Reversible Quaternary Cyclic Codes

Taher Abualrub
Department of Mathematics, American University of Sharjah, Sharjah-UAE
Irfan Siap
Adiyaman Education Faculty, Gaziantep University, Adiyaman, Turkey.

Abstract
In this paper we study reversible cyclic codes of any length \( n \) over the ring \( Z_4 \). First we find a set of generators for cyclic codes over \( Z_4 \) with no restrictions on the length \( n \). We then classify reversible cyclic codes with respect to their generators. Examples of reversible cyclic codes of lengths 8 and 9 with their minimum Hamming distance will be studied as well.

Key-Words: The ring \( Z_4 \), Linear Codes, Cyclic Codes, Quaternary Codes, Reversible Cyclic Codes

1 Introduction
Consider the ring \( Z_4 = \{0,1,2,3\} \). A linear code \( C \) of length \( n \) over \( Z_4 \) is defined to be an additive submodule of the \( Z_4 \)-module \( Z_4^n \). A cyclic code of length \( n \) over \( Z_4 \) is defined to be a submodule of \( Z_4^n \) that is invariant with respect to the shift operator that maps the element \((c_0,\ldots,c_{n-1})\) of \( Z_4^n \) to the element \((c_{n-1},c_0,\ldots,c_{n-2})\). For each element \((c_0,\ldots,c_{n-1})\) of \( Z_4^n \) we associate a polynomial \( c_0 + c_1 x + \cdots + c_{n-1} x^{n-1} \) in the ring \( R_n = Z_4[x]/(x^n - 1) \). In this case cyclic codes are defined to be ideals in \( R_n \). A code \( C \) is called reversible if it is invariant under a reversal of the digits in all its codewords, i.e., a cyclic code \( C \) is called reversible if for each codeword \( u = (u_0, u_1, \ldots, u_{n-1}) \in C \) then the reverse of \( u \), \( u^r = (u_{n-1}, u_{n-2}, \ldots, u_0) \) is also in \( C \). The Hamming distance between codewords \( u \) and \( v \), denoted by \( H(u, v) \), is simply the number of coordinates in which these two codewords differ. The Hamming weight of any codeword \( u = (u_0, u_1, \ldots, u_{n-1}) \), \( w(u) \) is the number of nonzero entries in \( u \). The Hamming distance of any linear code \( C \) is given by

\[
d(C) = \min \{ w(u) : u \in C \text{ and } u \neq 0 \}
\]

For each polynomial \( p(x) = p_0 + p_1 x + \cdots + p_r x^r \) with \( p_r \neq 0 \), we define the reciprocal of \( p(x) \) to be the polynomial \( p^*(x) = x^r p(1/x) = p_r + p_{r-1} x^{r-1} + \cdots + p_0 x^r \). Note that \( \deg p^*(x) = \deg p(x) \) and if \( p_0 \neq 0 \), then \( p(x) \) and \( p^*(x) \) always have the same degree. \( p(x) \) is called self-reciprocal if and only if \( p(x) = p^*(x) \).

Reversible cyclic codes over finite fields were studied first by [8]. It was shown that \( C = (f(x)) \) is reversible if and only if \( f(x) \) is a self-reciprocal polynomial.

In this paper we study reversible cyclic codes over \( Z_4 \) of any length \( n \). Such codes have applications in the subject of DNA computing. In particular this class of codes is important because it satisfies the following constraints.

- The Hamming constraint: For any two different codewords \( u, v \in C \), \( H(u, v) \geq d \).
- The reverse-constraint: For any two codewords \( u, v \in C \), \( H(u, v^r) \geq d \).

These two constraints will make non-desirable hybridization between a codeword and another codeword less likely to happen.

Also, this class of codes have some applications in constructing certain data storage and retrieval systems. We note that we put no restrictions on the length \( n \).

The rest of the paper is organized as follows. In section 2, we study cyclic codes of length \( n \) over \( Z_4 \) and we find a unique set of generators for them. In section 3, we study reversible cyclic codes over \( Z_4 \) and we put a set of constraints on their generator polynomials. Section 4 includes a list of all reversible cyclic codes of lengths 8 and 9. Section 5 Concludes the paper.

