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Dyck path statistics

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Abstract: The enumeration of Dyck paths according to the semilength and to various other parameters has been studied extensively. In this paper, the statistics "number of τ's" and "number of τ's at low level" are studied for every string τ of length up to four. The corresponding generating functions are evaluated and used in order to establish several enumerating results.

Key–Words: Catalan numbers, Dyck paths, Dyck words, generating functions, statistics, strings.

1 Introduction

A wide range of articles dealing with the occurrence of strings in Dyck paths appear frequently in the literature [2], [3], [5], [6], [7] and [11].

A *Dyck path* of semilength n is a lattice path of N^2 running from $(0,0)$ to $(2n,0)$, whose allowed steps are the up diagonal step $(1,1)$ and the down diagonal step $(1,-1)$. These steps are called *rise* and *fall* respectively.

It is clear that each Dyck path is coded by a word $\alpha = a_1 a_2 \cdots a_{2n} \in \{u,d\}^*$, called *Dyck word*, so that every rise (resp. fall) corresponds to the letter u (resp. d); (see Fig. 1).

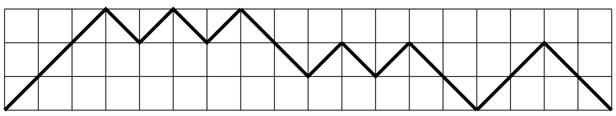

Figure 1: The Dyck path $uuudududdududduudd$

Throughout this paper we denote with $\mathcal{D}$ the set of all Dyck paths (or equivalently Dyck words). Furthermore, the subset of $\mathcal{D}$ that contains all the words α of semilength $l(\alpha) = n$ is denoted by $\mathcal{D}_n$. It is well-known that $|\mathcal{D}_n| = C_n$, where $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the n-th Catalan number (A00108 of [10]), with generating function $C(z)$, which satisfies the relation

$$zC^2(z) - C(z) + 1 = 0.$$

Furthermore, applying a version of Lagrange inversion formula [3] to the above equation, we obtain that

$$[z^n]C^s = \frac{s}{2n+s}\binom{2n+s}{n}.$$

A word $\tau \in \{u,d\}^*$, called in this context *string*, *occurs* in a Dyck path α if $\alpha = \beta\tau\gamma$, where $\beta, \gamma \in \{u,d\}^*$. If the string τ does not occur in α we say that α *avoids* τ.

For example, the Dyck path of Fig. 1 has three occurrences of udu, whereas it avoids ddd.

The statistic "number of occurrences of τ" has been studied by several authors, for various strings τ. The main tool used for the study of this statistic is the generating function $F(t,z)$ where t counts the occurrences of τ, and z counts the semilength of the Dyck path. In other words we have

$$F(t,z) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} a_{nk} t^k z^n,$$

where a_{nk} is the number of all $\alpha \in \mathcal{D}_n$ with k occurrences of τ. In particular, we will denote with $a_n = a_{n0}$ the number of all $\alpha \in \mathcal{D}_n$ that avoid τ.

The strings of length 2, namely uu, dd, ud and du, have been studied extensively; it has been proved (e.g., see [3]) that the corresponding statistics follow the Narayana distribution (A001263 of [10]). More generally, strings of length 2 have also been studied for k-colored Motzkin paths in [9].

The strings of length 3, namely uuu, uud, udu, udd, duu, dud, ddu and ddd, have also been studied extensively.

By symmetry with respect to a vertical axis, the statistics corresponding to each of the following pairs of strings: $\{duu, ddu\}$, $\{udu, dud\}$, $\{uuu, ddd\}$, $\{uud, udd\}$, are equidistributed.

The string $\tau = duu$ has been studied in [3] and it has been proved that the corresponding statistic follows the Touchard distribution, i.e.

$$a_{nk} = 2^{n-2k-1} C_k \binom{n-1}{2k}.$$

For a bijective proof of the above result see [1].

The string $\tau = udu$ has been studied independently in [7] and [11], and it has been proved that the corresponding statistic follows the Donaghey distribution, i.e.

$$a_{nk} = \binom{n-1}{k} M_{n-1-k},$$

where M_n is the n-th Motzkin number (A001006 of [10]). For a bijective proof of the above result see [1].

It is known (A092107 of [10]) that for $\tau = uuu$ the generating function satisfies the equation

$$z(t+z-tz)F^2 - (1-z+tz)F + 1 = 0.$$

Furthermore, the number of all Dyck paths that avoid τ is equal to M_n. For a bijective proof of the last statement see [1] or [11].

