Proceedings of the 9th WSEAS International Conference on Applied Mathematics, Istanbul, Turkey, May 27-29, 2006 (pp424-429)

Necessary Conditions for the optimal control of the double-membrane system

Ismail Kucuk

Department of Mathematics and Statistics,
American University of Sharjah
Sharjah, UAE

1 Abstract

Analytical solution of an optimal control problem for two coupled linear membranes subjected to pointwise actuators together with a quadratic cost functional is given. The membranes are simply supported along the boundaries. The basic control problem is to control excessive vibrations in the system. Necessary conditions of optimality are investigated. Numerical results are provided for the efficiency of the control mechanism.

2 Keywords

Optimal Control, Membrane, Integral Equations, Necessary Conditions

3 Introduction

The study of the dynamics of membrane-like structures is important to model the vibrations in different disciplines of engineering including biomedical devices. Modelling of complex mechanical structures is achieved by simple compounded two dimensional continuous systems that consist of plates and/or membranes.

The investigation of the transverse vibrations of an elastically connected complex systems was conducted in [12, 13] in 1960's. During the last decade, the transverse vibration of different forms of complex continuous systems have been studied in [7, 8, 9] that are just some of the works by Oniszczuk. In these papers, Oniszczuk has given the analytical solutions of natural frequencies for free and forced vibrations of different complex systems, and how the frequencies can be affected by different parameters. A well-investigated the transverse vibrations of the complex systems have been followed by the optimal control of such systems for different aspects: Sadek *et al* in [10] investigates the optimal control of two Euler-Bernoulli beams by means of Maximum principle for a modified energy functional. The controllability and stability of serially connected Timoshenko beams are investigated in [5]. The monographs [4] and [11] are some of the others that focus on control aspects of different vibrating systems. Kucuk and Sadek in [2, 3] introduced a new system for a complex system to control the vibrations by using calculus of variations.

In this paper, we focus on controlling the vibrations of double-membrane system that consists of two membranes and a Winkler elastic foundation. It is aimed to control the vibrations excessively in the system. Therefore, we implemented a finite number of control actuators in the domains of membranes. For the proposed new system, we find the optimal control actuators by using calculus of variations to derive the necessary conditions of optimality. Necessary conditions of optimality are obtained as Fredholm integral equations with degenerate kernel that lead to a system of linear equations. Then, we are able to write the optimal control actuators analytically as solutions of the system of linear equations. Finally, we illustrate the robustness of the developed theory by a numerical simulation for a system that has one actuator in each membrane. The details and different aspects of the present work can be found in [1].

4 Formulation of the Problem

The double-membrane system contains three layers: two thin, homogeneous membranes in which every point has an isotropic state of tension, and a massless homogeneous Winkler foundation to bind the two membranes. The transverse vibrations of the system is described by a set of two coupled non-homogeneous partial differential equations [7],

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$$m_1\ddot{w}_1 - N_1 \ \Delta w_1 + k(w_1 - w_2) = f_1(x,y,t) = \sum_{j=1}^{n_m} f_{1j}(t)\delta(x - x_j^{m_1}, y - y_j^{m_1}) \quad (1)$$

$$m_2\ddot{w}_2 - N_2 \ \Delta w_2 + k(w_2 - w_1) = f_2(x,y,t) = \sum_{j=1}^{n_m} f_{2j}(t)\delta(x - x_j^{m_2}, y - y_j^{m_2}) \quad (2)$$

where $w_1 = w_1(x,y,t)$ ($w_2 = w_2(x,y,t)$) is the transverse displacement in membrane 1 (membrane 2); $f_i(x,y,t)$ is a transverse continuous loading applied along the edges of the two membranes; N_i is the uniform constant tension per unit length for the membranes; k is the stiffness modulus of a Winkler elastic layer; $\delta(\ , \ )$ is the Dirac distribution; the locations of the actuators $(x_j^{m_1}, y_j^{m_1})$ and $(x_j^{m_2}, y_j^{m_2})$ are from $(0,a)\times(0,b)$; $\mathsf{f}_{ij}(t) \in L^2([0,t_f])$ is the amplitude (or influence) of distributed actuators, t_f is the terminal time; and

$$m_i = \rho_i h_i, \quad \dot{w}_i = \frac{\partial w_i}{\partial t}, \Delta w_i = \frac{\partial^2 w_i}{\partial x^2} + \frac{\partial^2 w_i}{\partial y^2},$$