2 Generators for Cyclic Codes
A cyclic code \( C \) in \( R_n = Z_4[x]/(x^n - 1) \) is an ideal in \( R_n \). Our goal in this section is to find a set of
generators for $C$ for any length $n$. Most previous work
on cyclic codes in $R_n$ was restricted to the case where
$n$ is odd [3, 6, 9]. Little work was done on cyclic codes in
$R_n$ for even length $n = 2^k$. It was shown that the ring $R_n$
is not a principal ideal ring. [2] has studied the case
where $n = 2e$ and $\gcd(e, n) = 1$ and has also shown
that $R_n$ is not a principal ideal ring. Our approach
in studying these codes will be general and doesn't depend on the length $n$.

Let $C$ be a cyclic code in $R_n$. Define $\varphi : C \rightarrow Z_2[x]/(x^n - 1)$ by $\varphi(x) = x \mod 2$.

$\varphi$ is a ring homomorphism with $\ker \varphi = \{2r(x) : r(x) \text{ is a binary polynomial in } C\}$. Let $J = \{r(x) : 2r(x) \in \ker \varphi\}$. It is easy to check that $J$
is an ideal in $Z_2[x]/(x^n - 1)$ and hence a cyclic code in
$Z_2[x]/(x^n - 1)$. So $J = (a(x))$ where $a(x) \mid (x^n - 1)$.
This implies that $\ker \varphi = (2a(x))$ with $a(x) \mid (x^n - 1) \mod 2$. The image of $\varphi$ is also an ideal and hence
a binary cyclic code that has a generator $g(x)$ with $g(x) \mid (x^n - 1) \mod 2$. This implies that $C = (g(x) + 2p(x), 2a(x))$ where $p(x)$ is a binary polynomial.

Note that with this construction $g(x)$, and $a(x)$ are binary polynomials divisors of $(x^n - 1) \mod 2$ rather
than mod 4.

**Claim 1** We may assume $\deg a(x) > \deg p(x)$, and $a(x) \not| g(x) \mod 2$.

**Proof.** Since

$$C = (g(x) + 2p(x), 2a(x))$$

$$= (g(x) + 2[p(x) + x'a(x)], 2a(x)),$$

then we may assume $\deg a(x) > \deg p(x)$. Since

$$2g(x) \in \ker \varphi = (2a(x)),$$

then $a(x) \mid g(x)$. If $g(x) = a(x)$, then $C = (g(x) + 2p(x))$.

**Claim 2** $a(x)|p(x)\left(\frac{x^n - 1}{g(x)}\right) \mod 2$.

**Proof.**

$$\varphi\left(\frac{x^n - 1}{g(x)}\right) = \varphi\left(x^{n-1} + 2p\right) = 0$$

So, $\left(2\frac{x^n - 1}{g(x)}\right) \in \ker \varphi = (2a(x))$ and hence,

$$a(x)\left(\frac{x^n - 1}{g(x)}\right) \mod 2.$$

**Claim 3** If $C = (g(x) + 2p(x), 2a(x)) = (h(x) + 2q(x), 2b(x))$ then $g(x) = h(x), a(x) = b(x)$
and $p(x) = q(x) \mod a(x)$.

**Proof.** From the construction of $C$ we have $J = \{r(x) : 2r(x) \in \ker \varphi\} = (a(x)) = (b(x))$. Hence
$a(x) = b(x)$.

Suppose $C = (g(x) + 2p(x), 2a(x)) = (h(x) + 2q(x), 2b(x))$. Note that $h(x) \in \ker (C) = (g(x))$. Hence $h = g(x)a(x)$ and $\deg h(x) \geq \deg g(x)$.