It seems that the string $\tau = uud$ has not been studied. For the evaluation of the generating function $F = F(t,z)$ in this case, we consider the first return decomposition of a non-empty Dyck path $\alpha = u\beta d\gamma$, where $\beta, \gamma \in \mathcal{D}$ and we observe that a new occurrence of uud appears in α (in addition to the ones contributed by β, γ) iff $\beta = ud\delta$, where $\delta \in \mathcal{D}$.

Thus, we have

$$F = 1 + z(tzF + F - zF)F,$$

which gives

$$z((t-1)z+1)F^2 - F + 1 = 0.$$

Furthermore, applying a version of Lagrange inversion formula [3] to the above equation, we obtain that

$$a_{nk} = \frac{1}{n+1}\binom{n+1}{k} \sum_{j=0}^{n-2k} \binom{k+j-1}{k-1}\binom{n+1-k}{n-2k-j}.$$

This number counts also the Dyck paths of semilength n with k long ascents (A091156 of [10]).

In section 2, several new results in the same direction are presented, referring to statistics for every string of length 4.

In section 3, the corresponding statistics are studied for the occurrence of strings at low level.

An extended version of this article, with detailed proofs, will be presented in a future work.

2 Strings of length 4

There exist sixteen strings τ of length 4, yielding ten cases to be studied, since by symmetry with respect to a vertical axis, the statistics "number of occurrences of τ" for some of them (given here in pairs) are equidistributed: $\{uuud, uddd\}$, $\{uudd\}$, $\{udud\}$, $\{dduu\}$, $\{uuuu, dddd\}$, $\{uudu, dudd\}$, $\{uduu, ddud\}$, $\{uddu, duud\}$, $\{duuu, dddu\}$, $\{dudu\}$.

The statistics corresponding to strings of the first four sets are known.

So the statistic of the string $\tau = uuud$ is equidistributed with the statistic "number of branch nodes at odd height" in ordered trees ([4], A091958 of [10]). The corresponding generating function satisfies the equation

$$(t-1)z^3F^3 + zF^2 - F + 1 = 0;$$

it also holds that

$$a_{nk} = \frac{1}{n+1}\binom{n+1}{k} \sum_{j=0}^{[\frac{n}{3}]-k} (-1)^j \binom{n+1-k}{j}\binom{2n-3k-3j}{n}.$$

Moreover, we note that the number of Dyck paths of semilength n that avoid $uuud$ is counted by the Motzkin numbers.

For $\tau = uudd$ the corresponding generating function (A098978 of [10]) satisfies the equation

$$zF^2 + ((t-1)z^2 - 1)F + 1 = 0.$$

Furthermore, it can be proved that

$$a_{nk} = \sum_{j=0}^{[\frac{n}{2}]-k} \frac{(-1)^j}{n-(j+k)}\binom{n-(j+k)}{j+k}\binom{2n-3(j+k)}{n-(j+k)-1}\binom{j+k}{k}.$$

For the avoiding sequence a_n in this case, see A086581 and A025242 of [10].

It is also known (A094507 of [10]) that for $\tau = udud$ the corresponding generating function satisfies the equation

$$z(1+(1-t)z)F^2 - (1+(1-t)z(z+1))F + 1 + (1-t)z = 0,$$

while the avoiding sequence counts also the irreducible stack sortable permutations (A078481 of [10]).

Finally, it is known (A114492 of [10]) that for $\tau = dduu$ the corresponding generating function satisfies the equation

$$z(t+(1-t)z)F^2 - (1+(1-t)(z-2)z)F + 1 - (1-t)z = 0.$$

We note that the avoiding sequence in this case is equal to the avoiding sequence corresponding to the string $\tau = uudd$, shifted by one unit.

We now come to study the remaining statistics.

2.1 The string $uuuu$

It can be proved that the generating function of Dyck paths according to the semilength and the number of $uuuu$'s satisfies the equation

$$(1-t)z^3F^3+z(t+z-tz)F^2+((1-t)z-1)F+1=0.$$

The first terms of the corresponding triangle formed by the coefficients of F, read by rows, are 1; 1; 2; 5; 13, 1; 36, 5, 1; 104, 21, 6, 1.