in which ρ_i is the mass density, and h_i is the thickness of the membranes and $i = 1,2$. The membranes are simply supported on the edges:

$$w_i(0,y,t) = w_i(a,y,t) = w_i(x,0,t) = w_i(x,b,t) = 0, \quad (3)$$

where a and b are the dimensions of the membranes. The initial conditions are assumed to be of the following form

$$w_i(x,y,0) = w_i^0(x,y), \dot{w}_i(x,y,0) = v_i^0(x,y) \quad (4)$$

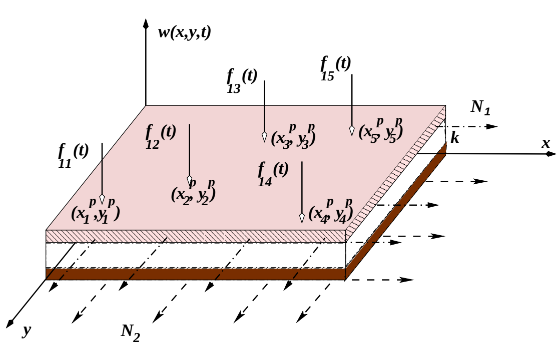

Figure 1: An elastically connected rectangular double-membrane system with the control parameters $f_{ik}, \ i = 1,2$ and $k = 1,\ldots,n_m$

The performance of the system under the influence of the applied control forces $f_{1j}(t)$ and $f_{2j}(t)$ in (1) and (2), respectively, are measured by the following performance index function

$$\mathcal{J}(f_{11},\ldots,f_{1n_m},f_{21},\ldots,f_{2n_m}) = \mathcal{J}(\mathsf{F})$$

where

$$\mathcal{J}(\mathsf{F}) = \frac{1}{2}\int_0^b\int_0^a \Big\{ \mu_1 w_1^2(x,y,t_f) + \mu_2 \dot{w}_1^2(x,y,t_f) + \mu_3 w_2^2(x,y,t_f) + \mu_4 \dot{w}_2^2(x,y,t_f) \Big\} dxdy + \frac{1}{2}\int_0^{t_f} \left(\sum_{i=1}^{n_m} \epsilon_i f_{1i}^2(t) + \sum_{i=1}^{n_m} \alpha_i f_{2i}^2(t) \right) dt. \quad (5)$$

Here in (5), $\mu_i \geq 0$ for $i = 1,2,3,4$ such that $\mu_1 + \mu_2 + \mu_3 + \mu_4 \neq 0$; $\epsilon_i \geq 0$, and $\alpha_i \geq 0, i = 1,\ldots,n_m$ are the weight factors that determine the influence of the distributed actuators. The first functional of (5) is the contribution of the modified energy due to the membrane 1 and membrane 2, and the other two functionals represent a contribution of the energy that accumulates over the control duration $[0,t_f]$.

The optimal control problem of interest can be stated as the following: Find an optimal $f_{ij}^{\star}(t) \in L^2([0,t_f])$ for $i = 1,2$ and $j = 1,\ldots,n_m$ such that

$$\mathcal{J}(\mathsf{F}^{\star}(t)) \leq \mathcal{J}(\mathsf{F}(t)), \quad \forall \mathsf{f}_{ij}(t) \in L^2([0,t_f])$$

subject to (1)-(4).

Existence and uniqueness of the optimal control of the system of the partial differential equations are thoroughly discussed in a classical work [6].

5 Solution of the vibration problem

Of great significance is the classical modal expansion (separation of variables in general), which transforms the basic optimal control problem into optimal control of lumped-parameter system, to solve vibrations of the considered system. The modal expansion assumes the solutions for vibrations in the form of

$$w_1(x,y,t) = \sum_{m,n}^{N} \overbrace{\varphi_m(x)\psi_n(y)}^{\Psi_{mn}(x,y)} T_{mn}^1(t) \quad (6)$$

$$w_2(x,y,t) = \sum_{m,n}^{N} \varphi_m(x)\psi_n(y) T_{mn}^2(t) \quad (7)$$

where $T_{mn}^1(t)$ and $T_{mn}^2(t)$ are unknown time functions, and N is the number of harmonics taken

