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A reformulation of the Ramsey model

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***Abstract*— In this paper we modify the Ramsey model by introducing the generalized logistic equation that describes more accurately population growth. We develop the model and study the stability properties, contrasting its long run equilibrium with the steady state of the traditional model.**

***Keywords*: Ramsey model; population growth; Richards law.**

I. INTRODUCTION

The main purpose of this short paper is to improve the classical Ramsey model of optimal economic growth ([2]) by modifying its population growth law. One of the fundamental characteristics of this model is the assumption that labour force L grows at a constant rate $r > 0$. If this is the case, for any initial level L_0, at time t the level of the labour force is $L(t) = L_0 e^{rt}$ and then labour force approaches infinity when t goes to infinity, which is clearly unrealistic. The exponential model does not accommodate growth reductions due to competition for environmental resources such as food and habitat. Verhulst [4] considered that a stable population would have a saturation level characteristic; this limit for the population size is usually called the *carrying capacity* of the environment (in this paper denoted by L_∞) and forms a numerical upper bound on the growth size. It is a very well known stylized fact that since the 1950s, population growth rate is decreasing and it is projected to decrease to 0 during the next six decades. This decrease is predominantly due to the aging of the population and, consequently, a dramatic increase in the number of deaths.

Then, as described by Maynard Smith [1], a more realistic law of growth of the labour force $L(t)$ must verify the following properties:

1) when population is small enough in proportion to environmental carrying capacity L_∞, then L grows at a constant rate $r > 0$.
2) when population is large enough in proportion to environmental carrying capacity L_∞, the economic resources become more scarce and population growth rate start decreasing to 0.

In this paper we assume that labour force $L(t)$ verify all these properties. In particular, in section 2 we introduce the Richards law that is a very general equation frequently used to describe and analyze population processes. Under this population growth law in section 3 we obtain a generalization of the Ramsey model. In section 4 we find the steady state and analyze the stability of the model. Finally, in section 5 we present some concluding remarks.

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II. THE RICHARDS POPULATION GROWTH LAW

To incorporate the carrying capacity L_∞ on the growth size, Verhulst (1838) introduced the logistic growth equation as an extension of the exponential model augmented by a multiplicative factor, $1 - \dfrac{L}{L_\infty}$, which represents the fractional deficiency of the current size from the saturation level L_∞:

$$\dot{L} = rL\left(1 - \frac{L}{L_\infty}\right) \tag{1}$$

The equation depends on two parameters: the upper asymptote L_∞ and the rate parameter r. The rate parameter determines the rate at which growth initially accelerates and the upper asymptote determines the long run population level. The curve is sigmoidal with inflection point at half the value of the upper asymptote, $\frac{L_\infty}{2}$. When we want to fit the equation to observations, this places an undesirable restriction on the shape of the curve and clearly limits the generality of the model. To avoid this constraint, many have chosen to use models with additional parameters that generalize the logistic equation. The Richards equation continues to be the most popular of these generalizotions since it was first proposed by Richards [3] to extend the Verhulst logistic growth equation to fit empirical data, but motivated by theoretical arguments. Richards law is the solution of the initial value problem:

$$\begin{cases} \dot{L} = rL\left(1 - \left(\frac{L}{L_\infty}\right)^\alpha\right) \\ L(0) = L_0 \end{cases} \tag{2}$$

where $r > 0$ is the intrinsic growth rate per capita, L_∞ is the carrying capacity, and α is a positive real number. The Richards equation has been popular for several reasons. It has an additional parameter α, which is a shape parameter that can make the equation equivalent to the logistic ($\alpha = 1$) and to other well known models like Grompertz and von Bertalanfy equations. Varying the shape parameter allows the point of inflection of the curve to be at any value between 0 and L_∞ and this make it more flexible to be fitted to data. The Richards equation always gave closer fits and more accurate estimates of the characteristics of other original sigmoid curves.

