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On eigenvalues and eigenvectors of subdirect sums

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Abstract: Some new properties of the eigenvalues of the subdirect sums are presented for the particular case of 1-subdirect sums. In particular, it is shown that if an eigenvalue λ is associated with certain blocks of matrix A or matrix B then λ is also an eigenvalue associated with the 1-subdirect sum $A \oplus_1 B$. Some results concerning eigenvectors of the k-subdirect sum $A \oplus_k B$ for an arbitrary positive integer k are also given.

Key–Words: Matrices, subdirect sum, eigenvalues, eigenvectors, overlapping blocks, domain decomposition.

1 Introduction

The concept of k-subdirect sum of matrices, introduced in [4], appears naturally in several contexts related to square matrices: e.g., matrix completion problems, analysis of matrix classes, overlapping subdomains in domain decomposition methods, global stiffness matrix in FEM, etc; see [4], [2], [1] and the references therein.

The characterization of the invertibility of the subdirect sum of two nonsingular matrices have been studied recently [1]. In the particular case of a 1-subdirect sum it is known that the 1-subdirect sum of two singular matrices is a singular matrix [4]. We present some new properties of the eigenvalues of 1-subdirect sums, which has relevance in some instances; see [4] and the references therein for details. We also give some results concerning eigenvectors of the subdirect sums.

2 Subdirect sums

Let A and B be two square matrices of order n_1 and n_2, respectively, and let k be an integer such that $1 \leq k \leq \min(n_1, n_2)$. Let A and B be partitioned into 2×2 blocks as follows,

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}, \quad (1)$$

where A_{22} and B_{11} are square matrices of order k. Following [4], we call the following square matrix of order $n = n_1 + n_2 - k$,

$$C = \begin{bmatrix} A_{11} & A_{12} & O \\ A_{21} & A_{22} + B_{11} & B_{12} \\ O & B_{21} & B_{22} \end{bmatrix} \quad (2)$$

the k-subdirect sum of A and B and denote it by $C = A \oplus_k B$.

2.1 Eigenvalues of 1-subdirect sums

The particular case of $k = 1$, i.e., a 1-subdirect sum, is important in certain applications (see [4] and the references therein). We denote as $\sigma(A)$ the spectrum (i.e., the set of eigenvalues) of a matrix A. Some first properties of the eigenvalues of 1-subdirect sums can be summarized as follows.

Theorem 1 *Let A and B be matrices of order n_1 and n_2, respectively, partitioned as in* (1) *and let $k = 1$. Let $C = A \oplus_1 B$. Then any of the following statements*

i) $\lambda \in \sigma(A) \cap \sigma(A_{11})$

ii) $\lambda \in \sigma(A_{11}) \cap \sigma(B_{22})$

iii) $\lambda \in \sigma(B) \cap \sigma(B_{22})$

implies that $\lambda \in \sigma(C)$.

Proof. Let us denote $a_{22} = A_{22}$ and $b_{11} = B_{11}$ to display that this quantities are matrices of order 1×1. A direct computation shows that

$$det(C \quad -\lambda I) = \\ = \begin{bmatrix} A_{11} - \lambda I & A_{12} & O \\ A_{21} & a_{22} + b_{11} - \lambda & B_{12} \\ O & B_{21} & B_{22} - \lambda I \end{bmatrix} \quad (3)$$

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and expanding the n_1th-row as a sum of the rows $(A_{21}, a_{22}-\lambda, 0)$ and $(0, b_{11}, B_{12})$ we have

$$det(C-\lambda I) =$$

$$det\begin{bmatrix} A_{11}-\lambda I & A_{12} & O \\ A_{21} & a_{22}-\lambda & 0 \\ O & B_{21} & B_{22}-\lambda I \end{bmatrix} +$$

$$det\begin{bmatrix} A_{11}-\lambda I & A_{12} & O \\ 0 & b_{11} & B_{12} \\ O & B_{21} & B_{22}-\lambda I \end{bmatrix},$$

which leads to

$$det(C-\lambda I) =$$

$$det(A-\lambda I)det(B_{22}-\lambda I)+$$

$$det(A_{11}-\lambda I)det\begin{bmatrix} b_{11} & B_{12} \\ B_{21} & B_{22}-\lambda I \end{bmatrix}, \tag{4}$$

