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On the uniqueness theorem for one nonstationary 3-D inverse heat conductivity problem in a layered domain

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Abstract: In this work, we consider the inverse 3-D heat transfer problem with any fixed $n \in \mathbb{N}$ discontinuous heat conductivity coefficients. At first with the help of conservative averaging method, we reduce the considered problem to the 1-D inverse problems. Then we prove the uniqueness theorem for obtained 1-D inverse problem.

Key-Words: Heat transfer problem, inverse problem, conservative averaging

1 Mathematical formulation of the initial problem

In this paper we consider the 3-D problem for the function $U(x,y,z,t)$ in the parallelepiped $D \overset{def}{\equiv} \{(x,y,z) \in \mathbb{R}^3 : x \in [0,x_n], y \in [0,Y], z \in [0,Z]\}$.

Let us the function $U(x,y,z,t)$ satisfies the following conditions and heat conductivity equation:

$$U_t(x,y,z,t) = k_j \cdot \Delta U(x,y,z,t),\ x \in (x_{j-1},x_j),\ y \in (0,Y),\ z \in (0,Z),\ x_0 = 0,\ j = \overline{1,n},\ t > 0, \quad (1)$$

$$U(x,y,z,t)\big|_{t=0} = 0, \quad (2)$$

$$U_x(x,y,z,t)\big|_{x=0} = 0, \quad (3)$$

$$\left[U(x,y,z,t) + C \cdot k_n \cdot U_x(x,y,z,t)\right]\big|_{x=x_n} = \nu(t), \quad (4)$$

$$U(x,y,z,t)\big|_{x=x_{j-0}} = U(x,y,z,t)\big|_{x=x_{j+0}},\quad j = \overline{1,n-1}, \quad (5)$$

$$k_j \cdot U(x,y,z,t)\big|_{x=x_{j-0}} = k_{j+1} \cdot U(x,y,z,t)\big|_{x=x_{j+0}},\quad j = \overline{1,n-1}, \quad (6)$$

$$U_y(x,y,z,t)\big|_{y=0} = U_y(x,y,z,t)\big|_{y=Y} = 0, \quad (7)$$

$$U_z(x,y,z,t)\big|_{z=0} = U_z(x,y,z,t)\big|_{z=Z} = 0. \quad (8)$$

From the problem (1)-(8) we have to determine the following data:

- the number $n \in N$ that means a number of discontinuity of heat conduction coefficient;
- the discontinuity points $x_j \quad (j = \overline{1,n-1})$;
- the heat conduction coefficients $k_j \quad (j = \overline{1,n})$;
- the constant C;
- the function $U(x,y,z,t)$, where $(x,y,z) \in D,\ t > 0$.

In addition, it is proposed that we have the some additional data. These data will be formulated in the next section of the present paper.

Now let us introduce the function (as in the conservative averaging method, see [9]-[11] and references there)

$$u(x,t) \overset{def}{\equiv} \frac{1}{Y \cdot Z} \cdot \int_0^Y dy \int_0^Z U(x,y,z,t)dz,$$

and we will integrate the basis equation (1), the boundary conditions (3)-(6) and also the initial condition (2) on variables y and z. Then we will use the conditions (7), (8) in obtained expression. For example, after above-mentioned procedure from (1) we get

$$u_t(x,t) = k_j \cdot u_{xx}(x,t) + \frac{1}{Y \cdot Z}\left\{\int_0^Z \frac{\partial U}{\partial y}\bigg|_{y=0}^{y=Y} dz + \int_0^Y \frac{\partial U}{\partial z}\bigg|_{z=0}^{z=Z} dy\right] = k_j \cdot u_{xx}(x,t),$$

from (2) we have

$$U(x,y,z,t)\big|_{t=0} = 0 \Rightarrow u(x,t)\big|_{t=0} = \frac{1}{Y \cdot Z} \cdot \int_0^Y dy \int_0^Z U(x,y,z,t)dz\big|_{t=0} =$$

$$= \frac{1}{Y \cdot Z} \cdot \int_0^Y dy \int_0^Z U(x,y,z,t)\big|_{t=0} dz = 0, \Rightarrow u(x,t)\big|_{t=0} = 0.$$