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in the calculations with some truncated error as a practical approach. The orthonormal sets of eigenfunctions, $\Big\{\varphi_m(x)\Big\}_{m=1}^{\infty}$, $\Big\{\psi_n(y)\Big\}_{n=1}^{\infty}$, of the operator

$$\mathcal{L}w = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}$$

are defined as

$$\begin{aligned}\Big\{\varphi_m(x)\Big\}_m &= \Big\{\sqrt{\frac{2}{a}}\sin(a_m x)\Big\}_{m=1}^{\infty} \\ \Big\{\psi_n(y)\Big\}_n &= \Big\{\sqrt{\frac{2}{b}}\sin(b_n y)\Big\}_{n=1}^{\infty}.\end{aligned} \tag{8}$$

where $a_m = a^{-1}m\pi$ and $b_n = b^{-1}n\pi$. By substituting the solutions (6) and (7) into (1) and (2), we obtain the following second order differential equations in time

$$\ddot{T}^1_{mn}(t) + G^1_{mn}T^1_{mn}(t) - G_{10}T^2_{mn}(t) = f_1(t), \tag{9a}$$

$$\ddot{T}^2_{mn}(t) + G^2_{mn}T^2_{mn}(t) - G_{20}T^1_{mn}(t) = f_2(t), \tag{9b}$$

where

$$\begin{aligned}G^i_{mn} &= \frac{N_i k_{mn} + k}{m_i}; G_{i0} = \frac{k}{m_i}; \\ f_1(t) &= m_1^{-1}\sum_{j=1}^{n_m} f_{1j}(t)\Psi_{mn}(x_j^{m_1}, y_j^{m_1}), \\ k_{mn} &= a_m^2 + b_n^2 \\ f_2(t) &= m_2^{-1}\sum_{j=1}^{n_m} f_{2j}(t)\Psi_{mn}(x_j^{m_2}, y_j^{m_2})\end{aligned}$$

with the proper form of initial conditions given by (4). We rewrite the coupled system of second order differential equations (9) in time as a coupled system of first order differential equations by introducing new variables $\mathsf{y_i}$. It follows immediately that the set of second order differential equations in (9) can be written as

$$\begin{aligned}\dot{\mathsf{y}}_1^{\mathsf{mn}}(t) &= \mathsf{y_2}^{mn}(t), \\ \dot{\mathsf{y}}_2^{mn}(t) &= -G^1_{mn}\mathsf{y_1}^{mn}(t) + G_{10}\mathsf{y_3}^{mn}(t) + f_1(t), \\ \dot{\mathsf{y}}_3^{\mathsf{mn}}(t) &= \mathsf{y_4}^{mn}(t), \\ \dot{\mathsf{y}}_4^{\mathsf{mn}}(t) &= -G^2_{mn}\mathsf{y_3}^{mn}(t) + G_{20}\mathsf{y_1}^{mn}(t) + f_2(t),\end{aligned} \tag{10}$$

or in a more compact form

$$\frac{d\mathsf{Y}}{dt} = \mathsf{AY} + \mathbf{F}(t) \tag{11}$$

where A is a 4×4 diagonalizable constant matrix. The matrices in the latter equation are of the following forms

$$\begin{aligned}\mathsf{Y} &= \begin{pmatrix}\mathsf{y_1^{mn}}\\ \mathsf{y_2^{mn}}\\ \mathsf{y_3^{mn}}\\ \mathsf{y_4^{mn}}\end{pmatrix}; \qquad \mathbf{F}(t) = \begin{pmatrix}0\\ f_1(t)\\ 0\\ f_2(t)\end{pmatrix} \\ \mathsf{A} &= \begin{pmatrix}0 & 1 & 0 & 0\\ -G^1_{mn} & 0 & G_{10} & 0\\ 0 & 0 & 0 & 1\\ G_{20}0 & -G^2_{mn} & 0 & \end{pmatrix}.\end{aligned} \tag{12}$$