and from this expression we have that each of the statements (i) and (ii) implies $\lambda \in \sigma(C)$. To prove (iii) the technique is the same but now we expand the n_1th-row of equation (3) as a sum of the rows $(A_{21}, a_{22}, 0)$ and $(0, b_{11}-\lambda, B_{12})$ to finally get

$$det(C-\lambda I) =$$

$$det\begin{bmatrix} A_{11}-\lambda I & A_{12} \\ A_{21} & a_{22} \end{bmatrix} det(B_{22}-\lambda I)+$$

$$det(A_{11}-\lambda I)det(B-\lambda I), \tag{5}$$

from which we conclude that statement (iii) implies $\lambda \in \sigma(C)$. □

Example 2 *Given the matrices*

$$A_{11} = \begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix}, \; A = \left[\begin{array}{ccc|c} & & & 5 \\ & A_{11} & & 4 \\ & & & 3 \\ \hline 1 & 0 & 1 & 5 \end{array}\right]$$

$$B = \left[\begin{array}{c|ccc} 3 & 1 & 0 & 1 \\ \hline 2 & 6 & 3 & 0 \\ 1 & -2 & 5 & -1 \\ 1 & -1 & 1 & 4 \end{array}\right]$$

the spectra of A and A_{11} are

$$\sigma(A) = \{-2, 1, 1, 7\}, \quad \sigma(A_{11}) = \{-2, 1, 3\}$$

and we obtain that the 1-subdirect sum

$$C = A \oplus_1 B = \left[\begin{array}{ccc|c|ccc} 1 & -1 & 4 & 5 & 0 & 0 & 0 \\ 3 & 2 & -1 & 4 & 0 & 0 & 0 \\ 2 & 1 & -1 & 3 & 0 & 0 & 0 \\ \hline 1 & 0 & 1 & 8 & 1 & 0 & 1 \\ \hline 0 & 0 & 0 & 2 & 6 & 3 & 0 \\ 0 & 0 & 0 & 1 & -2 & 5 & -1 \\ 0 & 0 & 0 & 1 & -1 & 1 & 4 \end{array}\right]$$

has the eigenvalues

$$\sigma(C) \approx \{9.8, \; -2, \; 1, \; 1.7\,, 4.9 \pm 2.4i, 4.7\}$$

and according to theorem 1, since the eigenvalues -2 and 1 are common to A and A_{11} they are also eigenvalues of C.

Example 3 *Given the matrix A of example 2 and matrix*

$$B = \left[\begin{array}{c|ccc} 4 & 1 & 2 & -1 \\ \hline 2 & & & \\ -2 & & B_{22} & \\ 1 & & & \end{array}\right]$$

with $B_{22} = \begin{bmatrix} 3 & 5 & -5 \\ -1 & -3 & 7 \\ -1 & -1 & 5 \end{bmatrix}$, *the spectra of B and B_{22} are $\sigma(B) \approx \{-1.5, 4.8, 2.8 \pm 1.0i\}$, $\sigma(B_{22}) = \{-2, 3, 4\}$, and we obtain that the 1-subdirect sum*

$$C = A \oplus_1 B = \left[\begin{array}{ccc|c|ccc} 1 & -1 & 4 & 5 & 0 & 0 & 0 \\ 3 & 2 & -1 & 4 & 0 & 0 & 0 \\ 2 & 1 & -1 & 3 & 0 & 0 & 0 \\ \hline 1 & 0 & 1 & 9 & 1 & 2 & -1 \\ \hline 0 & 0 & 0 & 2 & 3 & 5 & -5 \\ 0 & 0 & 0 & -2 & -1 & -3 & 7 \\ 0 & 0 & 0 & 1 & -1 & -1 & 5 \end{array}\right]$$

has the eigenvalues

$$\sigma(C) \approx \{9.9, -2, -1.7, 3.8, 1, 1, 3\},$$

and according to theorem 1, since the eigenvalues -2 and 3 are common to A_{11} and B_{22} they are also eigenvalues of C.

