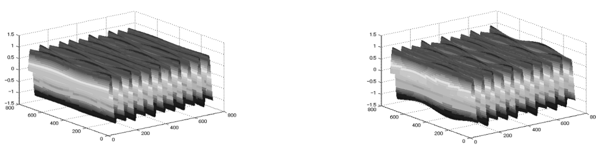

FIG. 2. The stable case

(2) Unstable cases

Fig.3 are the graphs of the approximate solution $\phi_\varepsilon^{(1)}(x,t,\mu)$ for $\mu^2=3.5\in \widehat{B}(2)$ such that $\eta(0)=1$, $\eta'(0)=0$ and $\eta(0)=0,\eta'(0)=1$.

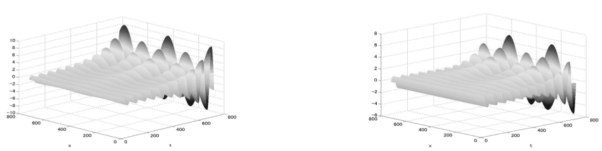

FIG. 3. The unstable case

By the both graphs, one can observe that those single component perturbations increase exponentially as $t\to\infty$. In comparison with Fig.1, the growth pattern is quite different, that is, the former is linearly and the latter is exponentially.

4. Conclusion

We observed the Benjamin-Feir type instability, which is a kind of modulation instability, of the nonhomoclinic solution (2) to the Sine-Gordon equation with respect to small perturbation of Fourier type. We investigated the problem by reducing it to consideration of the band structure of the spectrum of the Lamé equation. The asymptotic behavior of the approximate solution was observed by constructing the exact eigenfunctions for the simple periodic eigenvalues, and by constructing the numerical solutions for the continuous spectrum and the resolvent set.

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