Richards equation is a separable differential equation and has solution

$$L(t) = \frac{L_\infty}{\left[1 + e^{d - \alpha r t}\right]^{\frac{1}{\alpha}}} \tag{3}$$

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where $d = \ln\left[\left(\frac{L_\infty}{L_0}\right)^\alpha - 1\right]$ is a parameter that indirectly defines the value of t at which $L = \frac{L_\infty}{2}$.

The three main properties of the Richards logistic growth are:

1) $\lim_{t\to+\infty} L(t) = L_\infty$, the population tends to its carrying capacity.
2) The relative growth rate is:
$$n(t) = \frac{\dot{L}(t)}{L(t)} = r\frac{L_0 L_\infty (L_\infty^\alpha - L_0^\alpha)e^{\alpha r t}}{\left[L_0^\alpha e^{\alpha r t} + \left(L_\infty^\alpha - L_0^\alpha\right)\right]^{\frac{2}{\alpha}}} \tag{4}$$
When t is small, $n(t)$ is close to r and decreases monotonically to 0 as t tends to infinity.
3) The growth curve is sigmoidal and the inflection point is at the proportion $(1+\alpha)^{-\frac{1}{\alpha}}$ of the final size

III. The modified Ramsey model

There are three key elements to the model:

- The production function, i.e. how the inputs of capital K and labour L are transformed into output Y
- How the labour and capital change over time
- The utility function, i.e., what are the preferences of agents for consumption C

As usual, we shall assume that:

1) the production function $Y = F(K, L)$ satisfy the following conditions:
   a) $F(\lambda K, \lambda L) = \lambda F(K, L)$, $\forall \lambda, K, L \in R^+$
   b) $F(K, 0) = F(0, L) = 0$, $\forall K, L \in R^+$
   c) $\frac{\partial F}{\partial K} > 0$, $\frac{\partial F}{\partial L} > 0$, $\frac{\partial^2 F}{\partial K^2} < 0$, $\frac{\partial^2 F}{\partial L^2} < 0$
   d) $\lim_{K\to 0} \frac{\partial F}{\partial K} = \lim_{L\to 0} \frac{\partial F}{\partial L} = +\infty$; $\lim_{K\to+\infty} \frac{\partial F}{\partial K} = \lim_{L\to+\infty} \frac{\partial F}{\partial L} = 0$
2) the utility $u(c)$ of consumption verifies that $u'(c) > 0$ and $u''(c) < 0$
3) output is consumed or invested
$$Y(t) = C(t) + I(t) \tag{5}$$
and capital stock changes $\dot{K}(t)$ equal the gross investment $I(t)$ minus the capital depreciation $\delta K(t)$:
$$\dot{K}(t) = I(t) - \delta K(t). \tag{6}$$
4) the labour force $L(t)$ follows the Richards law:
$$\begin{cases} \dot{L} = rL\left(1 - \left(\frac{L}{L_\infty}\right)^\alpha\right) \\ L(0) = L_0 \end{cases} \tag{7}$$
The last assumption is the unique difference with the original Ramsey model. If $k = \frac{K}{L}$, $y = \frac{Y}{L}$ and $c = \frac{C}{L}$ are income, capital and consumption per worker respectively, then $f(k) = F\left(\frac{K}{L}, 1\right) = F(k, 1)$ is the production function in intensive form and we have that:

- $f(0) = 0$;
- $f'(k) > 0, \forall k \in R^+$
- $\lim_{k\to+\infty} f'(k) = 0$
- $\lim_{k\to 0^+} f'(k) = +\infty$
- $f''(k) < 0, \forall k \in R^+$

