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Homogenization in Nonlinear Chemical Reactive Flows

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Abstract: The goal of this paper is to study a nonlinear model for some chemical reactive flows involving diffusion, adsorption and chemical reactions which take place at the boundary and also in the interior of a periodically perforated material. The effective behavior of such flows is governed by another nonlinear boundary-value problem, which contains extra zero-order terms and captures the effect produced by the adsorption, diffusion and also by the nonlinear interior chemical reactions.

Key-Words: homogenization, reactive flows, adsorption, diffusion, variational methods.

1 Introduction

The aim of this paper is to study the effective behavior of chemical reactive flows involving diffusion, adsorption and chemical reactions which take place in a periodically perforated material.

Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 3$. We shall consider that Ω is an ε-periodic structure, consisting of two parts: a fluid phase Ω^ε and a solid skeleton (grains), $\Omega \setminus \overline{\Omega}^\varepsilon$; ε represents a small parameter related to the characteristic size of the grains. We assume that chemical substances are dissolved in the fluid part Ω^ε. They are transported by diffusion and also, by adsorption, they can change from being dissolved in the fluid to residing on the surface of the grains. Inside the fluid phase and on the boundary, chemical reactions take place:

$$\begin{cases} \dfrac{\partial v^\varepsilon}{\partial t}(t,x) - D\Delta v^\varepsilon(t,x) + \beta(v^\varepsilon(t,x)) = \\ \qquad = h(t,x), \quad x \in \Omega^\varepsilon,\, t > 0, \\ v^\varepsilon(t,x) = 0, \quad x \in \partial\Omega,\, t > 0, \\ v^\varepsilon(t,x) = v_1(x), \quad x \in \Omega^\varepsilon,\, t = 0, \end{cases} \tag{1}$$

$$-D\frac{\partial v^\varepsilon}{\partial \nu}(t,x) = \varepsilon f^\varepsilon(t,x), \quad x \in S^\varepsilon,\, t > 0, \tag{2}$$

$$\begin{cases} \dfrac{\partial w^\varepsilon}{\partial t}(t,x) + a w^\varepsilon(t,x) = f^\varepsilon(t,x), \\ \qquad x \in S^\varepsilon,\, t > 0, \\ w^\varepsilon(t,x) = w_1(x),\ x \in S^\varepsilon,\, t = 0, \end{cases} \tag{3}$$

$$f^\varepsilon(t,x) = \gamma(g(v^\varepsilon(t,x)) - w^\varepsilon(t,x)). \tag{4}$$

Hence, the model consists of a diffusion-reaction system in the fluid phase Ω^ε, a reaction system on the surface of the grains and a nonlinear boundary condition coupling them (see (2)). Here, ν is the exterior unit normal to Ω^ε, $a, \gamma > 0$, h is a given function representing an external source of energy, $v_1, w_1 \in H_0^1(\Omega)$ and S^ε is the boundary of our skeleton $\Omega \setminus \overline{\Omega^\varepsilon}$. The fluid is homogeneous and isotropic, with a constant diffusion coefficient $D > 0$. In (1)-(4), v^ε is the concentration of the solute in the fluid region, w^ε is the concentration of the solute on the surface of the skeleton $\Omega \setminus \overline{\Omega^\varepsilon}$, v_1 is the initial concentration of the solute and w_1 the initial concentration of the reactants on the surface S^ε.

We shall consider the case in which β is a continuous function, monotonously nondecreasing, with $\beta(0) = 0$ and g is a monotone smooth function satisfying the condition $g(0) = 0$. For more general functions g, see [4]. This situation is illustrated by two important practical examples:

$\beta(v) =\mid v \mid^{p-1} v$, $0 < p < 1$ (Freundlich kinetics),

$$g(v) = \frac{\alpha v}{1+\beta v}, \quad \alpha, \beta > 0 \text{ (Langmuir kinetics).}$$

Using the classical theory of semilinear monotone problems (see [1]), we know that there exists a unique weak solution $u^\varepsilon = (v^\varepsilon, w^\varepsilon)$ of system (1)-(4).

We shall consider only periodic structures obtained by removing periodically from Ω, with period εY (Y is a given hyper-rectangle in $\mathbb{R}^n$), an elementary hole F which has been appropriated rescaled and which is strictly included in Y.