from (6) we receive

$$k_j \cdot \frac{1}{Y \cdot Z} \cdot \left[\int_0^Y dy \int_0^Z U_x(x,y,z,t)dz\right] = k_j \cdot \frac{\partial}{\partial x}\left[\frac{1}{Y \cdot Z} \cdot \int_0^Y dy \int_0^Z U(x,y,z,t)dz\right] =$$

$$= k_j \cdot \frac{\partial u(x,t)}{\partial x} \Rightarrow k_j \cdot \frac{\partial u(x,t)}{\partial x}\bigg|_{x=x_j-0} = k_{j+1} \cdot \frac{\partial u(x,t)}{\partial x}\bigg|_{x=x_j+0}$$

and so on. As the final result, we receive the formulation of the 1-D problem (9)-(14), which we will consider in detail in the next section also.

Here, looking ahead, it should be noted that the obtained problem (9)-(14) is a exact (equivalent) statement of considered in previous section problem, and the problem (9)-(14) is not an approximate statement of initial 3-D problem.

Saying applied point of view we can interchange the 3-D problem by our 1-D problem, where the precision of this replacement (exchange) will depend on the precision of replacement of the boundary condition (4) (where in the general case $\nu = \nu(y,z,t)$ and $C = \dfrac{1}{h(y,z)}$) by some conditions, more exactly, by functions, which is not depend on both space coordinates.

As is known there are many real practical problems where mentioned approximate is generally accepted. If we will consider hydrodynamic problems where our layered body can be interpreted as a lamellate solid, where there is a fluid flow, then we can neglect a dependence of a heat change, which is an inverse value to a conditional thickness of the boundary layer (see [1] and references there). But in this case it is necessary to remember that such neglect is possible depending on the concrete conditions of considered real physical problem, for example, depending on a characteristic sizes (on Y, Z) and so on, but we have not to abused such neglect.

2 Formulations of obtained 1-D direct and corresponding 1-D inverse problem

So let us consider the following boundary problem for heat equation with the piecewise coefficient of heat conduction:

$$u_t(x,t) = k_j \cdot u_{xx};\quad x \in (x_{j-1}, x_j);\quad j = \overline{1,n};\quad x_0 = 0;\quad t > 0 \quad (9)$$

$$u(x,t)\big|_{t=0} = 0,\quad x \in \left[0 = x_0, x_n\right], \quad (10)$$

$$u_x(x,t)\big|_{x=0} = 0,\quad t > 0, \quad (11)$$

$$u(x,t)\big|_{x=x_n} + C \cdot k_n \cdot u_x(x,t)\big|_{x=x_n} = \nu(t),\quad t > 0, \quad (12)$$

where $0 < C < \infty$,

$$u(x,t)\big|_{x=x_j-0} = u(x,t)\big|_{x=x_j+0};\quad j = \overline{1,n-1};\quad t > 0, \quad (13)$$

$$k_j \cdot u_x(x,t)\big|_{x=x_j-0} = k_{j+1} \cdot u_x(x,t)\big|_{x=x_j+0};\ j = \overline{1,n-1}; t > 0. \quad (14)$$

In the problem (9)-(14) we assume that

- the function $\nu(t) \in C^2\left[0,+\infty\right)$ satisfies the condition $\left|\nu''(t)\right| \le \text{const}$ as $t \ge 0$;
- $\nu(0) = 0$;
- $k(x) = k_j > 0 \quad x \in \left[x_{j-1}, x_j\right],\quad j = \overline{1,n}.$

The direct problem (9)-(14) should be determined identically with the help of following initial data:

- the function $\nu(t)$;
- the constant C;
- the number $n \in N$;
- the parameters $k_j\ (j = \overline{1,n})$ and $x_j\ (j = \overline{1,n})$.

