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Approximate subdifferentials of marginal functions: the Lipschitzian case.

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Abstract. This paper establishes formulas for approximate subdifferentials of locally Lipschitzian marginal (or optimal value) functions which are not required to attain their infimum.

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Introduction.

It is well-known that a lot of problems in optimization and optimal control involve marginal functions and their subdifferentials since the sensitivity of these problems can be studied with the help of the behavior of the subdifferentials of some associated marginal functions. Generally the infimum defining the marginal function is required to be attained near the point of interest.
This paper is devoted to study how this condition can be removed. More general cases using more technical methods will be considered in a forthcoming second paper [1]. Here we will deal with locally Lipschitz marginal functions of the form

$$m(x) := \inf\{g(y) : y \in G(x)\}$$

where g is a real-valued function from a Banach space X into $I\!R$ and G is a multivalued mapping from X into a Banach space Y. The above infimum will not be required to be attained.

Note that, with appropriate modifications of our proofs, all the results in the paper have their analog ones with the limiting Fréchet subdifferentials (see Mordukhovich and Shao [14] for the definition) whenever the spaces X and Y are assumed to be Asplund.

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1. Preliminaries

In all this paper X, $I\!B_X$ and $\mathcal{F}(X)$ denote respectively a Banach space, the closed unit ball of X and the collection of all finite dimensional vector subspaces of X.

Let $f : X \longrightarrow I\!R \bigcup \{-\infty, +\infty\}$ be a function which is lower semicontinuous near $\bar{x} \in X$ with $| f(\bar{x}) | < \infty$. For each subset $S \subset X$, f_S will be the function given by $f_S(x) = f(x)$ if $x \in S$ and $f_S(x) = +\infty$ otherwise. Following Ioffe [9] the lower Dini ϵ -subdifferential (for $\epsilon \geq 0$) of f at x is defined by $\partial_\epsilon^- f(x) = \emptyset$ if $| f(x) | = +\infty$ and otherwise by:

$$\partial_\epsilon^- f(x) := \{x^* \in X^* :< x^*, v > \leq d^- f(x; v) + \epsilon \| v \|, \forall v \in X\}$$

where

$$d^- f(x; v) := \liminf_{u \to v, t \to 0^+} t^{-1}[f(x + tu) - f(x)]$$

and the approximate subdifferential of f at $\bar{x}$ is defined by

$$\partial_A f(\bar{x}) := \bigcap_{L \in \mathcal{F}(X)} \limsup_{x \to^f \bar{x}} \partial^- f_{x+L}(x) \tag{1.1}$$

where $\partial^- f(x)$ is $\partial_\epsilon^- f(x)$ for $\epsilon = 0$. Here, $x \to^f \bar{x}$ means $x \to \bar{x}$ and $f(x) \to f(\bar{x})$ and for a multivalued mapping M from X into X^*, $x^* \in \limsup_{x \to \bar{x}} M(x)$ means that there exist nets $(x_i, x_i^*) \in GrM$ converging to $(\bar{x}, x^*)$ with respect to the
$\| \; \| \times w^*-$topology.
One also has (see Ioffe [9])

$$\partial_A f(\bar{x}) = \bigcap_{L \in \mathcal{F}(X)} \limsup_{x \to^f \bar{x}, \epsilon \to 0^+} \partial_\epsilon^- f_{x+L}(x) \tag{1.2}$$

In the next proposition we are going to characterize the approximate subdifferential in terms of limits of some Fréchet subgradients. Recall (see Ioffe [10] and Mordukhovich and Shao [14]) that the Fréchet $\epsilon-$subgradient set $\partial_\epsilon^F(x)$ of f at x is the set of all $x^* \in X^*$ for which there exists somme $r > 0$ such that for every $u \in x + rI\!B_X$

$$< x^*, u - x > + f(x) \leq f(u) + \epsilon \| u - x \| .$$

If $\epsilon = 0$, one denotes $\partial^F f(x)$.

It is worth noting that the approximate subdifferential and normal cone used in the paper are infinite-dimensional extensions of the Mordukhovich subdifferential introduced in the paper [12], and the sequential extension of these constructions (called sometimes limiting Frechet normal cone and subdifferential) were first developed in the paper [13]. The other line of infinite-dimensional extensions of [12] was developed by Ioffe in [8], [9] and [10] .

1.1 Proposition. Let $f : X \to \overline{I\!R}$ be a lower semicontinuous function with $| f(\bar{x}) | < \infty$. Then
i)

$$\partial_A f(\bar{x}) = \bigcap_{L \in \mathcal{F}(X)} \limsup_{x \to^f \bar{x}, \epsilon \to 0_+} \partial^F_\epsilon f_{x+L}(x)$$

ii) and if f is k-Lipschitz near $\bar{x}$, then

$$\partial_A f(\bar{x}) = \bigcap_{L \in \mathcal{F}(X)} \limsup_{x \to \bar{x}, \epsilon \to 0^+} [\partial^F_\epsilon f_{x+L}(x) \cap (1+k) I\!B_{X^*}]$$

2. Subdifferentials of Lipschitz marginal functions.

In this section, we consider the marginal function

$$m(x) := \inf\{g(y) : y \in G(x)\}$$

where $g : Y \mapsto I\!R$ is a real valued function and G is a multivalued mapping from X into a Banach space Y. We will denote the graph of G by $GrG := \{(x, y) \in X \times Y : y \in G(x)\}$. It is known that a lot of apparently different types of marginal functions can be reduced to this form. In Hiriart-Urruty [5], Thibault [17] and references therein (for examples) one can find some reductions and several applications to the study of optimization problems.