Our goal in this paper is to obtain the asymptotic behavior, as ε tends to zero, of the solution u^ε in such domains. If we denote by $\star$ the convolution with respect to time and by $r(\rho) = e^{-(a+\gamma)\rho}$, then we prove that the solution v^ε, properly extended to the whole of Ω, converges to the unique solution v of the problem:

$$\begin{cases} \dfrac{\partial v}{\partial t}(t,x) - D\displaystyle\sum_{i,j=1}^{n} q_{ij}\frac{\partial^2 v}{\partial x_i \partial x_j}(t,x)+ \\ +\beta(v(t,x)) + F_0(t,x) = h(t,x), \\ \qquad t>0,\, x\in\Omega, \\ v(t,x) = 0, \quad t>0,\ x\in\partial\Omega, \\ v(t,x) = v_1(x), \quad t=0,\ x\in\Omega, \end{cases} \tag{5}$$

$$F_0(t,x) = \frac{|\partial F|}{|Y\setminus \overline{F}|}\gamma[g(v(t,x)) - w_1(x)e^{-(a+\gamma)t} - \\ -\gamma r(\cdot)\star g(v(\cdot,x))(t)]. \tag{6}$$

In (5), $Q = ((q_{ij}))$ is the classical homogenized matrix, whose entries are defined as follows:

$$q_{ij} = \delta_{ij} + \frac{1}{|Y\setminus\overline{F}|}\int_{Y\setminus\overline{F}} \frac{\partial\chi_j}{\partial y_i}dy \tag{7}$$

$$\begin{cases} -\Delta\chi_i = 0 \ \text{ in } \ Y\setminus\overline{F}, \\ \dfrac{\partial(\chi_i + y_i)}{\partial\nu} = 0 \ \text{ on } \ \partial F, \\ \chi_i \ \ Y- \ \text{periodic.} \end{cases} \tag{8}$$

Moreover, let us notice that the limit problem for the surface concentration w is

$$\begin{cases} \dfrac{\partial w}{\partial t}(t,x) + (a+\gamma)w(t,x) = \gamma g(v(t,x)), \\ \qquad t>0,\ x\in\Omega, \\ w(t,x) = w_1(x), \quad t=0,\, x\in\Omega. \end{cases}$$

Notice that the influence of the adsorption and chemical reactions taking place inside the fluid region and on the boundaries of the perforations is reflected by the appearance of the zero-order extra-terms.

In [5], the author proposes an homogenization strategy for the effective behavior of some chemical processes involving adsorption and reactions arising in porous media. Rigorous proofs of the convergence results are given in the case of linear adsorption rates and linear chemical reactions. The author leaves as an open question the case of a nonlinear adsorption rate. In [4] our goal was to study two well-known examples of such nonlinear models, namely the so-called Freundlich and Langmuir kinetics. In that paper we considered only the case of chemical reactions taking place on the boundary of the grains. In the present paper, which is a generalization of the above mentioned work, we deal with the case in which there are nonlinear reactions also inside the fluid region and this gives rise in the limit to a new zero-order extra-term.

A related work was completed by U. Hornung et al. (see [6]) using nonlinearities which are essentially different from the ones we consider in the present paper. The proof of these authors is also different, since it is mainly based on the technique of two-scale convergence, which proves to be a successful alternative for this kind of problems. However, we have decided to use the energy method, coupled with monotonicity methods and results from the theory of semilinear problems, because it offered us the possibility to cover the nonlinear cases of practical importance mentioned above.

The structure of our paper is as follows: in Chapter 2, after some necessary preliminaries, we formulate the main convergence result, the proof of which is given in Chapter 3. Finally, notice that throughout the paper, by C we shall denote a generic fixed strictly positive constant, whose value can change from line to line.

2 Preliminaries and the Main Result

Let Ω be a smooth bounded connected open subset of $\mathbb{R}^n$ $(n \geq 3)$ and let $Y = [0, l_1[\times...[0, l_n[$ be the representative cell in $\mathbb{R}^n$. Denote by F an open subset of Y with smooth boundary ∂F such that $\overline{F} \subset Y$. We shall refer to F as being *the elementary hole*. Also, let $[0, T]$, with $T > 0$, be the time interval of interest.