Now we will formulate the following corresponding inverse problem: having the following a priori information

- the length of segment (where is considered this problem) is known, i.e. x_n is known;
- the function $\nu(t),\ t \ge 0$ is known;

from the (9)-(14) we have to find the following unknown information:

- the number $n \in N$ that means a number of discontinuity of heat conduction coefficient;
- the discontinuity points $x_j\ (j = \overline{1,n-1})$;
- the heat conductivity coefficients $k_j\ (j = \overline{1,n})$;
- the constant C;
- the function $u(x,t)$ when $(x,t) \in \left[0, x_n\right] \times \left[0,+\infty\right]$;
- the additional information

$$u(x,t)\big|_{x=0} = T(t),\ t \ge 0; \quad (15)$$

the following important additional suppositions:

$$u_t(x,t) \in C\left\{\left[x_{j-1}, x_j\right] \times \left[0,+\infty\right)\right\},\ j = \overline{1,n}; \quad (16)$$

$$u_{xx}(x,t) \in C\left\{\left[x_{j-1}, x_j\right] \times \left[0,+\infty\right)\right\},\quad j = \overline{1,n}; \quad (17)$$

$$x_{j-1} < x_j,\ j = \overline{1,n}; \quad (18)$$

$$k_{j-1} \ne k_j,\quad j = \overline{2,n}; \quad (19)$$

$$u(x,t) \in C\left\{\left[0, x_n\right] \times \left[0,+\infty\right)\right\}; \quad (20)$$

$$u_x(x,t) \in C\left\{\left[x_{j-1}, x_j\right] \times \left[0,+\infty\right)\right\},\quad j = \overline{1,n}; \quad (21)$$

$$\left|u(x,t)\right| \le A_1 \cdot e^{A_2 \cdot t},\ x \in \left[0, x_n\right],\ t > 0,\ A_1 \text{ - const}; \quad (22)$$

$$\left|u_{xx}(x,t)\right| \le A_1 e^{A_2 t}, x \in (x_{j-1}, x_j), j = \overline{1,n}; t > 0, \quad (23)$$

A_2 - const.

Definition. The solution of the inverse problem (9)-(23) is called the number $n \in N$, $2 \cdot n$ parameters $\left\{x_j\left(j = \overline{1,n-1}\right); k_j\left(j = \overline{1,n}\right),\ C\right\}$ and the function $u(x,t)$ satisfying conditions (9)-(23).

The following theorem establishes the uniqueness of the solution of formulated above inverse problem (9)-(23).

3 Uniqueness theorem for the solution of the formulated inverse problem

Theorem 1. If two sets $\left\{u(x,t);\ n \in N;\ x_j\left(j = \overline{1,n-1}\right);\ k_j\left(j = \overline{1,n}\right);\ C\right\}$ and $\left\{\tilde{u}(x,t);\ m \in N;\ \tilde{x}_j\left(j = \overline{1,m-1}\right);\ \tilde{k}_j\left(j = \overline{1,m}\right);\ \tilde{C}\right\}$ are the solutions of the inverse problem (9)-(23), and if $x_n = \tilde{x}_m$ then the following statements are correct:

$$\left.\begin{array}{l} -\ n = m; \\ -\ x_j = \tilde{x}_j\ \left(j = \overline{1,n-1}\right); \\ -\ k_j = \tilde{k}_j\ \left(j = \overline{1,n}\right); \\ -\ C = \tilde{C}; \\ -\ u(x,t) = \tilde{u}(x,t),\quad (x,t) \in \left[0, x_n\right] \times \left[0,+\infty\right). \end{array}\right\} \quad (24)$$

Before to prove this theorem, we will try to interpret the physical significance of this theorem.

4 A physical interpretation of the main Theorem 1

Usually Newton boundary condition (so-called the third type boundary condition) for heat equation has a following form (see [2], [3]):

$$\left[k_n \cdot u_x + h \cdot u\right]\Big|_{x=x_n} = h \cdot \theta, \qquad (25)$$

where h is a coefficient of heat exchange on Newton, and θ is a temperature of environment. It means that the condition (12)

$$u(x,t)\big|_{x=x_n} + C \cdot k_n \cdot u_x(x,t)\big|_{x=x_n} = \nu(t), \quad t > 0, \quad 0 < C < \infty$$

coincides with the condition (25) if we will take $C = \frac{1}{h}$ and $\nu(t) = \theta(t)$. Then the physical matter of the Theorem 1 is following:

- If we change a constant $C = \frac{1}{h}$ (inversely proportional to the coefficient of heat exchange) and if we change backwards a temperature of environment as many the same then we can change no thermal or geometric properties of given layered material;
- If we cannot change (there are different technical, physical and others reasons for it) any characteristics of our layered material then changing the characteristics of heat exchange we have to change a temperature of environment inversely proportionally.