By the same means $g(x) = h(x)\beta(x) = g(x)a(x)\beta(x)$ and $\deg g(x) \geq \deg h(x)$. Since $g$, and $h$
are factors of $(x^n - 1) \mod 2$ and $(x^n - 1)$ factors uniquely over $\mathbb{Z}_2$ into a product of irreducible polynomials then $\alpha(x) = \beta(x) = 1$ and $g(x) = h(x)$. Since $g(x) + 2q(x) \in C$, then $g(x) + 2q(x) = [g(x) + 2p(x)] + 2a(x)m(x)$. This implies

$$2[q(x) - p(x)] = 2a(x)m(x)$$

Therefore $p(x) = q(x) \mod a(x)$.

**Claim 4** If $\gcd(a(x), g(x)) = 1$, then $C = (g(x), 2)$.

**Proof.** Suppose $\gcd(a(x), g(x)) = 1$, then $t(x)a(x) + s(x)g(x) = 1 \Rightarrow 2t(x)a(x) + 2s(x)g(x) = 2 \in C$.
Therefore $C = (g(x), 2)$.

**Claim 5** Suppose $n$ is odd, then $C = (g(x), 2a(x)) = (g(x) + 2a(x))$

**Proof.** Suppose $a(x)|g(x)$ and $a(x)|p(x)\left(\frac{x^n - 1}{g(x)}\right)$.

Then $g(x) = a(x)m_1(x)$ and $p(x)\left(\frac{x^n - 1}{g(x)}\right) = a(x)m_2(x)$ Since $n$ is odd then $(x^n - 1)$ factors uniquely as a product of distinct irreducible polynomials. This implies that $a(x)$ must be a factor of $p(x)$. But $p(x)$ has degree less than $a(x)$. Hence $p(x) = 0$ and $C = (g(x), 2a(x))$. Let $h(x) = g(x) + 2a(x)$. $2h(x) = 2g(x) \in (g(x) + 2a(x))$. Also, $\left(\frac{x^n - 1}{g(x)}\right)h(x) = 2\left(\frac{x^n - 1}{g(x)}\right)a(x) \in (g(x) + 2a(x))$. Since $n$ is odd then $\gcd\left(\frac{x^n - 1}{g(x)}, g(x)\right) = 1$, and hence there exist some binary polynomials $f_1(x)$, $f_2(x)$ such that

$$1 = \left(\frac{x^n - 1}{g(x)}\right)f_1(x) + g(x)f_2(x)$$

$$2a(x) = 2\left(\frac{x^n - 1}{g(x)}\right)a(x)f_1 + 2g(x)a(x)f_2$$

$$\in (g(x) + 2a(x))$. Hence $g(x) \in (g + 2a(x)$ and

$$C = (g(x), 2a(x)) = (g(x) + 2a(x))$$
We can summarize the above by the following theorem.

**Theorem 6** Let \( C \) be a cyclic code in \( R_n = Z_4[x]/(x^n - 1) \). Then

1. If \( n \) is odd then \( R_n \) is a principal ideal ring and \( C = \langle g(x), 2a(x) \rangle = \langle g(x) + 2a(x) \rangle \) where \( g(x) \) and \( a(x) \) are binary polynomials with \( a(x) | g(x) | (x^n - 1) \mod 2. \)

2. If \( n \) is not odd then
   (a) If \( g(x) = a(x) \), then \( C = \langle g(x) + 2p(x) \rangle \) where \( g(x) \) and \( p(x) \) are binary polynomials with \( g(x) | (x^n - 1) \mod 2 \), and
   \[
   g(x) | p(x) \left( \frac{x^n - 1}{g(x)} \right).
   \]
   (b) \( C = \langle g(x) + 2p(x), 2a(x) \rangle \) where \( g(x) \), \( a(x) \), and \( p(x) \) are binary polynomials with \( a(x) | g(x) | (x^n - 1) \mod 2, \)
   \[
   a(x) | p(x) \left( \frac{x^n - 1}{g(x)} \right) \\text{ and } \deg g(x) > \deg a(x) > \deg p(x).
   \]

3 **Reversible Cyclic Codes**

**Definition 7** A cyclic code of length \( n \) over \( Z_4 \) will be called reversible if \( \forall u \in C \), then \( u^* \in C. \)