Let ε be a real parameter taking values in a sequence of positive numbers converging to zero. For each ε and for any integer vector $k \in \mathbb{Z}^n$, set F_k^ε the translated image of εF by the vector $kl = (k_1 l_1, ..., k_n l_n)$. Also, let us denote by F^ε the set of all the holes contained in Ω, i.e. $F^\varepsilon = \bigcup \left\{ F_k^\varepsilon \mid \overline{F_k^\varepsilon} \subset \Omega,\ k \in \mathbb{Z}^n \right\}$. Set $\Omega^\varepsilon = \Omega \setminus \overline{F^\varepsilon}$. Ω^ε is a periodically perforated domain with holes of size of the same order as the period. Let $S^\varepsilon = \cup \{\partial F_k^\varepsilon \mid \overline{F_k^\varepsilon} \subset \Omega,\ k \in \mathbb{Z}^n\}$. So, $\partial \Omega^\varepsilon = \partial \Omega \cup S^\varepsilon$. We shall also denote $Y^* = Y \setminus \overline{F}$, $\theta = \frac{|Y^*|}{|Y|}$, where $|\omega|$ is the Lebesgue measure of any measurable subset ω of $\mathbb{R}^n$. Also, let:

$$H = L^2(\Omega),$$

with the classical scalar product and norm: $(u, v)_\Omega = \int_\Omega u(x) v(x) dx,\ \|u\|_\Omega^2 = (u, u)_\Omega$,

$$\mathcal{H} = L^2(0, T; H),$$

with $(u, v)_{\Omega, T} = \int_0^T (u(t), v(t))_\Omega dt$ and $u(t) = u(t, \cdot)$, $v(t) = v(t, \cdot)$,

$$V = H^1(\Omega),$$

with $(u, v)_V = (u, v)_\Omega + (\nabla u, \nabla v)_\Omega$,

$$\mathcal{V} = L^2(0, T; V),$$

with $(u, v)_\mathcal{V} = \int_0^T (u(t), v(t))_V dt$,

$$\mathcal{W} = \left\{ v \in \mathcal{V} \,|\, \frac{dv}{dt} \in \mathcal{V}' \right\}$$

where $\mathcal{V}'$ is the dual space of $\mathcal{V}$,

$$\mathcal{V}_0 = \{ v \in \mathcal{V} \,|\, v = 0 \text{ on } \partial\Omega \text{ a.e. on } (0, T) \}$$

and $\mathcal{W}_0 = \mathcal{V}_0 \bigcap \mathcal{W}$. Similarly, we define the spaces $V(\Omega^\varepsilon)$, $\mathcal{V}(\Omega^\varepsilon)$, $V(S^\varepsilon)$ and $\mathcal{V}(S^\varepsilon)$. For the space of test functions we use the notation $\mathcal{D} = C_0^\infty((0, T) \times \Omega))$.

We shall suppose that there exist a positive constant C and two exponents q_1, q_2, with $0 \leq q_1 < n/(n-2)$, $0 \leq q_2 < n/(n-2)$, such that

$$\left| \frac{dg}{dv} \right| \leq C(1 + |v|^{q_1}), \tag{9}$$

$$|\beta(v)| \leq C(1 + |v|^{q_2 + 1}). \tag{10}$$

The weak formulation of problem (1)-(4) is:

$$\begin{cases} \text{Find } v^\varepsilon \in \mathcal{W}_0(\Omega^\varepsilon), v^\varepsilon(0) = v_1|_{\Omega^\varepsilon} \text{ with} \\ -\left(v^\varepsilon, \dfrac{d\varphi}{dt} \right)_{\Omega^\varepsilon, T} + \varepsilon\, (f^\varepsilon, \varphi)_{\Omega^\varepsilon, T} + \\ + (\beta(v^\varepsilon), \varphi)_{\Omega^\varepsilon, T} = -D\, (\nabla v^\varepsilon, \nabla \varphi)_{\Omega^\varepsilon, T} + \\ + (h, \varphi)_{\Omega^\varepsilon, T}, \quad \forall \varphi \in \mathcal{W}_0(\Omega^\varepsilon) \end{cases} \tag{11}$$

and

$$\begin{cases} \text{Find } w^\varepsilon \in \mathcal{W}(S^\varepsilon), w^\varepsilon(0) = w_1|_{S^\varepsilon} \text{ with} \\ -\left(w^\varepsilon, \dfrac{d\varphi}{dt} \right)_{S^\varepsilon, T} + a\, (w^\varepsilon, \varphi)_{S^\varepsilon, T} = \\ = (f^\varepsilon, \varphi)_{S^\varepsilon, T}, \quad \forall \varphi \in \mathcal{W}(S^\varepsilon). \end{cases} \tag{12}$$

By classical existence results (see [1]), there exists a unique weak solution $u^\varepsilon = (v^\varepsilon, w^\varepsilon)$ of the system (11)-(12).