The avoiding sequence is given (A036765 of [10]) by the formula

$$a_n = \frac{1}{n+1}\sum_{j=0}^{[\frac{n}{2}]}\binom{n+1}{n-2j}\binom{n+1}{j}.$$

A natural generalization of the above, is to consider the string $\tau = u^r$, for any $r \geq 2$. It can be proved that the generating function of Dyck paths according to the semilength and the number of occurrences of u^r satisfies the equation

$$F = 1 + tzF^2 + (1-t)\sum_{i=1}^{r-1} z^i F^i.$$

2.2 The strings $uduu$ and $uudu$

It can be proved that the statistics corresponding to the strings $uudu$ and $uduu$ are equidistributed, with generating function satisfying the equation

$$z(1-(1-t)z)F^2 + ((1-t)z-1)F + 1 = 0.$$

Furthermore, we have that

$$a_{nk} = \sum_{j=0}^{[\frac{n-1}{2}]-k} \tfrac{(-1)^j}{n-j-k}\tbinom{j+k}{k}\tbinom{n-j-k}{j+k}\tbinom{2n-3j-3k}{n-j-k+1}.$$

The first terms of the triangle, read by rows, are 1; 1; 2; 4, 1; 9, 5; 22, 19, 1; 57, 66, 9; 154, 221, 53, 1.

We note that the strings $u^r du$, for $r \geq 1$, have been studied recently in [6].

2.3 The string $uddu$

It can be proved that the generating function corresponding to the string $\tau = uddu$ satisfies the equation

$$zF^3-((1-t)z+1)F^2+(1+2(1-t)z)F-(1-t)z = 0.$$

The first terms of the triangle, read by rows, are 1; 1; 2; 4, 1; 9, 5; 23, 17, 2; 63, 54, 15; 178, 177, 69, 5; 514, 594, 273, 49.

2.4 The string $duuu$

It can be proved that the generating function corresponding to the string $\tau = duuu$ satisfies the equation

$$tzF^3+(3(1-t)z-1)F^2-(3(1-t)z-1)F+(1-t)z=0.$$

The first terms of the triangle, read by rows, are 1; 1; 2; 5; 13, 1; 35, 7; 96, 36; 267, 159, 3.

The avoiding sequence a_n counts also the directed animals of size n, as well as the Grand-Dyck paths starting with u and avoiding udu (A005773 of [10]).

2.5 The string $dudu$

It can be proved that the statistic of the string $\tau = dudu$ is equidistributed with the statistic "number of ascents of length 1 that start at an odd level" (A102405 of [10]), with generating function that satisfies the equation

$$zF^2 + ((1-t)(z-1)z-1)F + (1-t)z + 1 = 0.$$

The first terms of the triangle, read by rows, are 1; 1; 2; 4, 1; 10, 3, 1; 26, 12, 3, 1; 72, 41, 15, 3, 1; 206, 143, 58, 18, 3, 1.

Finally, it can be proved that the avoiding sequence is given by the formula

$$a_n = \sum_{j=0}^{[\frac{n}{2}]} \frac{1}{n-j}\tbinom{n-j}{j}\sum_{i=0}^{n-2j}\tbinom{n-2j}{i}\tbinom{j+i}{n-2j-i+1}.$$

3 Strings at low level

We say that a string τ *occurs at height* 0 in a Dyck path if the string τ in this appearance meets the x-axis. An occurrence of a string τ at height 0 is usually referred to as a *low* occurrence of τ. For example, the path α of Fig. 1 has 2 low occurrences of uu and 1 low occurrence of ddu.

In this section, we study the statistic "number of low occurrences of τ" for various strings τ. The main tool used for the study of this statistic is the generating function $L(t,z)$, where t counts the low occurrences of τ, and z counts the semilength of the Dyck path. In other words we have

$$L(t,z) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} l_{nk}t^k z^n$$

where l_{nk} is the number of all $\alpha \in \mathcal{D}_n$ with k low occurrences of τ. In particular, we denote with $l_n = l_{n0}$ the number of all $\alpha \in \mathcal{D}_n$ that avoid low occurrences of τ.