**Lemma 8** Let \( f(x), g(x) \) be any two polynomials in \( Z_4 \), with \( \deg f(x) \geq \deg g(x) \). Then (see \([1]\) for the proof)

1. \( [f(x)g(x)]^* = f(x)^*g(x)^* \), and
2. \( [f(x) + g(x)]^* = f(x)^* + x^{\deg f - \deg g} g^*(x). \)

**Theorem 9** Let \( C = \langle f_0 + 2f_1 \rangle = \langle f_0, 2f_1 \rangle \) be a linear cyclic code of odd length \( n \) over \( Z_4 \). Then \( C \) is reversible cyclic code if and only if \( f_0 \) and \( f_1 \) are self-reciprocal.

**Proof.** Let \( c(x) \) be an element in \( C \). Then \( c(x) = f_0l_1 + 2f_1l_2 \) for some polynomials \( l_1, l_2 \) where \( \deg(l_1) \leq n - \deg(f_0) - 1 \), and \( \deg(l_2) \leq \deg(f_0) - \deg(f_1) - 1 \). We may assume \( \deg(c(x)) = n - 1 \). \( C \) is reversible if and only if \( c^*(x) \in C \) if and only if

\[
x^{n-1}c(1/x) = x^{n-1} \left[ l_1(1/x)f_0(1/x) + 2l_2(1/x)f_1(1/x) \right]
=
x^{n-\deg(f_0)-1}l_1(1/x)f_0^*(x)
+ 2x^{n-\deg(f_1)-1}l_2(1/x)f_1^*(x)
= l_1^*(x)f_0^*(x) + 2l_2^*(x)f_1^*(x).
\]

Hence, \( C = \langle f_0, 2f_1 \rangle \subseteq \langle f_0(x)^*, 2f_1(x)^* \rangle \). Similarly we get that \( \langle f_0(x)^*, 2f_1(x)^* \rangle \subseteq \langle f_0, 2f_1 \rangle \), and hence \( \langle f_0(x)^*, 2f_1(x)^* \rangle = \langle f_0, 2f_1 \rangle = C \). Therefore, \( C \) is reversible if and only if \( f_0 \) and \( f_1 \) are self-reciprocal.

**Theorem 10** Let \( C = \langle g(x) + 2p(x) \rangle \) or \( C = \langle 2a(x) \rangle \) be a cyclic code where \( n \) is even. Then \( C \) can not be reversible unless \( p(x) = 0. \)

**Proof.** Suppose \( C = \langle g(x) + 2p(x) \rangle \) is reversible. Then \( C \) mod\(2\) is a binary cyclic reversible code and hence \( g(x) \) is self-reciprocal. This implies

\[
(g(x) + 2p(x))^* = g(x)^* + 2x^i p^*(x)
= g(x) + 2x^i p^*(x)
= [g(x) + 2p(x)] m(x) \in C
\]

This implies that \( m(x) = 1, i = 0 \), \( \deg p(x) = \deg g(x) \), and \( p(x) = p^*(x) \). But \( \deg p(x) < \deg g(x) \).
Therefore, \( C \) can not be reversible.

**Theorem 11** Suppose \( C = \langle g(x) + 2p(x), 2a(x) \rangle \)
with \( a(x) | g(x) | (x^n - 1) \), \( a | p \left( \frac{x^n - 1}{g(x)} \right) \), and \( \deg g(x) > \deg a(x) > \deg p(x) \). \( C \) is a reversible cyclic code if and only if

1. \( g(x), a(x) \) are self-reciprocal
2. \( a(x) | (x^i p^*(x) + p(x)) \)

**Proof.** \( \Rightarrow \) Suppose \( C \) is reversible cyclic code. The binary codes \( \langle g(x) \rangle \), \( \langle a(x) \rangle \) must be reversible cyclic codes and hence \( g(x) \), and \( a(x) \) are self-reciprocal.
Since \( C \) is reversible then

\[
(g(x) + 2p(x))^* = g^*(x) + 2x^i p^*(x)
= g(x) + 2x^i p^*(x)
= (g(x) + 2p(x)) m_1(x) + 2a(x)m_2(x)
\]