Remark 2.1. *Let us notice that the solution of (3) can be written as*

$$w^\varepsilon(t, x) = w_1(x) e^{-(a+\gamma)t} +$$

$$+ \gamma \int_0^t e^{-(a+\gamma)(t-s)} g(v^\varepsilon(s, x)) ds$$

or, if we denote by $\star$ the convolution with respect to time and by $r(\rho) = e^{-(a+\gamma)\rho}$, as

$$w^\varepsilon(\cdot, x) = w_1(x) e^{-(a+\gamma)t} + \gamma r(\cdot) \star g(v^\varepsilon(\cdot, x)).$$

The solution v^ε of problem (11) being defined only on Ω^ε, we need to extend it to the whole of Ω to state our main convergence result. We recall the following extension result (see [2]-[3]):

Lemma 2.2. *There exists a linear continuous extension operator $P^\varepsilon \in \mathcal{L}(L^2(\Omega^\varepsilon); L^2(\Omega)) \cap$ $\cap \mathcal{L}(V^\varepsilon; H_0^1(\Omega))$ and a positive constant C, independent of ε, such that*

$$\|P^\varepsilon v\|_{L^2(\Omega)} \leq C \|v\|_{L^2(\Omega^\varepsilon)},$$

$$\|\nabla P^\varepsilon v\|_{L^2(\Omega)} \le C \|\nabla v\|_{L^2(\Omega^\varepsilon)},$$

for any $v \in V^\varepsilon$, *where* $\|v\|_{V^\varepsilon} = \|\nabla v\|_{L^2(\Omega^\varepsilon)}$ *and* $V^\varepsilon = \{v \in H^1(\Omega^\varepsilon) \mid v = 0 \text{ on } \partial\Omega\}$. ▪

The main result of this paper is given by:

Theorem 2.3. *One can construct an extension* $P^\varepsilon v^\varepsilon$ *of the solution* v^ε *of the problem (11) such that*

$$P^\varepsilon v^\varepsilon \rightharpoonup v \quad \text{weakly in } \mathcal{V},$$

where v *is the unique solution of the following limit problem:*

$$\begin{cases} \dfrac{\partial v}{\partial t}(t,x) + \beta(v(t,x)) + F_0(t,x) - \\ -D \sum\limits_{i,j=1}^{n} q_{ij} \dfrac{\partial^2 v}{\partial x_i \partial x_j}(t,x) = h(t,x), \\ \quad t > 0,\ x \in \Omega, \\ v(t,x) = 0, \quad t > 0,\ x \in \partial\Omega, \\ v(t,x) = v_1(x), \quad t = 0,\ x \in \Omega, \end{cases} \tag{13}$$

with

$$F_0(t,x) = \frac{|\partial F|}{|Y^\star|}\gamma[g(v(t,x)) - w_1(x)e^{-(a+\gamma)t} -$$

$$-\gamma r(\cdot) \star g(v(\cdot,x))(t)].$$

In (13), $Q = ((q_{ij}))$ *is the classical homogenized matrix, whose entries are defined by (7)-(8). The constant matrix* Q *is symmetric and positive-definite. Moreover, the limit problem for the surface concentration is:*

$$\begin{cases} \dfrac{\partial w}{\partial t}(t,x) + (a+\gamma)w(t,x) = \\ = \gamma g(v(t,x)), \quad t > 0,\ x \in \Omega, \\ w(t,x) = w_1(x), \quad t = 0,\ x \in \Omega \end{cases} \tag{14}$$

and obviously, w *can be written as*

$$w(t,x) = w_1(x)e^{-(a+\gamma)t} + \gamma r(t) \star g(v(t,x)). \ ▪$$

Remark 2.4. *The weak formulation of problem (13) is:*

$$\begin{cases} \textit{Find } v \in \mathcal{W}_0(\Omega), v(0) = v_1 \textit{ such that} \\ -\left(v, \dfrac{d\varphi}{dt}\right)_{\Omega,T} + (F_0 + \beta(v), \varphi)_{\Omega,T} = \\ = -D(Q\nabla v, \nabla\varphi)_{\Omega,T} + (h,\varphi)_{\Omega,T}, \\ \forall \varphi \in \mathcal{W}_0(\Omega). \ ▪ \end{cases} \tag{15}$$

3 Proof of the Main Result

In order to prove Theorem 2.3, let us first notice that there is at most one solution of the weak problem (15). Secondly, for describing the effective behavior of v^ε and w^ε, as $\varepsilon \to 0$, some a priori estimates on these solutions are required.