For strings of length 2 we have:

i. If $\tau = ud$, then

$$L(t,z) = \frac{1}{1-(t+C(z)-1)z},$$

with coefficients

$$l_{nk} = \sum_{j=0}^{\left[\frac{n-k}{2}\right]} \frac{j}{n-k-j}\binom{k+j}{k}\binom{2n-2k-2j}{n-k}.$$

ii. If $\tau = du$, then

$$L(t,z) = 1 + \frac{zC(z)}{1-tzC(z)},$$

with coefficients

$$l_{nk} = \frac{k+1}{2n-k-1}\binom{2n-k-1}{n}.$$

iii. If $\tau = uu$ or $\tau = dd$, then

$$L(t,z) = \frac{C(z)}{1+(1-t)z^2C^3(z)},$$

with coefficients

$$l_{nk} = \sum_{j=0}^{\left[\frac{n}{2}\right]-k} (-1)^j \tfrac{3j+3k+1}{n+j+k+1}\tbinom{j+k}{k}\tbinom{2n-j-k}{n+j+k}.$$

The first two of the above formulas are given in [3] and the third counts also the number of Dyck paths with prescribed length and number of large components (A097877 of [10]).

Among the strings of length 3 only for $\tau = udu$ the statistic "number of low udu's" is known [11], with

$$L(t,z) = 1 + \frac{zC(z)}{1+z(1-C(z)-t)},$$

and coefficients

$$l_{nk} = \tfrac{1}{n-k}\sum_{j=0}^{\left[\frac{n-k-1}{2}\right]} (2j+1)\tbinom{j+k}{k}\tbinom{2n-2k-2j-2}{n-k-1}.$$

3.1 The string uuu

For the evaluation of the generating function $L = L(t,z)$ for the case $\tau = uuu$, we consider the partition $\{\mathcal{A}_i\}$ of $\mathcal{D}$, where $\mathcal{A}_0 = \{\epsilon\}$ and $\mathcal{A}_i$ is the set of all Dyck paths with length of the first ascent equal to i, for every $i \geq 1$. Clearly, every element α of $\mathcal{A}_i$, for $i \geq 1$, can be written uniquely in the form $\alpha = u^i d\alpha_1 d\alpha_2 \cdots d\alpha_i$, where $a_j \in \mathcal{D}$, for every $j \in [i]$. Furthermore, the low occurrences of uuu in α, if $i \leq 2$ are the same to those of α_i, whereas for $i \geq 3$ we have a new low occurrence; thus we have that

$$\begin{aligned} L &= 1 + \sum_{i=1}^{2} z^i C^{i-1}(z) L + t\sum_{i=3}^{\infty} z^i C^{i-1}(z) L \\ &= 1 + \left(zC(z) - (1-t)\sum_{i=3}^{\infty} z^i C^{i-1}(z)\right) L \\ &= 1 + \left(zC(z) - (1-t)\tfrac{z^3C^2(z)}{1-zC(z)}\right) L \\ &= 1 + \left(zC(z) - (1-t)z^3C^3(z)\right) L. \end{aligned}$$

It follows that

$$L(t,z) = \frac{C(z)}{1+(1-t)z^3C^4(z)}.$$

If we expand the above generating function into a geometric series and use the formula that gives the powers of the Catalan generating function, we obtain that the number of all $\alpha \in \mathcal{D}_n$ with k low uuu's is equal to

$$l_{nk} = \sum_{j=0}^{\left[\frac{n}{3}\right]-k} (-1)^j \tfrac{4j+4k+1}{n+j+k+1}\tbinom{j+k}{k}\tbinom{2n-2j-2k}{n+j+k}.$$

The first terms of the triangle, read by rows, are 1; 1; 2; 4, 1; 9, 5; 22, 20; 58, 73, 1; 163, 257, 9; 483, 893, 54; 1494, 3097, 270, 1; 4783, 10779, 1221, 13.

The avoiding sequence l_n counts also the number of Dyck paths of semilength n with no peak at height 3 ([8], A059019 of [10]) as well as the number of Dyck paths of semilength n with no valleys at height 1 (A114489 of [10]).

3.2 The string duu

For the evaluation of the generating function $L = L(t,z)$ for the case $\tau = duu$ we consider the decomposition of a non-empty Dyck path $\alpha = \beta u\gamma d$, where $\beta, \gamma \in \mathcal{D}$ and we observe that a new occurrence of duu appears in α iff $\beta, \gamma \neq \epsilon$; thus we have that

$$L = 1 + zt(L-1)(C(z)-1) + zC(z) + z(L-1),$$

which, after some simple manipulations gives

$$L(t,z) = 1 + \frac{zC^2(z)}{1+(1-t)z^2C^3(z)}$$

and

$$l_{nk} = \sum_{j=0}^{\left[\frac{n-1}{2}\right]-k} (-1)^j \tfrac{3j+3k+2}{2n-j-k}\tbinom{j+k}{k}\tbinom{2n-j-k}{n+j+k+1}.$$

The first terms of the triangle, read by rows, are 1; 1; 2; 4, 1; 9, 5; 23, 18, 1; 65, 59, 8; 197, 190, 41, 1; 626, 618, 175, 11; 2056, 2047, 685, 73, 1.