This implies that \( m_1(x) = 1 \) and \( 2x^i p^*(x) + 2p(x) = 2a(x)m_2(x) \). Hence \( x^i p^*(x) + p(x) \in a(x) \Rightarrow a | (x^i p^* + p) \mod 2. \)

\( \Leftarrow \) For \( C \) to be reversible it is suffices to show \( (g(x) + 2p(x))^* \), and \( a^*(x) \) are in \( C \). Since \( a(x) \) is self-reciprocal then \( a^*(x) = a(x) \in C. \) Also,

\[
(g(x) + 2p(x))^* = g^*(x) + 2x^i p^*(x) = g(x) + 2x^i p^*(x)
= (g(x) + 2p(x)) + 2p(x) + 2x^i p^*(x)
\in C.
\]

Hence, \( C \) is reversible.
4 Examples

- Length $n = 8$. $(x^8 + 1) = (x + 1)^8$ over $\mathbb{Z}_2$. Let $f = (x + 1)$. The tables below give the generator polynomial for all reversible cyclic codes of length 8.

<table>
<thead>
<tr>
<th>1</th>
<th>Non-zero Generator polynomial (s) of $C$</th>
<th>$d(C)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$(\alpha f^i)$ where $\alpha = 1, 2$ and $1 \leq i \leq 4$</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>$(\alpha f^i)$ where $\alpha = 1, 2$ and $5 \leq i \leq 6$</td>
<td>4</td>
</tr>
<tr>
<td>4</td>
<td>$(\alpha f^i)$ where $\alpha = 1, 2$</td>
<td>8</td>
</tr>
<tr>
<td>5</td>
<td>$(x + 1, 2)$</td>
<td>1</td>
</tr>
<tr>
<td>6</td>
<td>$(x^2 + 1, 2)$, $(x^2 + 1, 2(x + 1))$</td>
<td>1, 2</td>
</tr>
<tr>
<td>7</td>
<td>$(x^2 + 1 + 2(x + 1))$</td>
<td>2</td>
</tr>
<tr>
<td>8</td>
<td>$(x^2 + x^2 + x + 1, 2)$</td>
<td>1</td>
</tr>
<tr>
<td>9</td>
<td>$(x^2 + x^2 + x + 1, 2(x + 1))$</td>
<td>2</td>
</tr>
<tr>
<td>10</td>
<td>$(x^2 + x^2 + x + 1 + 2, 2(x + 1))$</td>
<td>2</td>
</tr>
<tr>
<td>11</td>
<td>$(x^2 + x^2 + x + 1, 2(x^2 + 1))$</td>
<td>2</td>
</tr>
<tr>
<td>12</td>
<td>$(x^2 + x^2 + x + 1, 2(x + 1))$</td>
<td>1, 2</td>
</tr>
<tr>
<td>13</td>
<td>$(x^2 + x^2 + x + 1, 2(x + 1))$</td>
<td>2</td>
</tr>
<tr>
<td>14</td>
<td>$(x^4 + 1, 2(x + 1))$, $(x^4 + 1, 2(x + 1)^2)$</td>
<td>2, 2</td>
</tr>
<tr>
<td>15</td>
<td>$(x^4 + 1, 2 + 2(x + 1)^2)$</td>
<td>1</td>
</tr>
<tr>
<td>16</td>
<td>$(x^4 + 1 + 2x, 2(x + 1)^2)$</td>
<td>2</td>
</tr>
<tr>
<td>17</td>
<td>$(x^4 + 1 + 2(x + 1), 2(x + 1)^2)$</td>
<td>2</td>
</tr>
<tr>
<td>18</td>
<td>$(x^4 + x^4 + x + 1, 2)$</td>
<td>1</td>
</tr>
<tr>
<td>19</td>
<td>$(x^4 + x^4 + x + 1, 2(x + 1))$</td>
<td>2</td>
</tr>
<tr>
<td>20</td>
<td>$(x^4 + x^4 + x + 1 + 2, 2(x + 1))$</td>
<td>2</td>
</tr>
<tr>
<td>21</td>
<td>$(x^4 + x^4 + x + 1, 2(x^2 + 1))$</td>
<td>2</td>
</tr>
<tr>
<td>22</td>
<td>$(x^4 + x^4 + x + 1 + 2, 2(x^2 + 1))$</td>
<td>2</td>
</tr>
<tr>
<td>23</td>
<td>$(x^4 + x^4 + x + 1, 2(x^3 + x^2 + x + 1))$</td>
<td>2</td>
</tr>
<tr>
<td>24</td>
<td>$(x^4 + x^4 + x + 1 + 2(x + 1), 2(x^3 + x^2 + x + 1))$</td>
<td>2</td>
</tr>
<tr>
<td>25</td>
<td>$(x^4 + x^4 + x + 1 + 2(x^2 + 1), 2(x^3 + x^2 + x + 1))$</td>
<td>2</td>
</tr>
<tr>
<td>26</td>
<td>$(x^4 + x^4 + x + 1 + 2(x^2 + x), 2(x^3 + x^2 + x + 1))$</td>
<td>2</td>
</tr>
<tr>
<td>27</td>
<td>$(x^4 + x^4 + x + 1, 2(x^2 + 1))$</td>
<td>2</td>
</tr>
<tr>
<td>28</td>
<td>$(x^4 + x^4 + x + 1 + (x + 1), 2(x^2 + 1))$</td>
<td>2</td>
</tr>
<tr>
<td>29</td>
<td>$(x^4 + x^4 + x + 1 + (x^4 + x^2), 2(x^4 + 1))$</td>
<td>2</td>
</tr>
</tbody>
</table>