It is easy to find examples such that $\lambda \in \sigma(A) \cap \sigma(B)$ but $\lambda \notin \sigma(A \oplus_1 B)$. Indeed, although $\sigma(A) = \sigma(B)$ we can not ensure that $A \oplus_1 B$ shares eigenvalues with A and B, as we show in the next example.

Example 4 *Given the matrices*

$$A = \left[\begin{array}{cc|c} 2 & 1 & 1 \\ -8 & 3 & 9 \\ \hline 8 & 1 & -5 \end{array}\right], \quad B = \left[\begin{array}{c|cc} -3 & -13 & 10 \\ \hline -5 & -11 & 10 \\ -6 & -14 & 14 \end{array}\right]$$

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with the same spectrum, $\sigma(A) = \sigma(B) = \{2, 4, -6\}$, we have, notwithstanding, that the 1-subdirect sum

$$C = A \oplus_1 B = \left[\begin{array}{rr|r|rr} 2 & 1 & 1 & 0 & 0 \\ -8 & 3 & 9 & 0 & 0 \\ \hline 8 & 1 & -8 & -13 & 10 \\ \hline 0 & 0 & -5 & -11 & 10 \\ 0 & 0 & -6 & -14 & 14 \end{array}\right]$$

has not even one eigenvalue in common with A and B, since $\sigma(C) \approx \{-10.1, 0, 0.8, 4.6 \pm 1.3i\}$.

There is a particular case in which $\lambda \in \sigma(A) \cap \sigma(B)$ implies $\lambda \in \sigma(A \oplus_1 B)$. This happens when $\lambda = 0$ as we state in the following result.

Corollary 5 *Let A and B be matrices of order n_1 and n_2, respectively, partitioned as in* (1) *and let $k = 1$. If A and B are singular matrices then $C = A \oplus_1 B$ is also a singular matrix.*

Proof. From equation (4), or (5), making $\lambda = 0$ we have

$$det(C) = det(A_{11})det(B) + det(A)det(B_{22}) \quad (6)$$

and the proof follows. □

Remark. Expression (6) was already known in [4] but we have just obtained it as a particular case of the more general expressions (4) or (5).

The following examples show that theorem 1 does not hold when $k > 1$.

Example 6 *Given the matrices*

$$A = \left[\begin{array}{rrr|rr} & & & 1 & 5 \\ & A_{11} & & 1 & 4 \\ & & & -1 & 3 \\ \hline 1 & 2 & 3 & 9 & 2 \\ 2 & -1 & 1 & 2 & 1 \end{array}\right]$$

with $A_{11} = \begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix}$, *and*

$$B = \left[\begin{array}{rr|rr} 3 & 1 & 0 & 1 \\ 2 & 6 & 3 & 0 \\ \hline 1 & -2 & 5 & -1 \\ 1 & -1 & 1 & 4 \end{array}\right],$$

we have $\sigma(A) \approx \{10.4, 3, -2, -0.4, 1\}$ and $\sigma(A_{11}) = \{-2, 1, 3\}$, and therefore

$$\sigma(A_{11}) \cap \sigma(A) = \sigma(A_{11}) = \{1, -2, 3\},$$

but we obtain that the 2-subdirect sum

$$C = A \oplus_2 B = \left[\begin{array}{rrr|rr|rr} 1 & -1 & 4 & 1 & 5 & 0 & 0 \\ 3 & 2 & -1 & 1 & 4 & 0 & 0 \\ 2 & 1 & -1 & -1 & 3 & 0 & 0 \\ \hline 1 & 2 & 3 & 12 & 3 & 0 & 1 \\ 2 & -1 & 1 & 4 & 7 & 3 & 0 \\ \hline 0 & 0 & 0 & 1 & -2 & 5 & -1 \\ 0 & 0 & 0 & 1 & -1 & 1 & 4 \end{array}\right]$$

has the eigenvalues

$$\sigma(C) \approx \{14.9,\ -1.4,\ -0.3,\ 4.6 \pm 2.9i,\ 2.9,\ 4.7\},$$

and therefore we have $\lambda \in \sigma(A_{11}) \cap \sigma(A)$ but $\lambda \notin \sigma(A \oplus_2 B)$ and therefore part (i) of theorem 1 does not hold for $k > 1$.