5 Proof of the Theorem 1

Applying the Laplace transform

$$U(x,p) = \int_0^{+\infty} e^{-p \cdot t} \cdot u(x,t)dt$$

to the equalities (9), (11)-(14) we get that the function $U(x,p)$ is the solution of following problem:

$$pU(x,p) - k_j U''_x(x,p) = 0; x \in (x_{j-1}, x_j); j = \overline{1,n}; x_0 = 0, \qquad (26)$$

$$U'(x,p)\big|_{x=0} = 0, \qquad (27)$$

$$U(x,p)\big|_{x=x_n} + C \cdot k_n \cdot U'_x(x,p)\big|_{x=x_n} = \overline{\nu}(p), \qquad (28)$$

where

$$\overline{\nu}(p) \overset{def}{\equiv} \int_0^{+\infty} e^{-p \cdot t} \cdot \nu(t)dt\,,$$

$$U(x,p)\big|_{x=x_j-0} = U(x,p)\big|_{x=x_j+0}; \quad j = \overline{1,n-1}, \qquad (29)$$

$$k_j \cdot U'(x,p)\big|_{x=x_j-0} = k_{j+1} \cdot U'(x,p)\big|_{x=x_j+0}; \quad j = \overline{1,n-1}. \qquad (30)$$

Thus we have the problem (26)-(30), which decomposes into the following interconnection subproblems under each fixed $j = \overline{1,n}$. Moreover the condition (27) involves only in the first subproblem (i.e. under $j = 1$), and the condition (28) involves only in the last subproblem (i.e. under $j = n$), but the conjugating conditions (29), (30) involve in all subproblems except for the last subproblem (i.e. under $j = \overline{1,n-1}$).

Subproblem 1 ($j = 1$):

$$\begin{cases} p \cdot U(x,p) = k_1 \cdot U''_x(x,p), \quad x \in (x_0, x_1), \\ U'_x(0,p) = 0, \\ U(x_1 - 0, p) = U(x_1 + 0, p), \\ k_1 \cdot U'_x(x_1 - 0, p) = k_2 \cdot U'_x(x_1 + 0, p). \end{cases}$$

Subproblem 2 ($j = 2$):

$$\begin{cases} p \cdot U(x,p) = k_2 \cdot U''_x(x,p), \quad x \in (x_1, x_2), \\ U(x_2 - 0, p) = U(x_2 + 0, p), \\ k_2 \cdot U'_x(x_2 - 0, p) = k_3 \cdot U'_x(x_2 + 0, p). \end{cases}$$

Subproblem $n-1$ ($j = n-1$):

$$\begin{cases} p \cdot U(x,p) = k_{n-1} \cdot U''_x(x,p), \quad x \in (x_{n-2}, x_{n-1}), \\ U(x_{n-1} - 0, p) = U(x_{n-1} + 0, p), \\ k_{n-1} \cdot U'_x(x_{n-1} - 0, p) = k_n \cdot U'_x(x_{n-1} + 0, p). \end{cases}$$

Subproblem n ($j = n$):

$$\begin{cases} p \cdot U(x,p) = k_n \cdot U''_x(x,p), \quad x \in (x_{n-1}, x_n), \\ U(x_n, p) + C \cdot k_n \cdot U'_x(x_n, p) = \overline{\nu}(p). \end{cases}$$

The solution of the subproblem 1 is

$$U_1(x,p) = B_1^1 \cdot \left[e^{\sqrt{\frac{p}{k_1}} \cdot x} + e^{-\sqrt{\frac{p}{k_1}} \cdot x} \right],$$

where B_1^1 is a yet unknown constant.