The subdifferential that will be considered below is the (Ioffe) approximate subdifferential recalled in the first section. One of the important properties of this subdifferential appears in the fact that each of its elements is some limit of some Fréchet subgradients. The important role that proximal or Fréchet subgradients can play in the study of optimal value functions has been noticed for the first time (to the best of our knowledge) by Rockafellar (see his papers on the subject in the references of [15]).

Recall (see Ioffe [9] where the term "nucleus" is added) that the approximate coderivative $D_A^*G(\bar{x},\bar{y})$, for $\bar{y} \in G(\bar{x})$, is the multivalued mapping from Y^* into X^* defined by

$$x^* \in D_A^*G(\bar{x},\bar{y})(y^*) \Leftrightarrow (x^*,-y^*) \in I\!R_+\partial_A d(.;GrG)(\bar{x},\bar{y}).$$

We will write $\partial_A d(\bar{x},\bar{y};GrG)$ in place of $\partial_A d(.;GrG)(\bar{x},\bar{y})$. Here $d(.,S)$ denotes the distance function to a set S.

Note that, for $S \subset X$ and $S_{3r} := S \cap (\bar{x} + 3rI\!B_X)$ with $r > 0$ and $\bar{x} \in S$ it is not difficult to see that for $x \in \bar{x} + rI\!B_X$

$$d(x,S) = d(x,S_{3r})$$

and hence

$$\partial_A d(\bar{x},S) = \partial_A d(\bar{x},S_{3r}). \tag{2.1}$$

The following theorems are the main results of the paper.

2.1 <u>Theorem</u>. Suppose that g is locally Lipschitz and m is finite and locally $\gamma-$Lipschitz near $\bar{x}$. Then there exist nets $(x_i,y_i) \to (\bar{x},\bar{y})$ with $y_i \in G(x_i)$ and $g(y_i) \to m(\bar{x})$, $\epsilon_i \downarrow 0$, $y_i^* \in \partial_A g(y_i) + \epsilon_i I\!B_{Y^*}$, $x_i^* \in D_A^*G(x_i,y_i)(y_i^*)$ such that with respect to the weak-star topology

$$x^* = \lim x_i^*.$$

2.2 <u>Theorem</u>. Suppose that m is finite and $\gamma-$Lipschitz near $\bar{x}$ and there exists some neighborhood V of $\bar{x}$ such that g is $\beta-$Lipschits over some neighborhood of $G(V)$. Then there exists $\lambda > 0$ such that

$$\partial_A m(\bar{x}) \subset \{x^* \in X^* : \exists y^* \in \limsup_{g(y)\to m(\bar{x})} \partial_A g(y) \qquad \text{with}$$

$$(x^*,-y^*) \in \lambda \limsup_{x\to\bar{x},g(y)\to m(\bar{x}),y\in G(x)} \partial_A d(x,y,GrG)\}$$

and hence

$$\partial_A m(\bar{x}) \subset \bigcup\{ \limsup_{x\to\bar{x},g(y)\to m(\bar{x}),y\in G(x)} D_A^*G(\bar{x},\bar{y})(y^*) : y^* \in \limsup_{g(y)\to m(\bar{x})} \partial_A g(y)\}.$$

2.3 Corollary. Suppose that there exists some neighborhood V of $\bar{x}$ such that g is β-Lipschitz over some neighborhood of $G(V)$ and G is α-Lipschitz over V, that is for all x_1, x_2 in V

$$G(x_1) \subset G(x_2) + \alpha \parallel x_1 - x_2 \parallel \mathbb{B}_X.$$

Then, the conclusions of theorem 2.2 hold.

Before stating the next corollary, we recall that G is pseudo-Lipschitz at $(\bar{x}, \bar{y}) \in GrG$ if (see Aubin [2]) there exist $\alpha > 0, r > 0, s > 0$ such that for any x_1, x_2 in a neighborhood of $\bar{x}$

$$G(x_1) \cap (\bar{y} + s\mathbb{B}_Y) \subset G(x_2) + r \parallel x_1 - x_2 \parallel \mathbb{B}_X. \tag{2.10}$$

Rockafellar [16] showed that G is pseudo-Lipschitz at $(\bar{x}, \bar{y}) \in GrG$ iff $d(., G(.))$ is Lipschitz over a neighborhood of $(\bar{x}, \bar{y})$.
We can now state this second corollary which is in the line of some results in Thibault [17] and Jourani and Thibault [11].

2.4 Corollary. Let G be a multivalued mapping between X and Y which is pseudo-Lipschitz at $(\bar{x}, \bar{y}) \in GrG$. Then there exists some $\lambda > 0$ such that

$$\partial_A d(\bar{y}, G(.))(\bar{x}) \subset \bigcup_{y^* \in \boldsymbol{B}_{Y^*}} \{x^* \in X^* : (x^*, -y^*) \in \lambda \partial_A d(.; GrG)(\bar{x}, \bar{y})\}.$$

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