Following similar techniques to those used in [4], one can easily prove the following result:

Proposition 3.1. *Let* v^ε *and* w^ε *be the solutions of the problem (11)-(12). There exists a positive constant* C*, independent of* ε*, such that*

$$\|w^\varepsilon(t)\|^2_{S^\varepsilon} \le (\|w^\varepsilon(0)\|^2_{S^\varepsilon} + \frac{\gamma}{\delta}\|g(v^\varepsilon)\|^2_{S^\varepsilon,t})e^{\gamma\delta t},$$

$$\forall t \ge 0,\ \forall \delta > 0,$$

$$\left\|\frac{\partial w^\varepsilon}{\partial t}\right\|^2_{S^\varepsilon,t} \le C(\|w^\varepsilon(0)\|^2_{S^\varepsilon} + \|g(v^\varepsilon)\|^2_{S^\varepsilon,t}), \forall t \ge 0,$$

$$\|v^\varepsilon(t)\|^2_{\Omega^\varepsilon} \le C,\ \|\nabla v^\varepsilon(t)\|^2_{\Omega^\varepsilon,t} \le C,$$

$$\left\|\frac{\partial v^\varepsilon}{\partial t}(t)\right\|^2_{\Omega^\varepsilon} \le C. \ ▪$$

The proof of Theorem 2.3 will be divided into three steps.

First step. Let $v^\varepsilon \in \mathcal{W}_0(\Omega^\varepsilon)$ be the solution of the variational problem (11) and let $P^\varepsilon v^\varepsilon$ be the extension of v^ε inside the holes given by Lemma 2.2. Using our a priori estimates given by Proposition 3.1., we easily can see that there exists a constant C depending on T and the data, but independent of ε such that

$$\|P^\varepsilon v^\varepsilon(t)\|_\Omega + \|\nabla P^\varepsilon v^\varepsilon\|_{\Omega,t} + \|\partial_t(P^\varepsilon v^\varepsilon)(t)\|_\Omega \le C,$$

for all $t \le T$. Consequently, by passing to a subsequence, still denoted by $P^\varepsilon v^\varepsilon$, we can assume that there exists $v \in \mathcal{V}$ such that the following convergence properties hold:

$$P^\varepsilon v^\varepsilon \rightharpoonup v \quad \text{weakly in } \mathcal{V}, \tag{16}$$

$$\partial_t(P^\varepsilon v^\varepsilon) \rightharpoonup \partial_t v \quad \text{weakly in } \mathcal{H}, \tag{17}$$

$$P^\varepsilon v^\varepsilon \to v \quad \text{strongly in } \mathcal{H}. \tag{18}$$

It remains to obtain the equation satisfied by v.

Second step. In order to get the limit equation satisfied by v we have to pass to the limit in (11). For getting the limit of the second term in the left hand side of (11), let us introduce, for any $h \in L^{s'}(\partial T)$, $1 \leq s' \leq \infty$, the linear form μ_h^ε on $W_0^{1,s}(\Omega)$ defined by

$$\langle \mu_h^\varepsilon, \varphi \rangle = \varepsilon \int_{S^\varepsilon} h(\frac{x}{\varepsilon})\varphi d\sigma \quad \forall \varphi \in W_0^{1,s}(\Omega),$$

with $1/s + 1/s' = 1$. It is proved in [2] that

$$\mu_h^\varepsilon \to \mu_h \quad \text{strongly in } (W_0^{1,s}(\Omega))', \tag{19}$$

where

$$\langle \mu_h, \varphi \rangle = \mu_h \int_\Omega \varphi dx, \; \mu_h = \frac{1}{|Y|} \int_{\partial F} h(y) d\sigma.$$

In the particular case in which $h \in L^\infty(\partial F)$ or even h is constant, we have

$$\mu_h^\varepsilon \to \mu_h \quad \text{strongly in } W^{-1,\infty}(\Omega). \tag{20}$$

We shall denote by μ^ε the above introduced measure in the particular case in which $h = 1$.