The solution of the subproblem 2 is

$$U_2(x,p) = B_1^2 \cdot e^{\sqrt{\frac{p}{k_2}} \cdot x} + B_2^2 \cdot e^{-\sqrt{\frac{p}{k_2}} \cdot x},$$

where B_1^2 and B_2^2 are a yet unknown constants, and they have a connection with the constant B_1^1 by the following relation:

$$B_1^2 = e^{-\sqrt{\frac{p}{k_2}} \cdot x_1} \cdot B_1^1 \cdot \left[ch\left(\sqrt{\frac{p}{k_1}} \cdot x_1\right) + \sqrt{\frac{k_1}{k_2}} \cdot sh\left(\sqrt{\frac{p}{k_1}} \cdot x_1\right) \right],$$

$$B_2^2 = e^{\sqrt{\frac{p}{k_2}} \cdot x_1} \cdot B_1^1 \cdot \left[ch\left(\sqrt{\frac{p}{k_1}} \cdot x_1\right) - \sqrt{\frac{k_1}{k_2}} \cdot sh\left(\sqrt{\frac{p}{k_1}} \cdot x_1\right) \right].$$

Thus we can determine the solutions of all subproblems $3 - n$, but this way is too difficult and for this it is necessary more computations to find a corresponding constants, which involve in the solutions of each subproblem. Besides the relation connecting these constants with the constant B_1^1 will become complicated after passage from the present subproblem to the next subproblem. Therefore having the initial problem (26)-(30) we will use another way, precisely from (26)-(30) we will select the following problem for consideration in detail.

Let us we have the function $\mathrm{V}(x,p)$ that is the solution of the problem (26), (27), (29), (30), i.e. we have our initial problem without condition (28), but in addition we suppose that

$$\mathrm{V}(0,p)=1. \qquad (31)$$

It is not difficult to show that

$$\mathrm{V}(x_n,p)+C\cdot k_n\cdot \mathrm{V}'_x(x_n,p)\neq 0 \quad \text{if} \quad \operatorname{Re} p\geq 0.$$

Since the functions $U(x,p)$ and $\mathrm{V}(x,p)$ satisfy the condition (27) then we can write

$$\mathrm{V}(x,p)=U(0,p)\cdot \mathrm{V}(x,p).$$

Then from (28) we get

$$U(0,p)=\frac{\overline{\nu}(p)}{\mathrm{V}(x_n,p)+C\cdot k_n\cdot \mathrm{V}'_x(x_n,p)}.$$

Since (see to the condition of the Theorem 1) the function $\tilde{u}(x,t)$ is also the solution of the inverse problem (9)-(15) then if we realize the similar transform for the function $\tilde{u}(x,t)$ we receive

$$\tilde{U}(0,p)=\frac{\overline{\nu}(p)}{\tilde{\mathrm{V}}(\tilde{x}_m,p)+\tilde{C}\cdot \tilde{k}_m\cdot \mathrm{V}'_x(\tilde{x}_m,p)},$$

where $x_n=\tilde{x}_m$ and the functions $\tilde{U}(x,p)$, $\tilde{\mathrm{V}}(x,p)$ are determined by analogy with the accordingly functions $U(x,p)$, $\mathrm{V}(x,p)$.

From the additional information (15) it follows that $U(0,p)=\tilde{U}(0,p)$. Therefore we can write

$$\mathrm{V}(x_n,p)+Ck_nV'_x(x_n,p)=\tilde{\mathrm{V}}(\tilde{x}_m,p)+\tilde{C}\tilde{k}_m\,\tilde{\mathrm{V}}'_x(\tilde{x}_m,p). \qquad (32)$$

It is clear that the relation (32) is correct for all values of the parameter p, because all functions $\mathrm{V}(x,p)$, $\mathrm{V}'_x(x,p)$, $\tilde{\mathrm{V}}(\tilde{x},p)$ and $\tilde{\mathrm{V}}'_x(\tilde{x},p)$ are an integer functions under the fixed variables x, $\tilde{x}$. We will consider in detail the function $\mathrm{V}(x,p)$ only under real and negative values of the parameter p, i.e. under $\mathbb{R}\ni p=-\lambda$, $\lambda>0$.