The avoiding sequence l_n counts also the partial partial sums of the Catalan numbers (A014137 of [10]).

3.3 The string uud

For the evaluation of the generating function $L = L(t,z)$ for the case $\tau = uud$ we consider the first return decomposition of a non-empty Dyck path $\alpha = u\beta d\gamma$, where $\beta, \gamma \in \mathcal{D}$ and we observe that a new occurrence of uud appears in α iff $\beta = ud\delta$, where $\delta \in \mathcal{D}$; thus we have that

$$L = 1 + z(tzC(z) + C(z) - zC(z))L,$$

which gives

$$L(t,z) = \frac{C(z)}{1+(1-t)z^2C^2(z)}$$

and

$$l_{nk} = \frac{1}{n+1}\sum_{j=0}^{\left[\frac{n}{2}\right]-k}(-1)^j(2j+2k+1)\binom{j+k}{k}\binom{2n-2j-2k}{n}.$$

The first terms of the triangle, read by rows, are 1; 1; 1, 1; 2, 3; 6, 7, 1; 19, 18, 5; 61, 53, 17, 1; 200, 168, 54, 7; 670, 552, 176, 31, 1.

The avoiding sequence l_n counts also the number of Dyck paths of semilength n that start with a pyramid followed by a pyramid-free Dyck path (A035929 of [10]).

3.4 Strings of length 4

We conclude by giving, without proofs, the generating function $L(t,z)$ for every string of length 4.

i. For $\tau = uuuu$ we have that

$$L(t,z) = \frac{C(z)}{1+(1-t)z^4C^5(z)},$$

with coefficients

$$l_{nk} = \sum_{j=0}^{\left[\frac{n}{4}\right]-k}(-1)^j\,\frac{5j+5k+1}{n+j+k+1}\binom{j+k}{k}\binom{2n-3j-3k}{n+j+k}.$$

The first terms of the triangle, read by rows, are 1; 1; 2; 5; 13, 1; 36, 6; 105, 27; 319, 110; 1002, 427, 1; 3235, 1616, 11; 10685, 6034, 77.

More generally, it can be proved that for $\tau = u^r$, where $r \in N^*$ we have

$$L(t,z) = \frac{C(z)}{1+(1-t)z^rC^{r+1}(z)},$$

with coefficients

$$l_{nk} = \sum_{j=0}^{\left[\frac{n}{r}\right]-k}(-1)^j\frac{(r+1)(j+1)+1}{n+j+k+1}\binom{j+k}{k}\binom{2n-(r-1)(j+k)}{n+j+k}.$$

ii. For $\tau = uudd$ it is known (A1144086 in [10]) that

$$L(t,z) = \frac{C(z)}{1+(1-t)z^2C(z)}.$$

Furthermore, it can be proved that the corresponding coefficients are equal to

$$l_{nk} = \sum_{j=0}^{\left[\frac{n}{2}\right]-k}(-1)^j\,\frac{j+k+1}{n-j-k+1}\binom{j+1}{k}\binom{2n-3j-3k}{n-j-k}.$$

The first terms of the triangle, read by rows, are 1; 1; 1, 1; 3, 2; 10, 3, 1; 31, 8, 3; 98, 27, 6, 1; 321, 88, 16, 4; 1078, 287, 54, 10, 1; 3686, 960, 183, 28, 5.

iii. For $\tau = uddu$ we have that

$$L(t,z) = 1 + \frac{zC^2(z)}{1+(1-t)z^2C^2(z)},$$

with coefficients

$$l_{nk} = \frac{2(k+1)}{n+1}\sum_{j=0}^{\left[\frac{n-1}{2}\right]-k}(-1)^j\binom{j+k+1}{k+1}\binom{2n-2j-2k-1}{n}.$$

The first terms of the triangle, read by rows, are 1; 1; 2; 4, 1; 10, 4; 29, 12, 1; 90, 36, 6; 290, 114, 24, 1; 960, 376, 86, 8; 3246, 1272, 303, 40, 1.

iv. For $\tau = dduu$ we have that

$$L(t,z) = \frac{C(z)(1+(1-t)z^2C^2(z))}{1+(1-t)(1-z)z^2C^3(z)}.$$

The first terms of the triangle, read by rows, are 1; 1; 2; 5; 13, 1; 36, 6; 106, 25, 1; 327, 94, 8.

v. For $\tau = uudu$ it is known that

$$L(t,z) = \frac{C(z)}{1+(1-t)z^3C^3(z)}.$$

This result is given in [6] in a more general setup.