Moreover, if $z^\varepsilon \in H_0^1(\Omega)$ is such that $z^\varepsilon \rightharpoonup z$ weakly in $H_0^1(\Omega)$, then

$$\langle \mu_h^\varepsilon, z^\varepsilon|_{\Omega^\varepsilon} \rangle \to \mu_h \int_\Omega z dx. \tag{21}$$

Let us prove now that for any $\varphi \in C_0^\infty(\Omega)$ and for any $z^\varepsilon \rightharpoonup z$ weakly in $H_0^1(\Omega)$, we get

$$\varphi g(z^\varepsilon) \rightharpoonup \varphi g(z) \quad \text{weakly in } W_0^{1,\overline{q_1}}(\Omega), \tag{22}$$

where $\overline{q_1} = \dfrac{2n}{q_1(n-2)+n}$.

To prove (22), let us first note that

$$\sup \|\nabla g(z^\varepsilon)\|_{L^{\overline{q_1}}(\Omega)} < \infty. \tag{23}$$

Indeed, from (9) we get

$$\int_\Omega \left| \frac{\partial g}{\partial x_i}(z^\varepsilon) \right|^{\overline{q_1}} dx \leq C \int_\Omega \left(1 + |z^\varepsilon|^{q_1 \overline{q_1}}\right) \left| \frac{\partial z^\varepsilon}{\partial x_i} \right|^{\overline{q_1}} dx$$

$$\leq C(1 + (\int_\Omega |z^\varepsilon|^{q_1 \overline{q_1} \gamma} dx)^{1/\gamma})(\int_\Omega |\nabla z^\varepsilon|^{\overline{q_1} \delta} dx)^{1/\delta},$$

where we took γ and δ such that $\overline{q_1}\delta = 2$, $1/\gamma + 1/\delta = 1$ and $q_1 \overline{q_1} \gamma = 2n/(n-2)$. Notice that, since $0 \leq q_1 < n/(n-2)$, we have $\overline{q_1} > 1$. Since

$$\sup \|z^\varepsilon\|_{L^{\frac{2n}{n-2}}(\Omega)} < \infty,$$

we get immediately (23). Hence, to get (22), it remains only to prove that

$$g(z^\varepsilon) \to g(z) \quad \text{strongly in } L^{\overline{q_1}}(\Omega). \tag{24}$$

But this is just a consequence of the following well-known result (see [4]):

Theorem 3.2. *Let $G : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function, i.e.*

a) for every z the function $G(\cdot, z)$ is measurable with respect to $x \in \Omega$.

b) for every (a.e.) $x \in \Omega$, the function $G(x, \cdot)$ is continuous with respect to z.

Moreover, if we assume that there exists a positive constant C such that

$$|G(x,z)| \leq C\left(1 + |z|^{r/t}\right),$$

with $r \geq 1$ and $t < \infty$, then the map $z \in L^r(\Omega) \mapsto G(x, z(x)) \in L^t(\Omega)$ is continuous in the strong topologies. ▪

Indeed, since

$$|g(z)| \leq C(1 + |z|^{q_1+1}),$$

applying the above theorem for $G(x,z) = g(z)$, $t = \overline{q_1}$ and $r = (2n/(n-2)) - r'$, with $r' > 0$ such that $q_1 + 1 < r/t$ and using the compact injection $H^1(\Omega) \hookrightarrow L^r(\Omega)$ we easily get (24).

Finally, from (20) (with $h = 1$) and (22) written for $z^\varepsilon = P^\varepsilon v^\varepsilon(t)$, we conclude

$$\langle \mu^\varepsilon, \varphi g(P^\varepsilon v^\varepsilon(t)) \rangle \to \frac{|\partial F|}{|Y|} \int_\Omega \varphi g(v(t)) dx, \forall \varphi \in \mathcal{D}.$$

Using Lebesgue's convergence theorem, the above pointwise convergence, the a priori estimates obtained in Proposition 3.1. and the growth condition (9), we get

$$\lim_{\varepsilon \to 0} \varepsilon \gamma(g(v^\varepsilon), \varphi)_{S^\varepsilon, T} = \frac{|\partial F|}{|Y|} \gamma(g(v), \varphi)_{\Omega, T}. \tag{25}$$

In a similar manner, one can easily prove that

$$\beta(z^\varepsilon) \to \beta(z) \quad \text{strongly in } L^{\overline{q_2}}(\Omega),$$

for any $z^\varepsilon \rightharpoonup z$ weakly in $H_0^1(\Omega)$ and, hence,

$$\lim_{\varepsilon \to 0} (\beta(v^\varepsilon), \varphi)_{\Omega^\varepsilon, T} = \frac{|Y^*|}{|Y|} (\beta(v), \varphi)_{\Omega, T}. \tag{26}$$

This ends the second step of the proof.