Example 7 *Given the matrices*

$$A = \left[\begin{array}{rrr|rr} & & & 1 & -1 \\ & A_{11} & & 3 & 2 \\ & & & 1 & 2 \\ \hline 1 & 3 & 1 & 2 & 3 \\ -1 & 2 & 2 & 1 & 1 \end{array}\right]$$

and

$$B = \left[\begin{array}{rr|rrr} 1 & -1 & 2 & -3 & 2 \\ -1 & 1 & 1 & 1 & -1 \\ \hline 2 & 1 & & & \\ -3 & 1 & & A_{11} & \\ 2 & -1 & & & \end{array}\right],$$

with A_{11} given in example 6, we have $\sigma(A_{11}) \cap \sigma(B_{22}) = \sigma(A_{11})$, and a computation gives $\sigma(A_{11}) \cap \sigma(A \oplus_2 B) = \emptyset$. Then we conclude that statement (ii) of theorem 1 does not hold for $k > 1$.

Example 8 *Let A be the matrix given in example 6 and let $B = \begin{bmatrix} A_{22}^T & A_{12}^T \\ A_{21}^T & A_{11}^T \end{bmatrix}$. A computation shows that $\sigma(B_{22}) \cap \sigma(B) = \sigma(B_{22}) = \sigma(A_{11}^T)$, and $\sigma(A_{11}) \cap \sigma(A \oplus_2 B) = \emptyset$. Then we conclude that statement (iii) of theorem 1 does not hold for $k > 1$.*

We have seen that theorem 1 allows to obtain some of the eigenvalues of the 1-subdirect sum when certain conditions of the eigenvalues of A, B and some submatrices are satisfied. The following result, which relates the eigenvalues of a matrix and a submatrix shall be useful to further explode theorem 1.

Theorem 9 *Let A be an square matrix of order n and let $\lambda \in \sigma(A)$. Let A_m be a principal submatrix of A of order m. Let $g \geq 1$ be a positive integer. If the geometric multiplicity of λ is at least g and $m > n - g$ then it holds that $\lambda \in \sigma(A_m)$.*

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Proof. See [5], p. 60.

Combining theorems 1 and 9 we can state the following interesting result.

Theorem 10 *Let A and B be matrices of order n_1 and n_2, respectively, partitioned as in* (1) *and let $k = 1$. If $\lambda \in \sigma(A) \cup \sigma(B)$ and the geometric multiplicity of λ is at least* 2 *(with reference to any of both matrices) then λ is an eigenvalue of the* 1*-subdirect sum $C = A \oplus_1 B$.*

Proof. From theorem 1 and theorem 9. In detail: If $\lambda \in \sigma(A)$ and its geometric multiplicity is greater or equal to 2 then, by theorem 9 applied to principal submatrix A_{11} of order $m = n_1 - k = n_1 - 1$ we have that $m > n_1 - g$ since $m = n_1 - 1 > n_1 - 2$. Therefore we conclude that λ is also an eigenvalue of A_{11}. From theorem 1 we conclude that λ is also an eigenvalue of the 1-subdirect sum C. If $\lambda \in \sigma(B)$, from theorem 9 we conclude that λ is also an eigenvalue of B_{22} and from theorem 1 we conclude that λ is also an eigenvalue of the 1-subdirect sum C. □

Example 11 *Given the matrices*

$$A = \left[\begin{array}{rr|r} 2 & -1 & 3 \\ -2 & 3 & 3 \\ \hline 2 & 1 & 1 \end{array}\right], \quad B = \left[\begin{array}{r|rr} 1 & 2 & 1 \\ \hline -1 & -3 & 1 \\ 1 & 3 & 3 \end{array}\right]$$

their spectra are given by $\sigma(A) = \{-2, 4, 4\}$ and

$$\sigma(B) \approx \{-3.0,\ 0.4,\ 3.7\}$$

and the eigenvalue $\lambda = 4$ of A has geometric multiplicity equal to 2 *with the associated eigenspace spanned by eigenvectors $[-1, 2, 0]^T$ and $[3, 0, 2]^T$. Therefore, according to theorem 10, the* 1*-subdirect sum*

$$C = A \oplus_1 B = \left[\begin{array}{rr|r|rr} 2 & -1 & 3 & 0 & 0 \\ -2 & 3 & 3 & 0 & 0 \\ \hline 2 & 1 & 2 & 2 & 1 \\ \hline 0 & 0 & -1 & -3 & 1 \\ 0 & 0 & 1 & 3 & 3 \end{array}\right]$$

has $\lambda = 4$ as an eigenvalue. In fact we obtain:

$$\sigma(C) \approx \{-2.7,\ -2.2,\ 4.7,\ 4,\ 3.2\}$$

In certain cases, imposing more conditions to the eigenvalues and eigenvectors we can obtain more results concerning the eigenvalues and eigenvectors of the k-subdirect sums for an arbitrary k. We do this in the next section.

