Proceedings of the 9th WSEAS International Conference on Applied Mathematics, Istanbul, Turkey, May 27-29, 2006 (pp524-529)

Six New Sequences of Orthogonal Polynomials

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Abstract: Orthogonal polynomials appeared in many branches of mathematics. Numerical computation is one of them. In this paper, We win introduce six new sequences of orthogonal polynomials on the closed interval $[0,1]$. We will show the orthogonal polynomials up to degree seven of each sequence. The application of these orthogonal polynomials will be illustrated with an example.

Key-Word: Orthogonal-polynomial Gauss-Quadrature-Formula Weight-function Definite-integral

1. Introduction

The orthogonal polynomial of degree n on the interval [0,1] with respect to the weight function $w(x)$ is in the form

$$p_n(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n$$

with the proper that

$$\int_a^b w(x)p_n(x)q_m(x)dx = 0$$

for all polynomials $q_m(x)$ of degree $m \le n-1$.

In this paper, We will introduce five formulas of the type of open formula. These five formulas are one point formula, two points formula, three points formula, four points formula and five points formula.

The sequence of orthogonal polynomials $\{p_0(x), p_1(x), p_2(x), ..., p_n(x), ...\}$ is said to be the sequence of orthogonal polynomials on the interval [0,1] with respect to the weight function $w(x)$ if all of orthogonal polynomials $p_k(x)$'s are the orthogonal polynomials on the interval [0,1] with respect to the weight function $w(x)$. Some classical sequences of orthogonal polynomials are the Legendre orthogonal polynomials, the Tschevbyshev orthogonal polynomials, the Laguerre orthogonal polynomials and the Hermite orthogonal polynomials.

2. Formulation

The six sequences of orthogonal polynomials are as follow.

First Sequence

Let $w(x) = \frac{1}{\sqrt{1+x}}, a = 0$ and $b = 1$. The first eight orthogonal polynomials are;

$p_0(x) = 1$

$p_1(x) = x - 0.4711045485$

$p_2(x) = x^2 - 0.9764397632x + 0.15467933503$

$p_3(x) = x^3 - 1.4769751911x^2 + 0.57693340739x - 0.046099089423$

$p_4(x) = x^4 - 1.9769618518x^3 +$
$1.2512119581x^2 - 0.27188144735x +$
0.013120269904

$p_5(x) = x^5 - 2.4768206795x^4 +$
$2.1760008549x^3 - 0.80365139197x^2 +$
$0.1123756970x - 0.0035350040845$

$p_6(x) = x^6 - 2.9766566076x^5 +$
$3.3509423298x^4 - 1.7666226663x^3 +$
$0.43522687160x^2 - 0.02689821012x +$
0.00098950881186

$p_7(x) = x^7 - 3.4763141997x^6 +$
$4.7753691994x^5 - 3.2851985082x^4 +$
$1.1811015873x^3 - 0.20961887295x^2 +$
$0.0152494827311x - 0.0002658386289.$

Second Sequence

Let $w(x) = \sqrt{1+x}, a = 0$ and $b = 1$. The first eight orthogonal polynomials are;

$p_0(x) = 1$

$p_1(x) = x - 0.52778738810$

$p_2(x) = x^2 - 1.0223373158x +$
0.17890348444

$p_3(x) = x^3 - 1.5211028612x^2 +$
$0.62204754883x - 0.05399304445$

$p_4(x) = x^4 - 2.0205190117x^3 +$
$1.3174668835x^2 - 0.29897599862x$
0.015472074567

$p_5(x) = x^5 - 2.5201258044x^4 +$
$2.2634997887x^3 - 0.86070462696x^2 +$
$0.12538483514x - 0.0043050312156$

$p_6(x) = x^6 - 3.0198783996x^5 +$
$3.4598689786x^4 - 1.8644599402x^3 +$
$0.47243127924x^2 - 0.048109945956x +$
0.0011756233879

$p_7(x) = x^7 - 3.5214142186x^6 +$
$4.9116340750x^5 - 3.4414138058x^4 +$
$1.2653495969x^3 - 0.230998851713x^2 +$
$0.017428805873x - 0.00031920953445.$

Third Sequence

Let $w(x) = \frac{1}{\sqrt{1-x}}, a = 0$ and $b = 1$. The first eight orthogonal polynomials are;

$p_0(x) = 1$

$p_1(x) = x - 0.6666666667$

$p_2(x) = x^2 - 1.1428571429x +$
0.22857142857

$p_3(x) = x^3 - 1.6363636364x^2 +$
$0.7272727272x - 0.069264069317$

$p_4(x) = x^4 - 2.1333333333x^3 +$
$1.4769230786x^2 - 0.35804195892x +$
0.019891219999