Third step. Let ξ^ε be the gradient of v^ε in Ω^ε and let us denote by $\widetilde{\xi^\varepsilon}$ its extension with zero to the whole of Ω. Obviously, $\widetilde{\xi^\varepsilon}$ is bounded in $(\mathcal{H}(\Omega))^n$ and hence there exists $\xi \in (\mathcal{H}(\Omega))^n$ such that

$$\widetilde{\xi^\varepsilon} \rightharpoonup \xi \quad \text{weakly in } (\mathcal{H}(\Omega))^n. \tag{27}$$

Let us see which is the equation satisfied by ξ. Take $\varphi \in \mathcal{D}$. From (11) we get

$$-\left(\chi_{\Omega^\varepsilon} P^\varepsilon v^\varepsilon, \frac{d\varphi}{dt}\right)_{\Omega,T} + D(\widetilde{\xi^\varepsilon}, \nabla\varphi)_{\Omega,T} +$$

$$+\varepsilon(f^\varepsilon, \varphi)_{S^\varepsilon,T} + (\beta(v^\varepsilon), \varphi)_{\Omega^\varepsilon,T} = (\chi_{\Omega^\varepsilon} h, \varphi)_{\Omega,T}.$$

Now, we can pass to the limit, with $\varepsilon \to 0$, in all the terms of the above equation. We have:

$$-\frac{|Y^\star|}{|Y|}\left(v, \frac{d\varphi}{dt}\right)_{\Omega,T} + D(\xi, \nabla\varphi)_{\Omega,T} +$$

$$+\frac{|Y^*|}{|Y|}(\beta(v), \varphi)_{\Omega,T} + \frac{|Y^\star|}{|Y|}(F_0, \varphi)_{\Omega,T} =$$

$$= \frac{|Y^*|}{|Y|}(h, \varphi)_{\Omega,T} \quad \forall\varphi \in \mathcal{D}(\Omega).$$

Hence ξ verifies

$$\frac{|Y^*|}{|Y|}\frac{\partial v}{\partial t} - D\operatorname{div}\xi + \frac{|Y^*|}{|Y|}F_0 + \frac{|Y^*|}{|Y|}\beta(v) =$$

$$= \frac{|Y^*|}{|Y|}h, \ t > 0, \ x \in \Omega. \tag{28}$$

It remains now to identify ξ. Following a standard procedure and using (7)-(8) (see, for instance, [2] and [4]), we get

$$D\frac{|Y^*|}{|Y|}\sum_{i,j=1}^{n} q_{ij}\frac{\partial^2 v}{\partial x_i \partial x_j} = D\operatorname{div}\xi = \frac{|Y^*|}{|Y|}\frac{\partial v}{\partial t} +$$

$$+\frac{|Y^*|}{|Y|}F_0 + \frac{|Y^*|}{|Y|}\beta(v) - \frac{|Y^*|}{|Y|}h,$$

which means that v satisfies, for $t > 0$, $x \in \Omega$,

$$\frac{\partial v}{\partial t} - D\sum_{i,j=1}^{n} q_{ij}\frac{\partial^2 v}{\partial x_i \partial x_j} + F_0(t,x) + \beta(v) = h.$$

Since $v \in \mathcal{W}_0(\Omega)$ (i.e. $v = 0$ on $\partial\Omega$) and v is uniquely determined, the whole sequence $P^\varepsilon v^\varepsilon$ converges to v and Theorem 2.3 is proved. ▪

4 Conclusions

The general question which made the object of this paper was the effective behavior of some nonlinear chemical reactive flows involving diffusion, different types of adsorption rates and chemical reactions which take place in a periodically perforated material. Using a variational method we were able to prove that the influence of the adsorption and chemical reactions taking place inside the fluid region and on the boundaries of the perforations is reflected by the appearance of zero-order extra-terms in the equations governing the homogenized problem.

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