3 Eigenvectors of subdirect sums

In this section we analyze some properties of the eigenvalues and eigenvectors of the k-subdirect sums. We first consider the case when $\lambda = 0$ is an eigenvalue of A or B and then we impose certain conditions to their eigenvectors. After that we extend to the case of general λ.

The following result shows how to obtain an eigenvector of the k-sub-direct sum assuming certain conditions for two blocks of matrix A.

Lemma 12 *Let A and B be matrices of order n_1 and n_2, respectively, partitioned as in* (1)*, with $1 \le k \le \min(n_1, n_2)$. Let $x_1 \in \mathbf{C}^{(n_1-k)\times 1}$ be a nonzero vector such that $x_1 \in ker A_{11}$ and $x_1 \in ker A_{21}$. Then it holds that vector $x = \begin{bmatrix} x_1 \\ O_{n_2\times 1} \end{bmatrix}$ is an eigenvector of the k-subdirect sum $C = A \oplus_k B$ corresponding to the eigenvalue $\lambda = 0$.*

Proof. We only have to show that $Cx = 0$. A direct computation gives

$$Cx = \begin{bmatrix} A_{11} & A_{12} & O \\ A_{21} & A_{22} + B_{11} & B_{12} \\ O & B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ O_{k\times 1} \\ O_{(n_2-k)\times 1} \end{bmatrix} =$$

$$= \begin{bmatrix} A_{11}x_1 \\ A_{21}x_1 \\ O_{(n_2-k)\times 1} \end{bmatrix},$$

and since $x_1 \in ker A_{11}$ and $x_1 \in ker A_{21}$ we get

$$Cx = \begin{bmatrix} O_{(n_1-k)\times 1} \\ O_{n_2\times 1} \end{bmatrix} = 0x. \quad \square$$

We have an analogous result when focusing on matrix B.

Lemma 13 *Let A and B be a matrices of order n_1 and n_2, respectively, partitioned as in* (1)*, with $1 \le k \le \min(n_1, n_2)$. Let $x_2 \in \mathbf{C}^{(n_2-k)\times 1}$ be a nonzero vector such that $x_2 \in ker B_{11}$ and $x_2 \in ker B_{12}$. Then it holds that vector $x = \begin{bmatrix} O_{n_1\times 1} \\ x_2 \end{bmatrix}$ is an eigenvector the k-subdirect sum $C = A \oplus_k B$ corresponding to the eigenvalue $\lambda = 0$.*

Proof. A direct computation gives $Cx = 0$. □

Remark. Note that conditions $x_1 \in ker A_{11}$ and $x_1 \in ker A_{21}$ imply that the first $n_1 - k$ columns of A are a linearly dependent set and therefore the same columns of C are also a linearly dependent set and then the k-subdirect sum C is a singular matrix. What lemma 12 offers is an explicit eigenvector associated with the eigenvalue $\lambda = 0$ of C. The same argument can be made for conditions $x_2 \in ker B_{11}$ and $x_2 \in ker B_{12}$ in lemma 13.

As a direct consequence of lemmas 12 and 13 we have the following.

Theorem 14 *Let A and B be matrices of order n_1 and n_2, respectively, and let k be an integer such that $1 \leq k \leq \min(n_1, n_2)$. Let A and B be partitioned as in (1). Let x_1 and x_2 be nonzero vectors such that $x_1 \in ker A_{11}$, $x_1 \in ker A_{21}$, $x_2 \in ker B_{11}$ and $x_2 \in ker B_{12}$. Then it holds that vector $x = \begin{bmatrix} x_1 \\ O_{k\times 1} \\ x_2 \end{bmatrix}$ is an eigenvector of the k-subdirect sum $C = A \oplus_k B$ corresponding to the eigenvalue $\lambda = 0$.*

Proof. It is easy to show that $Cx = O$. □

Now we can make a step forward leaving the condition of null eigenvalue.

