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Complete convergence for negatively dependent random variables $^{(b)}$

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Abstract: *In this paper, we extend complete convergence of weighted sums* $T_n = \sum_{k=1}^{n} a_{nk} X_k$ where $\{X_n, n \geq 1\}$ is a sequence of negatively dependent random variables and a_{nk}, $n \geq 1$,$k \geq 1$ is an array of real numbers.

Key Words: Negatively dependent; Complete convergence;Weighted sums;

1.Introduction

Let $\{X_n, n \geq 1\}$ be a sequence of independent and identically random variables. In 1947 Hsu and Rabbins proved that if $E[X] = 0$ and $E[X^2] < \infty$, then $\frac{1}{n}\sum_{k=1}^{n} X_k$ converges to 0 completely. Recently, the strong convergence of weighted sums for the case of independent random variables has been discussed by Wu (1999), Hu and et. (2000, 2003) proved the complete convergence theorem for arrays of independent random variables and Amini and Bozorgnia (2003) studied complete convergence of the sequence $\frac{1}{n}\sum_{k=1}^{n} X_k$, via. exponential bounds in the case of negatively dependent and identically random variables. Zarei (2006) extended some results of Chow (1966) for negatively dependent and identically distributed random variables. In this paper, we study complete convergence of weighted sums $T_n = \sum_{k=1}^{n} a_{nk} X_k$ where $\{X_n, n \geq 1\}$ is a sequence of negatively dependent random variables and a_{nk}, $n \geq 1$, $k \geq 1$ is an array of real numbers where $a_{nk} = 0$ if $k > n$, $|a_{nk}| \leq C A_n$ for $A_n = \sum_{k=1}^{\infty} a_{nk}^2$ and some $0 < C < \infty$. In fact we omit the condition identically distributed random variables. The material in this note is closely related to Zarei (2006) and chow(1966).
In this paper we define $X^+ = \max\{0, X\}$ and $X^- = \max\{0, -X\}$ and I_A denotes indicator function A. To prove the main result we need to the following definitions and lemmas.

Definition 1. The random variables $X_1, \ldots, X_n$ are said to be negatively dependent(ND) if we have

$$P[\bigcap_{j=1}^{n}(X_j \leq x_j)] \leq \prod_{j=1}^{n} P[X_j \leq x_j],$$

and

$$P[\bigcap_{j=1}^{n}(X_j > x_j)] \leq \prod_{j=1}^{n} P[X_j > x_j],$$

for all $x_1, ..., x_n \in R$. An infinite sequence $\{X_n, n \geq 1\}$ is said to be ND if every finite subset $X_{i_1}, ..., X_{i_n}$ is ND.

Definition 2. The random variables $\{X_n, n \geq 1\}$ of random variables converges to zero completely if for every $\varepsilon > 0$

$$\sum_{n=1}^{\infty} P[|X_n| > \varepsilon] < \infty.$$

The next two lemmas will be need in the proof of the strong law of large numbers in the next section.

Lemma 1. Let $\{X_n, n \geq 1\}$ be a sequence of ND random variables and $\{f_n, n \geq 1\}$ be a sequence of Borel functions all of which are monotone increasing (or all are monotone decreasing).Then $\{f_n(X_n), n \geq 1\}$ is a sequence of ND random variables.

Lemma 2. Let $X_1, ...X_n$ be a finite sequence of ND random variables and $t_1, ..., t_n$ be all nonnegative (or nonpositive) then

$$E[e^{\sum_{i=1}^{n} t_i X_i}] \leq \prod_{i=1}^{n} E[e^{t_i X_i}].$$

2. Strong Convergence of weighted sums

Let $\{X_n, n \geq 1\}$ be a sequence of ND random variables and $a_{nk},\ n \geq 1,\ k \geq 1$ be an array of real numbers and $A_n = \sum_{k=1}^{\infty} a_{nk}^2$. The flowing theorem is a extension of Hsu and Rabbin's theorem.