To solve the coupled system of differential equations in (11), we first define a new matrix B whose columns are the eigenvectors $\mathsf{v_1}, \mathsf{v_2}, \mathsf{v_3}$, and $\mathsf{v_4}$ of A. Then we introduce a new dependent variable X with

$$\mathsf{Y}(t) = \mathsf{BX}(t) \tag{13}$$

Substituting this new variable for Y into (11) leads to

$$\begin{aligned}\dot{\mathsf{X}}(t) &= (\mathsf{B}^{-1}\mathsf{AB})\mathsf{X}(t) + \mathsf{B}^{-1}\mathbf{F}(t) \\ &= \mathsf{DX}(t) + \mathbf{G}(t)\end{aligned} \tag{14}$$

where $\mathsf{D} = \mathsf{B}^{-1}\mathsf{AB}$ is diagonal matrix with the eigenvalues of A are on the main diagonal, and $\mathbf{G}(t) = \mathsf{B}^{-1}\mathbf{F}(t)$ defined as

$$\mathbf{G}(t) = \tilde{\beta}\begin{pmatrix}1\\ 1\\ -1\\ -1\end{pmatrix} f_1(t) + \begin{pmatrix}\tilde{\Delta}_1\\ \tilde{\Delta}_1\\ \tilde{\Delta}_2\\ \tilde{\Delta}_2\end{pmatrix} f_2(t) \tag{15}$$

where $\tilde{\beta}, \tilde{\Delta}_1$ and $\tilde{\Delta}_2$ are some constants.

Equation (14) is a system of four uncoupled differential equations for $\mathsf{X}_i^{mn}(t)$, $i = 1, 2, 3, 4$. In scalar form, we observe the following equations

$$\frac{d\mathsf{X}_i^{mn}(t)}{dt} = \lambda_i \mathsf{X}_i^{mn}(t) + \mathrm{G}_i(t) \tag{16}$$

with the proper form of the initial conditions given in (4), and the solutions of (10) are of the following form:

$$\mathsf{X}_i^{mn}(t) = e^{\lambda_i t}\int_0^t e^{-\lambda_i s}\mathrm{G}_i(s)ds + c_i e^{\lambda_i t} \tag{17}$$

where c_i are constants to be determined.

The deflections in membrane 1 and membrane 2 are obtained as

$$w_1(x, y, t) = \sum_{mn}^{N}\Psi_{mn}(x, y)\sum_{j=1}^{4}\mathsf{b}_{1j}\mathsf{X}_j^{mn}(t), \tag{18}$$

$$w_2(x, y, t) = \sum_{mn}^{N}\Psi_{mn}(x, y)\sum_{j=1}^{4}\mathsf{b}_{3j}\mathsf{X}_j^{mn}(t), \tag{19}$$

where $\mathsf{B} = [\mathsf{b}_{ij}]_{4\times 4}$ whose columns are the eigenvectors of A in (12), and $\mathsf{X}_j^{mn}(t)$ is defined in (17).

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6 Necessary Conditions

The necessary conditions of optimality for the optimal controls are obtained by using calculus of variation. The first variation of the performance index function (5) with respect to $f_{ik},\ k=1,\dots,n_m$ and $i=1,2$ leads to coupled integral equations for each actuators. Solving the integral equations gives the optimal actuators.

To derive the necessary conditions of optimality for the applied actuators in the domain of the membrane 1 and membrane 2, we first fix the location of the actuators, and weight factors in the performance index function (5) and differentiate (5) with respect to $f_{ik},\ k=1,\dots,n_m$ and $i=1,2$. The two differentiations of (5) lead to the coupled integral equations that returns the optimal actuators.

After substituting the solutions (18) and (19) for w_1 and w_2, respectively, into (5), the performance index can be rewritten as

$$\mathcal{J}(F)=\frac{1}{2}\sum_{mn}^{N}\sum_{i=1}^{4}\mu_i\left(\sum_{j=1}^{4}\mathsf{b}_{ij}\mathsf{X}_j^{mn}(t_f)\right)^2+ \frac{1}{2}\int_0^{t_f}\left(\sum_{i=1}^{n_m}\epsilon_i f_{1i}^2(t)+\sum_{i=1}^{n_m}\alpha_i f_{2i}^2(t)\right)dt. \tag{20}$$

Taking the first variation of (20) with respect to f_{1k} leads to

$$\boldsymbol{\delta}_{f_{1k}}\mathcal{J}(F)=\int_0^{t_f}\left\{\sum_{mn}^{N}\sum_{i=1}^{4}\mu_i\left(\sum_{j=1}^{4}\mathsf{b}_{ij} \mathsf{X}_j^{mn}(t_f)\right)\sum_{j=1}^{4}\mathsf{b}_{ij}e^{\lambda_j(t_f-s)}\bar{G}_{1j} \Psi_{mn}(x_k^{m_1},y_k^{m_1})+\epsilon_k f_{1k}(s)\right\} \Delta f_{1k}(s)ds=0. \tag{21}$$