$p_5(x) = x^5 - 2.6315788645x^4 +$
$2.4767800066x^3 - 0.99071194801x^2 +$
$0.15241721079x - 0.0055424433613$

$p_6(x) = x^6 - 3.1304319523x^5 +$
$3.7267005049x^4 - 2.0921797149x^3 +$
$0.55381121119x^2 - 0.059073036280x +$
0.0015146870282

$p_7(x) = x^7 - 3.6295789224x^6 +$
$5.2265046236x^5 - 3.7872417751x^4 +$
$1.4427195010x^3 - 0.27334748661x^2 +$
$0.02143793410x - 0.00040831051774.$

Forth Sequence

Let $w(x) = \sqrt{1-x}, a = 0$ and $b = 1$. The first eight orthogonal polynomials are;

$p_0(x) = 1$

$p_1(x) = x - 0.4$

$p_2(x) = x^2 - 0.8888888889x +$
0.126984126698

$p_3(x) = x^3 - 1.3846153846x^2 +$
$0.50349650346x - 0.037296037291$

$p_4(x) = x^4 - 1.8823529404x^3 +$
$1.1294117635x^2 - 0.23167420770x$
0.010530645789

$p_5(x) = x^5 - 2.3809523655x^4 +$
$2.0050125012x^3 - 0.70765146281x^2 +$
$0.094353526787x - 0.0029031853618$

$p_6(x) = x^6 - 3.8799998326x^5 +$
$3.1304343707x^4 - 1.5900617502x^3 +$
$0.37659352319x^2 - 0.035444089580x +$
0.00078764619650

$p_7(x) = x^7 - 3.3793037757x^6 +$
$4.5057279963x^5 - 3.0038103386x^4 +$

$1.0448000352x^3 - 0.17190780617x^2 + 0.012568897842x - 0.00021122403206.$

Fifth Sequence

Let $w(x) = \frac{1}{\sqrt{1+x^2}}, a = 0$ and $b = 1$. The first eight orthogonal polynomials are;

$p_0(x) = 1$

$p_1(x) = x - 0.5309976652$

$p_2(x) = x^2 - 1.0234523093x + 0.1773933380$

$p_3(x) = x^3 - 1.52300003888x^2 + 0.6222069705x - 0.0533317048$

$p_4(x) = x^4 - 2.0227637825x^3 + 1.3191134400x^2 - 0.2985481776x + 0.0152586613$

$p_5(x) = x^5 - 2.5226535516x^4 + 2.2667965962x^3 - 0.8612858775x^2 + 0.1250047782x - 0.004243780$

$p_6(x) = x^6 - 3.0225935016x^5 + 3.4648487766x^4 - 1.8668649143x^3 + 0.4723294550x^2 - 0.0478919716x + 0.0011577601$

$p_7(x) = x^7 - 3.5225450748x^6 + 4.9130663975x^5 - 3.4403639879x^4 + 1.2630157827x^3 - 0.2298830754x^2 + 0.0172345366x - 0.0003118397.$

Sixth Sequence

Let $w(x) = \sqrt{1+x^2}, a = 0$ and $b = 1$. The first eight orthogonal polynomials are;

$p_0(x) = 1$

$p_1(x) = x - 0.4699636663$

$p_2(x) = x^2 - 0.9763049327x + 0.1565496839$

$p_3(x) = x^3 - 1.4769718145x^2 + 0.5783057699x - 0.0468693681$

$p_4(x) = x^4 - 1.9772251490x^3 + 1.252823131x^2 - 0.2733506531x + 0.0133737275$

$p_5(x) = x^5 - 2.4773413906x^4 + 2.1781537053x^3 - 0.8061026564x^2 +$

$p_6(x) = x^6 - 2.9774060935x^5 + 3.3538442639x^4 - 1.7704776001x^3 + 0.4373512645x^2 - 0.0431352995x + 0.0010116466$

$p_7(x) = x^7 - 3.4775224362x^6 + 4.7799443380x^5 - 3.2918678641x^4 + 1.1857046505x^3 - 0.2111263103x^2 + 0.01545005x - 0.000273320.$

3. Gauss-Quadrature Formula

Gauss-Quadrature formula is the formula for finding the definite integral which is in the form

$$\int_a^b w(x) f(x) dx \cong \sum_{k=1}^{n} A_k f(x_k)$$

where x_k's are the roots of the orthogonal of degree n, $p_n(x)$, on the interval $[a, b]$ with respect to the weight function $w(x)$ and

$$A_k = \frac{1}{p_n'(x_k)} \int_a^b \frac{w(x) p_n(x)}{x - x_k} dx .$$

The following tables are the table of the points x_k's and the values A_k's of the above orthogonal polynomials.