Theorem. Let X_n be negatively dependent random variables with $EX_n = 0$. Let $a_{nk} = 0$ if $k > n^\lambda$ for some $1 \leq \lambda < \infty$ and $|a_{nk}| \leq CA_n$ for some $0 < C < \infty$. If for some $0 < \alpha \leq 1$, $A_n \leq Cn^{-\alpha}$ and $E[|X_n|^{(1+\lambda)/\alpha}(\log^+ |X_n|)^2] \leq C$, then

$$\sum_{n=1}^{\infty} P[|T_n| \geq \varepsilon] < \infty \qquad (1)$$

for every $\varepsilon > 0$, where $T_n = \sum_{k=1}^{\infty} a_{nk} X_k$.

Proof: We can assume that $C \geq 1$ and $E|X_n|^{(1+\lambda)/\alpha} \leq C$, for each n. For $0 < \beta < \alpha$ and $N = 2, 3, ...$, we define
if $a_{nk} \geq 0$

$$\begin{aligned} X_k^{'} &= X_k I_{[X_k \leq n^\beta]}, \\ Y_k &= X_k I_{[X_k \leq n^\beta]} + n^\beta I_{[X_k > n^\beta]}, \\ X^{''} &= X_k I_{[X_k \geq \varepsilon n^\alpha/(NC^2)]}, \end{aligned}$$

and if $a_{nk} < 0$

$$\begin{aligned} X_k^{'} &= X_k I_{[X_k \geq -n^\beta]}, \\ Y_k &= X_k I_{[X_k \geq -n^\beta]} - n^\beta I_{[X_k < -n^\beta]}, \\ X^{''} &= X_k I_{[X_k \leq -\varepsilon n^\alpha/(NC^2)]}. \end{aligned}$$

And put

$$\begin{aligned} X_k^{'''} &= X_k - X_k^{'} - X_k^{''}, \\ T_n^{'} &= \sum_{k=1}^{\infty} a_{nk} X_k^{'}, \\ T_n^{''} &= \sum_{k=1}^{\infty} a_{nk} X_k^{''}, \\ T_n^{'''} &= \sum_{k=1}^{\infty} a_{nk} X_k^{'''}, \\ U_n &= \sum_{k=1}^{\infty} a_{nk} Y_k, \\ U_n^{(1)} &= \sum_{k=1}^{\infty} a_{nk} Y_k I_{[B^c]}, \\ U_n^{(2)} &= \sum_{k=1}^{\infty} a_{nk} Y_k I_{[B]}, \end{aligned}$$

where $B = \{a_{nk} \geq 0\}$. If a random variable $X \leq 1$ a.e., then obviously $E \exp[X] \leq \exp[EX + EX^2]$. Let $0 < t < C^{-1} n^{-\beta}$, then $ta_{nk} Y_k / A_n \leq 1$ and $E(a_{nk} Y_k) \leq 0$. Hence

$$\begin{aligned} E \exp[ta_{nk} X_k^{'}/A_n] &\leq \exp[ta_{nk} Y_k/A_n] \\ &\leq \exp[t^2 a_{nk}^2 A_n^{-2} EY_k^2] \\ &\leq \exp[t^2 a_{nk}^2 A_n^{-2} EX_k^2] \\ &\leq \exp[Ct^2 a_{nk}^2 A_n^{-2}]. \end{aligned}$$

Since $\{X_n, n \geq 1\}$ is a sequence ND random variables, by Cauchy Schwartz inequality and Lemmas 1 and 2 we have

$$\begin{aligned} E \exp[tT_n^{'}/A_n] &\leq E \exp[tU_n/A_n] \\ &\leq [E \exp[2tU_n^{(1)}/A_n] E \exp[2tU_n^{(2)}/A_n]]^{1/2} \\ &\leq \exp[t^2 K/A_n], \end{aligned}$$

where K=2C. Therefore

$$\begin{aligned} P[T_n^{'} \geq \varepsilon] &= P[tT_n^{'} A_n^{-1} \geq t\varepsilon A_n^{-1}] \\ &\leq \exp[-t(\varepsilon - tK)/A_n]. \end{aligned}$$