Lemma 15 *Let A and B be matrices of order n_1 and n_2, respectively, partitioned as in (1), with $1 \leq k \leq \min(n_1, n_2)$. Let $x_1 \in \mathbf{C}^{(n_1-k)\times 1}$ be a nonzero vector such that x_1 is an eigenvector of A_{11} associated with the eigenvalue λ and such that $x_1 \in ker A_{21}$. Then it holds that vector $x = \begin{bmatrix} x_1 \\ O_{n_2\times 1} \end{bmatrix}$ is an eigenvector of the k-subdirect sum $C = A \oplus_k B$ corresponding to the eigenvalue λ.*

Proof. As in the proof of lemma 12 it is easy to show that $Cx = \lambda x$. □

The counterpart, focusing on matrix B, is the following.

Lemma 16 *Let A and B be a matrices of order n_1 and n_2, respectively, partitioned as in (1), with $1 \leq k \leq \min(n_1, n_2)$. Let $x_2 \in \mathbf{C}^{(n_2-k)\times 1}$ be a nonzero vector such that x_2 is an eigenvector of B_{22} associated with the eigenvalue λ and such that $x_2 \in ker B_{12}$. Then it holds that vector $x = \begin{bmatrix} O_{n_1\times 1} \\ x_2 \end{bmatrix}$ is an eigenvector of the k-subdirect sum $C = A \oplus_k B$ corresponding to the eigenvalue λ.*

Proof. A direct calculation gives $Cx = \lambda x$. □

It is easy to see that if $\begin{bmatrix} x_1 \\ O_{k\times 1} \end{bmatrix} \in \mathbf{C}^{n_1\times 1}$ is an eigenvector of A corresponding to the eigenvalue λ then it implies that x_1 is an eigenvector of A_{11} associated with λ and also that $x_1 \in ker A_{21}$. Therefore as a consequence of lemmas 15 and 16 we have the following.

Theorem 17 *Let A and B be matrices of order n_1 and n_2, respectively, and let k be an integer such that $1 \leq k \leq \min(n_1, n_2)$. Let A and B be partitioned as in (1). Let $C = A \oplus_k B$. Let us assume that A and B have the same eigenvalue λ, let*

$$\begin{bmatrix} x_1 \\ O_{k\times 1} \end{bmatrix} \in \mathbf{C}^{n_1\times 1}$$

be an eigenvector of A corresponding to the eigenvalue λ and let

$$\begin{bmatrix} O_{k\times 1} \\ x_2 \end{bmatrix} \in \mathbf{C}^{n_2\times 1}$$

be an eigenvector of B corresponding to the eigenvalue λ. Then it holds that

$$x = \begin{bmatrix} x_1 \\ O_{k\times 1} \\ x_2 \end{bmatrix}$$

is an eigenvector of C corresponding to the eigenvalue λ.

Proof. It is easy to show that $Cx = \lambda x$. □

Example 18 *Given the matrices*

$$A = \left[\begin{array}{ccc|cc} & & & 3 & 2 \\ & A_{11} & & 5 & -1 \\ & & & 1 & 3 \\ \hline 4 & 5 & 1 & 2 & 1 \\ -2 & 0 & 2 & 1 & 4 \end{array}\right]$$

with $A_{11} = \begin{bmatrix} 2 & 1 & 1 \\ -8 & 3 & 9 \\ 8 & 1 & -5 \end{bmatrix}$, we have that $x_1^T = [1, -1, -1]$ is an eigenvector of A_{11} associated with the eigenvalue $\lambda = 2$, and $[x_1, 0, 0]^T$ is an eigenvector of A associated with the same eigenvalue. Given the matrix

$$B = \left[\begin{array}{cc|ccc} 1 & 2 & 1 & 1 & 0 \\ -1 & -3 & 1 & 0 & 1 \\ \hline 1 & 3 & & & \\ 2 & -1 & & B_{22} & \\ -4 & 4 & & & \end{array}\right],$$