It follows easily that the corresponding coefficients are equal to

$$l_{nk} = \frac{1}{n+1}\sum_{j=0}^{\left[\frac{n}{3}\right]-k}(-1)^j(3j+3k+1)\binom{j+k}{k}\binom{2n-3j-3k}{n}.$$

The first terms of the triangle, read by rows, are 1; 1; 2; 4, 1; 10, 4; 28, 14; 85, 46, 1; 271, 151, 7; 893, 502, 35; 3013, 1697, 151, 1; 10351, 5828, 607, 10.

It can be proved bijectively that the statistics "number of low $uudu$'s", "number of low $uuud$'s" and "number of low $uduu$'s" are equidistributed.

vi. For $\tau = duuu$ we have that

$$L(t,z) = 1 + \frac{zC^2(z)}{1+(1-t)z^3C^4(z)},$$

with coefficients

$$l_{nk} = \sum_{j=0}^{\left[\frac{n-1}{3}\right]-k}(-1)^j\,\frac{2j+2k+1}{n-j-k+1}\binom{j+k}{k}\binom{2n-2j-2k-1}{n-j-k}.$$

The first terms of the triangle, read by rows, are 1; 1; 2; 5; 13, 1; 36, 6; 105, 27; 320, 108, 1; 1011, 409, 10; 3289, 1508, 65; 10957, 5491, 347, 1.

vii. For $\tau = udud$ we have that

$$L(t,z) = \frac{(1+(1-t)z)C(z)}{1+(1-t)z(1+zC(z))}.$$

The first terms of the triangle, read by rows, are 1; 1; 1, 1; 4, 1; 11, 2, 1; 33, 6, 2, 1; 105, 17, 7, 2, 1; 343, 56, 19, 8, 2, 1; 1148, 185, 64, 21, 9, 2, 1.

viii. For $\tau = dudu$ we have that

$$L(t,z) = 1 + zC(z) + \frac{z^2C^3(z)}{1+(1-t)zC(z)},$$

with coefficients
$l_{nk} = \delta_{0k}c_{n-1} + \sum_{j=0}^{n-k-2} (-1)^j \frac{k+j+3}{n+1}\binom{j+k}{k}\binom{2n-k-j-2}{n}$.

The first terms of the triangle, read by rows, are 1; 1; 2; 4, 1; 11, 2, 1; 32, 7, 2, 1; 99, 22, 8, 2, 1; 318, 73, 26, 9, 2, 1; 1051, 246, 90, 30, 10, 2, 1.

References:

[1] D. Callan, Two bijections for Dyck path parameters, Preprint, (2004), 4pp, http://www.arxiv.org/abs/math.CO/0406381.

[2] A. Denise and R. Simion, Two combinatorial statistics on Dyck paths, *Discrete Math.* **137** (1995), 155–176.

[3] E. Deutsch, Dyck path enumeration, *Discrete Math.* **204** (1999), 167–202.

[4] E. Deutsch, A bijection on ordered trees and its consequences, *J. Combin. Theory A* **90** (2000), 210–215.

[5] T. Mansour, Counting peaks at height k in a Dyck path, *J. Integer Seq.* **5** (2002), Article 02.1.1.

[6] T. Mansour, Statistics on Dyck paths, *J. Integer Seq.* **9** (2006), Article 06.1.5.

[7] D. Merlini, R. Sprungoli and M. Verri, Some statistics on Dyck paths, *J. Statist. Plann. and Infer.* **101** (2002), 211–227.

[8] P. Peart and W.-J. Woan, Dyck paths with no peaks at height k, *J. Integer Seq.* **4** (2001), Article 01.1.3.

[9] A. Sapounakis and P. Tsikouras, Counting peaks and valleys in k-colored Motzkin paths, *Electronic J. Combin.* **12** (2005), R16.

[10] N. J. A. Sloane, Online Encyclopedia of Integer Sequences, published electronically at http://www.research.att.com/ njas/sequences/.

[11] Y. Sun, The statistic “number of udu’s” in Dyck paths, *Discrete Math.* **237** (2004), 177–186.