Table 1: Reversible Cyclic Codes of Length 8.
<table>
<thead>
<tr>
<th></th>
<th>Generator polynomial (s) of $C$</th>
<th>$d(C)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>30</td>
<td>$(x^7 + x^6 + x^5 + x^2 + x + 1, 2(x^3 + x^2 + x + 1), 2(x^4 + 1))$</td>
<td>2</td>
</tr>
<tr>
<td>31</td>
<td>$(x^6 + x^4 + x^3 + 1, 2(x+1)), ((x^6 + x^4 + x^3 + 1), 2(x+1))$</td>
<td>2, 2</td>
</tr>
<tr>
<td>32</td>
<td>$(x^6 + x^3 + x^2 + 1) + 2, 2(x+1))$</td>
<td>2</td>
</tr>
<tr>
<td>33</td>
<td>$(x^6 + x^4 + x^3 + 1), 2(x^2 + 1))$</td>
<td>2</td>
</tr>
<tr>
<td>34</td>
<td>$(x^6 + x^4 + x^3 + 1) + 2, 2(x^2 + 1))$</td>
<td>2</td>
</tr>
<tr>
<td>35</td>
<td>$(x^6 + x^4 + x^3 + 1) + 2x, 2(x^2 + 1))$</td>
<td>2</td>
</tr>
<tr>
<td>36</td>
<td>$(x^6 + x^3 + x^2 + 1) + 2(x+1), 2(x^3 + 1))$</td>
<td>2</td>
</tr>
<tr>
<td>37</td>
<td>$(x^6 + x^4 + x^3 + 1), 2(x^3 + x^2 + x + 1))$</td>
<td>2</td>
</tr>
<tr>
<td>38</td>
<td>$(x^6 + x^3 + x^2 + 1) + 2(x+1), 2(x^3 + x^2 + x + 1))$</td>
<td>2</td>
</tr>
<tr>
<td>39</td>
<td>$(x^6 + x^3 + x^2 + 1), 2(x^2 + 1))$</td>
<td>2</td>
</tr>
<tr>
<td>40</td>
<td>$(x^6 + x^4 + x^3 + 1) + 2(x^2 + 1), 2(x^3 + 1))$</td>
<td>2</td>
</tr>
<tr>
<td>41</td>
<td>$(x^6 + x^4 + x^3 + 1) + 2(x^3 + x), 2(x^4 + 1))$</td>
<td>2</td>
</tr>
<tr>
<td>42</td>
<td>$(x^6 + x^4 + x^3 + 1) + 2(x^3 + x^2 + x + 1), 2(x^4 + 1))$</td>
<td>2</td>
</tr>
<tr>
<td>43</td>
<td>$(x^6 + x^4 + x^3 + 1), 2(x^3 + x^2 + x + 1))$</td>
<td>4</td>
</tr>
<tr>
<td>44</td>
<td>$(x^6 + x^4 + x^3 + 1) + 2(x^3 + 1), 2(x^3 + x^2 + x + 1))$</td>
<td>4</td>
</tr>
<tr>
<td>45</td>
<td>$(x^6 + x^4 + x^3 + 1), 2(x^2 + 1))$</td>
<td>1</td>
</tr>
<tr>
<td>46</td>
<td>$(x^6 + x^4 + x^3 + 1) + 2(x^2 + 1), 2(x^3 + 1))$</td>
<td>2</td>
</tr>
<tr>
<td>47</td>
<td>$(x^6 + x^4 + x^3 + 1) + 2(x^3 + x), 2(x^4 + 1))$</td>