Since (21) is true for all variations of Δf_{1k}, we observe the following for fixed $k=1,\dots,n_m$

$$\sum_{mn}^{N}\sum_{i=1}^{4}\mu_i\left[\sum_{j=1}^{4}\mathsf{b}_{ij}\left\{\int_0^{t_f}e^{\lambda_j(t_f-r)}\left[\bar{G}_{1j}\sum_{q=1}^{n_m}f_{1q}(r) \Psi_{mn}(x_q^{m_1},y_q^{m_1})+\bar{G}_{2j}\sum_{q=1}^{n_m}f_{2q}(r)\Psi_{mn}(x_q^{m_2},y_q^{m_2})\right]dr +c_je^{\lambda_j t_f}\right\}\right]\times\left[\sum_{j=1}^{4}\mathsf{b}_{ij}e^{\lambda_j(t_f-s)}\bar{G}_{1j}\Psi_{mn}(x_k^{m_1},y_k^{m_1})\right] +\epsilon_k f_{1k}(s)=0 \tag{22}$$

where $\bar{G}_{1j}$ and $\bar{G}_{2j}$ are the terms from the column coefficient matrices of $f_1(t)$ and $f_2(t)$ given in (15).

A similar result is obtained from the variation of $\mathcal{J}(F)$ with respect to f_{2k} for fixed $k=1,\dots,n_m$. The coupled nonhomogeneous Fredholm integral equations with degenerate kernel can be observed for each fixed k as

$$\sum_{mn}^{N}\left[\sum_{q=1}^{n_m}\sum_{i=1}^{4}\int_0^{t_f}\left(K_{imn}^{qk}(s)f_{1q}(r)+ \overline{K}_{imn}^{qk}(s)f_{2q}(r)\right)e^{\lambda_i(t_f-r)}dr+P_{mn}^k(s)\right] +\epsilon_k f_{1k}(s)=0, \tag{23a}$$

$$\sum_{mn}^{N}\left[\sum_{q=1}^{n_m}\sum_{i=1}^{4}\int_0^{t_f}\left(L_{imn}^{qk}(s)f_{1q}(r)+ \overline{L}_{i,mn}^{qk}(s)f_{2q}(r)\right)e^{\lambda_i(t_f-r)}dr+\overline{P}_{mn}^k(s)\right] +\sigma_k f_{2k}(s)=0. \tag{23b}$$

where $P(s),K(s),\overline{K}(s)$ and others are known $4n_m\times 1$ column matrices.

The integral equations in (23) are transformed into system of linear equations in the following compact form

$$\begin{aligned}(A+IE)\,\mathcal{C}+\qquad\overline{A}\,\overline{\mathcal{C}}+H&=O,\\ B\,\mathcal{C}+(\overline{B}+IS)\overline{\mathcal{C}}+\overline{H}&=O.\end{aligned} \tag{24}$$

The solution of the system of linear equations for the unknown $\mathcal{C}$ and $\overline{\mathcal{C}}$ in (24) writes the optimal actuators f_{1q} and f_{2q} as

$$\begin{aligned}f_{1k}(s)&=-\frac{1}{\epsilon_k}\sum_{mn}^{N}\left\{P_{mn}^k(s)+ \sum_{q=1}^{n_m}\sum_{i=1}^{4}(K_{imn}^{qk}(s)c_i^q+\overline{K}_{imn}^{qk}(s)\bar{c}_i^q)\right\},\\ f_{2k}(s)&=-\frac{1}{\sigma_k}\sum_{mn}^{N}\left\{\overline{P}_{mn}^k(s)+ \sum_{q=1}^{n_m}\sum_{i=1}^{4}(L_{imn}^{qk}(s)c_i^q+\overline{L}_{imn}^{qk}(s)\bar{c}_i^q)\right\}\end{aligned} \tag{25}$$

where $k=1,\dots,n_m$. The optimal control parameters derived in (25) are for a fixed membrane tensions and location of the actuators.