From (5) and (6) we obtain
$$\frac{Y(t)}{L(t)} = \frac{C(t)}{L(t)} + \frac{\dot{K}(t)}{L(t)} + \frac{\delta K(t)}{L(t)} \tag{8}$$
i.e.
$$y(t) = c(t) + \frac{\dot{K}(t)}{L(t)} + \delta k(t) \tag{9}$$
But
$$\dot{k} = \frac{L\dot{K} - K\dot{L}}{L^2} = \frac{\dot{K}}{L} - k\frac{\dot{L}}{L} = \frac{\dot{K}}{L} - kr\left(1 - \left(\frac{L}{L_\infty}\right)^\alpha\right) \tag{10}$$
and then
$$y = c + \dot{k} + \left(r\left(1 - \left(\frac{L}{L_\infty}\right)^\alpha\right) + \delta\right)k \tag{11}$$
We therefore have the condition
$$\dot{k} = f(k) - c - \left[r\left(1 - \left(\frac{L}{L_\infty}\right)^\alpha\right) + \delta\right]k \tag{12}$$
Then the per capita problem of optimal growth is to maximize the discounted value of utility subject to equations (7) and (12); i.e.
$$\begin{cases} \max_c \int_0^\infty u(c)e^{-\rho t}dt \\ \dot{k} = f(k) - c - \left[r\left(1 - \left(\frac{L}{L_\infty}\right)^\alpha\right) + \delta\right]k \\ \dot{L} = rL\left(1 - \left(\frac{L}{L_\infty}\right)^\alpha\right) \\ k(0) = k_0 > 0,\ L(0) = L_0 > 0 \\ 0 \le c \le f(k). \end{cases} \tag{13}$$
The current value Hamiltonian of the problem is
$$\mathcal{H}_c = u(c) + \lambda\left[f(k) - c - \left[r\left(1 - \left(\frac{L}{L_\infty}\right)^\alpha\right) + \delta\right]k\right] + \mu rL\left(1 - \left(\frac{L}{L_\infty}\right)^\alpha\right) \tag{14}$$
with first order conditions
$$\begin{cases} \frac{\partial H}{\partial c} = u'(c) - \lambda = 0 \\ \dot{\lambda} = -\lambda\left\{f'(k) - \left[r\left(1 - \left(\frac{L}{L_\infty}\right)^\alpha\right) + \delta + \rho\right]\right\} \\ \dot{k} = f(k) - c - \left[r\left(1 - \left(\frac{L}{L_\infty}\right)^\alpha\right) + \delta\right]k \\ \dot{L} = rL\left(1 - \left(\frac{L}{L_\infty}\right)^\alpha\right) \end{cases} \tag{15}$$
If we differentiate the condition $u'(c) = \lambda$ with respect to t we have that
$$u''(c)\dot{c} = \dot{\lambda} \tag{16}$$
and then it is
$$u''(c)\dot{c} = -\lambda\left\{f'(k) - \left[r\left(1 - \left(\frac{L}{L_\infty}\right)^\alpha\right) + \delta + \rho\right]\right\} \tag{17}$$
and using that $u'(c) = \lambda$, this condition can be expressed as:
$$-\frac{u''(c)}{u'(c)}\dot{c} = f'(k) - \left[r\left(1 - \left(\frac{L}{L_\infty}\right)^\alpha\right) + \delta + \rho\right] \tag{18}$$
But $\sigma(c) = -\frac{cu''(c)}{u'(c)}$ is the measure of relative risk aversion and then it is
$$\dot{c} = \frac{\left\{f'(k) - \left[r\left(1 - \left(\frac{L}{L_\infty}\right)^\alpha\right) + \delta + \rho\right]\right\}c}{\sigma(c)} \tag{19}$$

From this and (15), we obtain the equations of motion for the modified Ramsey model which describes how consumption and capital per worker varies over time:

$$\begin{cases} \dot{c} = \frac{c}{\sigma(c)} \left\{ f'(k) - \left[r \left(1 - \left(\frac{L}{L_\infty} \right)^\alpha \right) + \delta + \rho \right] \right\} \\ \dot{k} = f(k) - c - \left[r \left(1 - \left(\frac{L}{L_\infty} \right)^\alpha \right) + \delta \right] k \\ \dot{L} = rL \left(1 - \left(\frac{L}{L_\infty} \right)^\alpha \right) \end{cases} \tag{20}$$