<td>2</td>
</tr>
<tr>
<td>48</td>
<td>$(x^6 + x^4 + x^3 + 1) + 2(x^3 + x^2 + x + 1), 2(x^4 + 1))$</td>
<td>2</td>
</tr>
<tr>
<td>49</td>
<td>$(x^6 + x^4 + x^3 + 1) + 2(x^2 + 1), 2(x^3 + 1))$</td>
<td>2</td>
</tr>
<tr>
<td>50</td>
<td>$(x^6 + x^4 + x^3 + 1) + 2(x^3 + x^2 + x + 1))$</td>
<td>2</td>
</tr>
<tr>
<td>51</td>
<td>$(x^6 + x^4 + x^3 + 1) + 2(x^3 + x^2 + x + 1))$</td>
<td>2</td>
</tr>
<tr>
<td>52</td>
<td>$(x^6 + x^4 + x^3 + 1) + 2(x^2 + 1))$</td>
<td>2</td>
</tr>
<tr>
<td>53</td>
<td>$(x^6 + x^4 + x^3 + 1) + 2(x^3 + x^2 + x + 1), 2(x^4 + 1))$</td>
<td>2</td>
</tr>
<tr>
<td>54</td>
<td>$(x^6 + x^4 + x^3 + 1) + 2(x^3 + x^2 + x + 1))$</td>
<td>2</td>
</tr>
<tr>
<td>55</td>
<td>$(x^6 + x^4 + x^3 + 1) + 2(x^3 + x^2 + x + 1))$</td>
<td>4</td>
</tr>
<tr>
<td>56</td>
<td>$(x^6 + x^4 + x^3 + 1) + 2(x^3 + x^2 + x + 1))$</td>
<td>4</td>
</tr>
<tr>
<td>57</td>
<td>$(x^6 + x^4 + x^3 + 1) + 2(x^3 + x^2 + x + 1))$</td>
<td>4</td>
</tr>
</tbody>
</table>

Table 2: More Reversible Cyclic Codes of Length 8.
• Length \( n = 9 \). We know that
\[(x^9+1) = (x^6 + x^3 + 1)(x^2 + x + 1)(x+1) \text{ over } \mathbb{Z}_2.
\]
Since the length \( n \) is odd and all the factors above are self-reciprocal polynomials then by Theorem 9, all the 27 cyclic codes of this length are reversible.

5 Conclusion Remarks

In this paper we studied reversible cyclic codes of length \( n \) over \( \mathbb{Z}_4 \). We found a unique set of generators for these codes as ideals in the ring \( R_n = \mathbb{Z}_4[x]/(x^n - 1) \). A list of all reversible cyclic codes of lengths 8 and 9 is included in section 4. Open problems include the study of reversible negacyclic codes over \( \mathbb{Z}_4 \). Also it will be interesting to study these codes over \( \mathbb{Z}_{2^n} \).

References