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7 Numerical Simulations

In this section, we consider an example of a double-membrane continuous system that consists of two membranes bounded through a Winkler elastic foundation. For the simplicity of the analysis, it is assumed that the double-membrane system is subjected to the initial conditions (4) of the form :

$$w_i(x,y,0) = \Phi_{11}(x,y), \qquad \dot{w}_i(x,y,0) = 0 \quad i = 1,2 \tag{26}$$

where $\Phi_{11}(x,y)$ is given by (6). The initial conditions given in (26) allow us to study the behavior of the fundamental mode of the system.

In numerical simulations, the following parameters are used to characterize the physical and geometrical properties of the double-membrane system: $a = 1m,\ b = 2m,\ k = 2 \times 10^2,\ N_1 = N_2 = 50kg/m^2,\ h_1 = h_2 = 1 \times 10^{-2}m,\ m_1 = m_2 = 0.2,\ t_f = 18s,\ \rho_1 = \rho_2 = 20kg/m^3,\ (x_1^{m_1}, y_1^{m_1}) = (0.3, 1.1), (x_1^{m_2}, y_1^{m_2}) = (0.8, 1.5)$

The energy of the system before any actuator implemented is 529.4; however, the controlled system reaches the minimal energy 0.00002 when only one actuator applied to membrane 1. This drastic achievement in energy results in completely damped out vibrations in the first membrane that can be observed in Figure 2 for the deflection and Figure 3 for the velocity. The deflection and velocity of the membrane 1 in Figure 2 and Figure 3,respectively, are observed at $(x,y) = (0.5,1)$. Addition to riding of the vibration in the first membrane, the same success is observed in the second membrane as well.

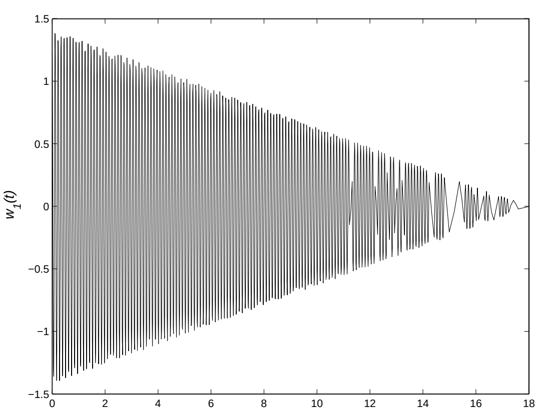

Figure 2: The deflection of *Membrane 1* after the control.

8 Remarks and Conclusion

Vibrations of two parallel rectangular membranes connected by a Winkler elastic foundation are studied to prevent any undesirable resonance in the complex continuous system. The two membranes that are homogeneous, isotropic, and thin are subjected to simply supported boundary conditions. The proposed new system damps out the undesirable vibrations by applying point-wise controllers in the domain of the membranes.

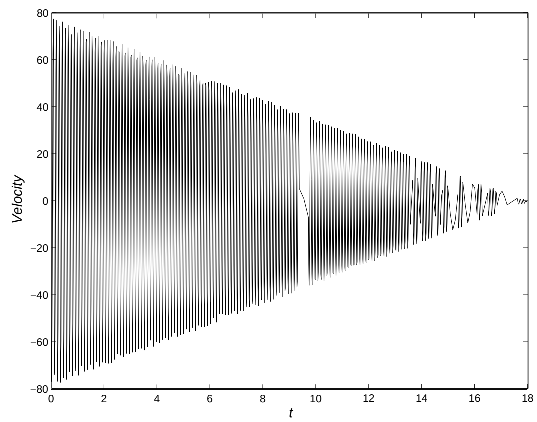

Figure 3: The velocity of *Membrane 1* after the control.

The modal expansion technique to evaluate the optimal control of the distributed parameter system is of great advantage to reduce the problem to the optimal control of the lumped parameter system (LPS). Calculus of variation is used to derive the necessary conditions of optimality for the point-wise control of the LPS, and the coupled nonhomogeneous Fredholm integral equations with degenerate kernel are obtained for the necessary conditions. Then, the integral equations are transformed into the system of linear equations. Finally, the optimal point-wise controllers are given explicitly as the solutions of the system of linear equations.

In short, we conclude that the proposed mechanism is an effective way to determine the optimal controllers. As a result, the performance index function becomes almost zero at the terminal time and the vibrations are damped out completely.

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