If put $t = n^{-\beta} K^{-1}$ for sufficiently large n we have

$$P[T_n^{'} \geq \varepsilon] \leq exp[-2\varepsilon n^{\alpha-\beta}/K^2].$$

Since $0 < \beta < \alpha$,

$$\sum_{n=1}^{\infty} P[T_n^{'} \geq \varepsilon] < \infty. \qquad (2)$$

Now put $m = [n^\lambda]$, the integral part of n^λ, and $Z_n = (NC^2|X_n|/\varepsilon)^{\alpha^{-1}}$. Then for $n > 3$ we have

$$\begin{aligned} P[T_n^{''} \geq \varepsilon] &\leq \sum_{k=1}^{m} P[(NC^2|X_k|\varepsilon)^{\alpha^{-1}} \geq n] \\ &\leq \sum_{k=1}^{m} P[Z_k^{1+\lambda} \log^2 Z_k \geq n^{1+\lambda} \log^2 n] \\ &\leq \sum_{k=1}^{m} n^{-(1+\lambda)} \log^{-2} n . E[Z_k^{1+\lambda} \log^2 Z_k]. \end{aligned}$$

By condition $E[|X_n|^{(1+\lambda)/\alpha}(\log^+|X_n|)^2] \leq C$, there exists a constant $C^* = C(N,\ C,\ \lambda,\ \varepsilon,\ \alpha) < \infty$ such that $E[Z_k^{1+\lambda} \log^2 Z_k] \leq C^*$. Hence

$$\begin{aligned} P[T^{''} \geq \varepsilon] &\leq C^* \sum_{k=1}^{m} n^{-(1+\lambda)} \log^{-2} n \\ &\leq C^* n^{-1} \log^{-2} n. \end{aligned}$$

Then

$$\sum_{n=1}^{\infty} P[T_n^{''} \geq \varepsilon] < \infty. \qquad (3)$$

Since $a_{nk} X_k^{'''} \leq \varepsilon/N$, $T_n^{'''} \geq \varepsilon$ implies that there are at least N non-zero $X_k^{'''}$ for $k = 1, 2, ..., m$ (say $X_{l_1}, ..., X_{L_N}$). Hence

$$P[T_n^{'''} \geq \varepsilon] \leq \binom{m}{N} P[|X_{l_1}| > n^\beta, ..., |X_{l_N}| > n^\beta].$$

Now we define

$$A_i = [|X_{l_i}| > n^\beta].$$

Hence

$$P[A_1 \bigcap A_2 \bigcap ... \bigcap A_N] = P[A_1]P[A_2|A_1]...P[A_N|A_1, ..., A_{N-1}].$$

Since for $2 \leq i \leq N$,

$$P[A_i|A_1, ..., A_{i-1}] = L_i P[A_1],$$

where $L_i,\ 2 \leq i \leq N$ are positive real numbers. Hence

$$\begin{aligned} P[T_n^{'''} \geq \varepsilon] &\leq \binom{m}{N} L^N P[|X_{l_1}| > n^\beta]^N \\ &\leq L^N \binom{m}{N} P[|X_{l_1}|^{(1+\lambda)/\alpha} > n^{\beta(1+\lambda)/\alpha}]^N, \end{aligned}$$

where $L = \max_{2 \leq i \leq N} L_{l_i}$. By Chebichov's inequality,

$$\begin{aligned} P[T_n^{'''} \geq \varepsilon] &\leq (CL)^N \binom{m}{N} n^{-N\beta(1+\lambda)/\alpha} \\ &= O(n^{N(\lambda-\beta(1+\lambda)/\alpha)}). \end{aligned}$$

Choose β so near to α such that $\lambda - \beta(1+\lambda)/\alpha < 0$, and then choose N so large such that $N(\lambda - \beta(1+\lambda)/\alpha) \leq -2$. then

$$\sum_{n=1}^{\infty} P[T_n^{'''} \geq \varepsilon] \leq \sum_{n=1}^{\infty} O(n^{-2}) < \infty.$$

Therefore

$$\sum_{n=1}^{\infty} P[T_n^{'''} \geq \varepsilon] < \infty. \qquad (4)$$

From (2), (3) and (4), we have

$$\sum_{n=1}^{\infty} P[T_n \geq \varepsilon] < \infty.$$

By symmetry, we obtain $\sum_{n=1}^{\infty} P[T_n^{'''} \leq -\varepsilon] < \infty$. Hence the proof is complete. □

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