n	x_k	A_k
1	0.4711045485	0.8284271245
2	0.1989461051 0.7774936581	0.4387283037 0.3896988211
3	0.2042632302 0.5798240091 0.9080161415	0.2532837718 0.3655546621 0.209588691
4	0.1350003060 0.4074525791 0.7162705667 0.9411242235	0.1622822800 0.2806057761 0.2565691935 0.1289698751
5	0.0955716029 0.2978335287 0.5547190456 0.7975180625 0.9591489125	0.1121607421 0.2127526404 0.2333767179 0.1832077546 0.0869292698
6	0.0710768112 0.2258215399 0.4348406875 0.6674445037	0.0819118792 0.1641207608 0.1986837076 0.1856145092

	0.8489051729	0.1356667546
	0.9700198341	0.0624295133
7	0.0546849180	0.0623539322
	0.1760572976	0.1294149823
	0.3465366686	0.1665991398
	0.5409289198	0.1712453148
	0.7295537332	0.1479382074
	0.8829891292	0.1039205379
	0.9770269159	0.0469550103

Table 1 $w(x) = \frac{1}{\sqrt{1+x}}$

n	x_k	A_k
1	0.5277873881	1.2189514165
2	0.2241322275	0.5741649645
	0.7982050888	0.6447864520
3	0.1184235688	0.3060376742
	0.5117563184	0.5438439011
	0.8909229740	0.3690698413
4	0.0723475823	0.1868232043
	0.3387227404	0.3811738364
	0.6773793390	0.4160510651
	0.9320693500	0.2349033107
5	0.0485688699	0.1253495504
	0.2367409581	0.2703476962
	0.5073960242	0.3482420422
	0.7737413563	0.3134367503
	0.9536785960	0.1615153774
6	0.0347942284	0.0896975460
	0.1735215012	0.1988953569
	0.3869239588	0.2766270470
	0.6248520500	0.2958656669
	0.8334065116	0.2402585354
	0.9663801496	0.1176072643
7	0.0261921641	0.0673006632
	0.1324197976	0.1515257681
	0.3024813597	0.2195612116
	0.5058434690	0.2558893219
	0.7072781820	0.2469547427
	0.8727008320	0.1887423136
	0.9744984160	0.0892953955

Table 2 $w(x) = \sqrt{1+x}$

n	x_k	A_k
1	0.6666666667	2.0
2	0.2584442528	0.6957096904
	0.8844128901	1.3004290310
3	0.1305006052	0.3426489848
	0.5628021472	0.7215231465
	0.9430608842	0.9358278687
4	0.0778433918	0.2024570737
	0.3653225244	0.4447620672
	0.7238156858	0.6274132938
	0.9663517323	0.7253675654
5	0.0515060664	0.1333426703
	0.2516653451	0.2989026756
	0.5384026082	0.4381727212
	0.8121684159	0.5385334550
	0.9778364290	0.5910484778
6	0.0365385445	0.0943502361
	0.1825713237	0.2138780900
	0.4072489704	0.3201563722
	0.6550569000	0.4063350509
	0.8646996657	0.4669854980
	0.9843165481	0.4982947532
7	0.0272418051	0.0702329353
	0.1379991702	0.1603079586
	0.3157240386	0.2430326938
	0.5276156286	0.3144070640
	0.7345106011	0.3710810764
	0.8981639350	0.4104034198
	0.9883237436	0.4305348522

Table 3 $w(x) = \frac{1}{\sqrt{1-x}}$

n	x_k	A_k
1	0.4	0.6666666667
2	0.1788380868	0.3891106684
	0.7100508021	0.2775559982
3	0.0991941707	0.2332816246
	0.4501315007	0.3076023677
	0.8352897132	0.1257826743
4	0.0626657505	0.1523625355
	0.3010519874	0.2525273453
	0.6237754848	0.1960962659
	0.8948597178	0.0656805200
5	0.0430686910	0.1065419861
	0.2131199395	0.1976337600
	0.4668780439	0.1986308016
	0.7305392041	0.1256731557
	0.9273464871	0.0381873474
6	0.0313837035	0.0784269074
	0.1580131383	0.1551301598
	0.3574725142	0.1784600916
	0.5873872891	0.1470105936
	0.7988542264	0.0836026421
	0.9468889612	0.0240362722
7	0.0238705697	0.0600377689
	0.1215176978	0.1236299164
	0.2805440997	0.1541921447

	0.4752172889	0.1464883995
	0.6739894390	0.1084110941
	0.8446439559	0.0578425762
	0.959520725	0.0160647669