IV. Equilibria and stability

The equilibria of the model are the solution of the system

$$\begin{cases} \frac{c}{\sigma(c)} \left\{ f'(k) - \left[r \left(1 - \left(\frac{L}{L_\infty} \right)^\alpha \right) + \delta + \rho \right] \right\} = 0 \\ f(k) - c - \left[r \left(1 - \left(\frac{L}{L_\infty} \right)^\alpha \right) + \delta \right] k = 0 \\ rL \left(1 - \left(\frac{L}{L_\infty} \right)^\alpha \right) = 0 \end{cases} \tag{21}$$

If we exclude the trivial solutions obtained by considering $c = 0, k = 0$ or $L = 0$, then the model has a unique positive equilibrium $\left(\hat{c}, \hat{k}, \hat{L}\right)$ verifying:

$$\begin{cases} \hat{L} = L_\infty \\ f'(\hat{k}) = \delta + \rho \\ \hat{c} = f(\hat{k}) - \delta \hat{k} \end{cases} \tag{22}$$

Note that the parameters of the population law do not enter in the formula of $\hat{c}$ and $\hat{k}$. The steady state values of consumption and capital per worker depend only on the parameters of technology and on the discount factors δ and ρ. In particular we note that the equilibrium values $\hat{c}$ and $\hat{k}$ are not affected by the intrinsic rate of population growth $n(t)$ this is an important difference with the classical Ramsey model, where an increase in the intrinsic rate of population growth leads to lower levels of these variables. Note also that when we substitute exponential growth of population by the Richards law, the equilibrium values are improved.

To study the stability of the equilibrium $\left(\hat{c}, \hat{k}, \hat{L}\right)$ we consider the linear approximation of the system around this point. The Jacobian matrix of the linear approximation is given by

$$J_G \left(\hat{c}, \hat{k}, \hat{L}\right) = \begin{pmatrix} 0 & \dfrac{f''(\hat{k})\hat{c}}{\sigma(\hat{c})} & \dfrac{\alpha r \hat{c}}{\sigma(\hat{c})} \\ -1 & \rho & \alpha r \hat{k} \\ 0 & 0 & -\alpha r \end{pmatrix} \tag{23}$$

Then the characteristic polynomial of this matrix is

$$-\left(X^2 - \rho X + \frac{f''(\hat{k})\hat{c}}{\sigma(\hat{c})} \right) (X + \alpha r) \tag{24}$$

and, being $\delta > 0$ and $\dfrac{f''(\hat{k})\hat{c}}{\sigma(\hat{c})} < 0$, this polynomial has two negative roots and one positive root. Then the steady state $\left(\hat{c}, \hat{k}, \hat{L}\right)$ is a saddle point and the stable transitional path is a two dimensional locus.

V. Concluding remarks.

In this paper we have developed an improved version of the Ramsey growth model suggesting a more realistic approach by considering that population growth is strictly increasing and bounded and that its rate of growth is strictly decreasing to zero. In particular we use the Richards equation to model population growth. Dynamics of the model is described by a system of three differential equations that have a unique non trivial steady state. We show that this steady state is unstable. In particular, we linearize the system and we show that the equilibrium is a saddle point with two dimensional stable transitional path.

The paper shows that when population verifies the more realistic conditions, the intrinsic rate of population growth $n(t)$ plays no role in determining the long run equilibrium levels of per capita consumption, capital and output, while with exponential population growth an increase in the intrinsic rate of population growth leads to lower levels of these variables. It also shows that equilibrium per capita levels of consumption, capital and output are greater than those of the classical model. Thus, in the long run, economic growth is improved if labour force growth rate decreases. This is a motivation for policy makers to have an efficient population growth rate.

In the equilibrium of the classical Ramsey model, aggregate capital and consumption tends unrealistically to infinity as t tends to infinity, because population grows to infinity. This situation is improved in our model, where in equilibrium, aggregate capital and consumption tends to the finite values $\hat{L}\hat{c}$ and $\hat{L}\hat{k}$. Both equilibrium are saddle paths and then it will by a central planner that has to indicate the saddle path conducing the economy to the equilibrium.

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