with $B_{22} = \begin{bmatrix} 3 & 6 & -5 \\ 1 & 7 & -4 \\ 1 & 6 & -3 \end{bmatrix}$, we have that $x_2^T = [1, -1, 1]$ is an eigenvector of B_{22} associated with the eigenvalue $\lambda = 2$, and $[0, 0, x_2]^T$ is an eigenvector of B associated with the same eigenvalue. Therefore, *according to theorem 17, we get that the 2-subdirect sum* $C = A \oplus_2 B =$

$$= \left[\begin{array}{ccc|cc|ccc} 2 & 1 & 1 & 3 & 2 & 0 & 0 & 0 \\ -8 & 3 & 9 & 5 & -1 & 0 & 0 & 0 \\ 8 & 1 & -5 & 1 & 3 & 0 & 0 & 0 \\ \hline 4 & 5 & 1 & 3 & 3 & 1 & 1 & 0 \\ -2 & 0 & 2 & 0 & 1 & 1 & 0 & 1 \\ \hline 0 & 0 & 0 & 1 & 3 & 3 & 6 & -5 \\ 0 & 0 & 0 & 2 & -1 & 1 & 7 & -4 \\ 0 & 0 & 0 & -4 & 4 & 1 & 6 & -3 \end{array}\right]$$

has the eigenvector $[1, -1, -1, 0, 0, 1, -1, 1]^T$ *associated with the eigenvalue* $\lambda = 2$.

In a similar fashion as in the preceding theorems we can state the following result.

Theorem 19 *Let A and B be matrices of order n_1 and n_2, respectively, and let k be an integer such that $1 \le k \le \min(n_1, n_2)$. Let A and B be partitioned as in* (1). *Let $C = A \oplus_k B$. Let $w \in \mathbf{C}^{k\times 1} \in ker A_{12} \cap ker B_{21}$ such that w is a common eigenvector of A_{22} and B_{11} associated with the eigenvalue λ. Then it holds that*

$$x = \begin{bmatrix} O_{(n_1-k)\times 1} \\ w \\ O_{(n_2-k)\times 1} \end{bmatrix}$$

is an eigenvector of C corresponding to the eigenvalue 2λ.

Proof. It is easy to show that $Cx = 2\lambda x$, and x is nonzero since w, being an eigenvector, is nonzero □

The following example shows a couple of matrices that satisfy the above theorem.

Example 20 *Given the matrices*

$$A = \left[\begin{array}{c|cc} 4 & -1 & -1 \\ \hline 3 & 26 & -4 \\ 2 & -6 & 24 \end{array}\right], B = \left[\begin{array}{cc|c} 32 & 2 & 5 \\ 15 & 45 & -4 \\ \hline -1 & -1 & 3 \end{array}\right]$$

we have that $w^T = [1, -1]$ *satisfies* $A_{12}w = B_{21}w = [0, 0]^T$ *and is an eigenvector of* $A_{22} = \begin{bmatrix} 26 & -4 \\ -6 & 24 \end{bmatrix}$ *and of* $B_{11} = \begin{bmatrix} 32 & 2 \\ 15 & 45 \end{bmatrix}$ *associated with the eigenvalue* $\lambda = 30$, *and according to the theorem 19 we obtain that* $[0, 1, -1, 0]^T$ *is an eigenvector of the subdirect sum*

$$C = A \oplus_2 B = \left[\begin{array}{c|cc|c} 4 & -1 & -1 & 0 \\ \hline 3 & 58 & -2 & 5 \\ 2 & 9 & 69 & -4 \\ \hline 0 & -1 & -1 & 3 \end{array}\right]$$

associated with the eigenvalue $\lambda = 60$.

4 Conclusions and open issues

Some new properties of the eigenvalues of the subdirect sums have been presented for the particular case of 1-subdirect sums and some results concerning the eigenvectors of the k-subdirect sums have also been given. Several numerical examples illustrate the results. The characterization of the eigenvalues of the k-subdirect sum for general k is still an open problem. Some recent results on eigenvalue inclusion sets (see [3], [6]) may help outline some properties of the eigenvalues of the k-subdirect sum $C = A \oplus_k B$ in terms of the spectra of matrices A and B.

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