Table 4 $w(x)=\sqrt{1-x}$

n	x_k	A_k
1	0.5309976652	1.1477935747
2	0.2210883743	0.5358430511
	0.8023639350	0.6119505236
3	0.1162858009	0.2891038080
	0.5133956005	0.5033299843
	0.8933189873	0.3553597823
4	0.0710914682	0.1786775386
	0.3381120474	0.3512635205
	0.6799252579	0.3895324751
	0.9336350090	0.2283200405
5	0.0954840272	0.1209662749
	0.2355290535	0.2508646114
	0.5085089505	0.3207202074
	0.7759675012	0.2972290388
	0.9548402718	0.1580134395
6	0.0343070657	0.0851762636
	0.1720591917	0.1934604315
	0.3890126914	0.2405142322
	0.6210277770	0.2883510506
	0.8421390033	0.2249841592
	0.9640477726	0.1153074375
7	0.0257937474	0.0656730956
	0.1311433199	0.1432560463
	0.3014895668	0.2020365663
	0.5062299305	0.2351217586
	0.7086365759	0.2318378479
	0.8739788305	0.1819149536
	0.9752731038	0.0879533063

Table 5 $w(x)=\dfrac{1}{\sqrt{1+x^2}}$

N	x_k	A_k
1	0.4699636663	0.8813735872
2	0.2022448825	0.4687223003
	0.7740600502	0.4126512869
3	0.1092977949	0.2673101166
	0.4868318107	0.3956641601
	0.8808422089	0.2183993053
4	0.0678345926	0.1693304129
	0.3221774801	0.3041149290
	0.6599082002	0.2749687503
	0.9273048762	0.1329594948
5	0.0460398287	0.1160525478
	0.2261561871	0.2287920095
	0.4915842851	0.2537321224
	0.7623195614	0.1937894909
	0.9512415283	0.0890074164
6	0.0332404393	0.0842410462
	0.1665436383	0.1748097707
	0.3745844495	0.2165103552
	0.6120702823	0.2002244676
	0.8258727200	0.1419545688
	0.9650945641	0.0636306786
7	0.0251083272	0.0638402913
	0.1273823530	0.1366261926
	0.2927935326	0.1808261471
	0.4938380488	0.1865650683
	0.6982797175	0.1579689428
	0.8654225718	0.1078426909
	0.9749328658	0.0477042542

Table 6 $w(x)=\sqrt{1+x^2}$

4 Examples

There will be one example in this section. We will use the above forty-two formulas to find the numerical value of the definite integral of this example.

Example 1

Find the value of the definite integral of $\int_0^1 e^x dx$. The exact value of this integral is $e-1 \cong 1.7182818285$.

The numerical results of six sequences of orthogonal polynomials are in following table 1 to table 6.

Seq.	Cal. value	Error
1	1.6094377560	0.1088440724
2	1.6717452055	0.0465366229
3	2.2490495458	0.5307677173
4	1.2839582688	0.4343235596
5	1.7239956217	0.0057137933
6	1.5581046763	0.1601771522

Table 1 One Point

Seq.	Cal. value	Error
1	17166955350	0.0015862934
2	1.7175241521	0.0007576763
3	1.8495870765	0.1313052480
4	1.5619658430	0.1563159855

5	1.7174373640	0.00084446450
6	1.7170219221	0.00125990640

Table 2 Two Points

Seq.	Cal. value	Error
1	1.7174486963	0.00083313212
2	1.7178204304	0.00046139811
3	1.9565479956	0.23826616714
4	2.4553418617	0.73706003321
5	1.7182729217	0.00000890676
6	1.7182733379	0.00000849056

Table 3 Three Points

Seq.	Cal. value	Error
1	1.7174687850	0.00081304346
2	1.7178451667	0.00043666177
3	1.7504438839	0.03216205540
4	1.6679239491	0.05035787938
5	1.7182976508	0.00001582234
6	1.7182817265	0.00000010194

Table 4 Four Points

Seq.	Cal. value	Error
1	4.5469097570	2.82862792850
2	1.7177401328	0.00054169566
3	1.7390685189	0.02078669041
4	1.6841009484	0.03418088002
5	1.7182817422	0.00000008621
6	1.7182818367	0.00000000826

Table 5 Five Points

Seq.	Cal. value	Error
1	1.7175256096	0.00056218813
2	1.7178798885	0.00040193997
3	1.8424288684	0.12414704000
4	1.6935541559	0.02472767255
5	1.7182807388	0.00000108964
6	1.7182776358	0.00000419278

Table 6 Six Points

Seq.	Cal. value	Error
1	1.7174597543	0.0008220742
2	1.7182226383	0.0000591902
3	1.7290520761	0.0107702477
4	1.6995598772	0.0187219512
5	1.7182818284	0.0000000000
6	1.7181937670	0.0000880615

Table 7 Seven Points

5. Conclusion

The results from the example, indicated that the first, second, fifth and sixth sequences of the orthogonal polynomials give the expected results but not the third and forth sequences of the orthogonal polynomials. We strongly recommend the fifth sequence and the sixth sequence.

References:

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