Let us designate by $B^1_j(\lambda)$ and $B^2_j(\lambda)$ the constants defining the function $\mathrm{V}(x,\lambda)$ in the interval (x_{j-1},x_j). The defined above function $\mathrm{V}(x,\lambda)$ is the solution of the problem (26), (27), (29), (30), and it has the following form:

$$\mathrm{V}(x,\lambda)=B^1_j(\lambda)\cdot e^{i\cdot\sqrt{\frac{\lambda}{k_j}}\cdot(x_j-x_{j-1})}+B^2_j(\lambda)\cdot e^{-i\cdot\sqrt{\frac{\lambda}{k_j}}\cdot(x_j-x_{j-1})}.$$

Now taking into account in this expression the formulae (29) and (30) we receive the following system with respect to unknown constants B^1_{j+1} and B^2_{j+1}

$$\sqrt{k_{j+1}}\left[B^1_{j+1}(\lambda)-B^2_{j+1}(\lambda)\right]=\sqrt{k_j}\left[B^1_j(\lambda)e^{i\sqrt{\frac{\lambda}{k_j}}(x_j-x_{j-1})}-B^2_j(\lambda)e^{-i\sqrt{\frac{\lambda}{k_j}}(x_j-x_{j-1})}\right],$$

$$B^1_{j+1}(\lambda)+B^2_{j+1}(\lambda)=B^1_j(\lambda)e^{i\cdot\sqrt{\frac{\lambda}{k_j}}(x_j-x_{j-1})}+B^2_j(\lambda)e^{-i\cdot\sqrt{\frac{\lambda}{k_j}}(x_j-x_{j-1})},$$

or introducing designations

$$\begin{cases}\xi_j=\dfrac{x_j-x_{j-1}}{\sqrt{k_j}}, & j=\overline{1,n},\\[2ex] \eta_j=\sqrt{\dfrac{k_j}{k_{j+1}}}, & j=\overline{1,n-1},\end{cases}$$

we can rewrite this system in the following equivalent form:

$$\begin{cases}B^1_{j+1}(\lambda)+B^2_{j+1}(\lambda)=B^1_j(\lambda)\cdot e^{i\cdot\sqrt{\lambda}\cdot\xi_j}+B^2_j(\lambda)\cdot e^{-i\cdot\sqrt{\lambda}\cdot\xi_j},\\[2ex] B^1_{j+1}(\lambda)-B^2_{j+1}(\lambda)=\eta_j\cdot\left[B^1_j(\lambda)\cdot e^{i\cdot\sqrt{\lambda}\cdot\xi_j}-B^2_j(\lambda)\cdot e^{-i\cdot\sqrt{\lambda}\cdot\xi_j}\right].\end{cases}$$

Solving this system relatively $B^1_{j+1}(\lambda)$ and $B^2_{j+1}(\lambda)$ we will get

$$\begin{cases}B^1_{j+1}(\lambda)=\left(\dfrac{1+\eta_j}{2}e^{i\sqrt{\lambda}\xi_j}\right)B^1_j(\lambda)+\left(\dfrac{1-\eta_j}{2}e^{-i\sqrt{\lambda}\xi_j}\right)B^2_j(\lambda),\\[2ex] B^2_{j+1}(\lambda)=\left(\dfrac{1-\eta_j}{2}e^{i\sqrt{\lambda}\xi_j}\right)B^1_j(\lambda)+\left(\dfrac{1+\eta_j}{2}e^{-i\sqrt{\lambda}\xi_j}\right)B^2_j(\lambda).\end{cases} \qquad (33)$$

From the conditions $\mathrm{V}'_x(0,p)=0$, $\mathrm{V}(0,p)=1$ and also from (33) we receive that

$$\left.\begin{array}{l}B^1_1(\lambda)=B^2_1(\lambda)=\dfrac{1}{2},\\[1ex] B^2_j(\lambda)=B^1_j(\lambda), \quad j=\overline{1,n}.\end{array}\right\} \qquad (34)$$

Conversely from (33) and (34) we get

$$\begin{cases}B^1_j(\lambda)=\dfrac{1}{2}\cdot\displaystyle\sum_{l=1}^{N(j)}\theta_l(j)\cdot e^{i\cdot\sqrt{\lambda}\cdot\tau_l(j)},\\[2ex] B^2_j(\lambda)=\dfrac{1}{2}\cdot\displaystyle\sum_{l=1}^{N(j)}\theta_l(j)\cdot e^{-i\cdot\sqrt{\lambda}\cdot\tau_l(j)},\end{cases}$$

where

$N(j)\leq 2^{j-1}$;

$\tau_{l_1}(j)\neq\tau_{l_2}(j)$ under $l_1\neq l_2$;

$\theta_l(j),\tau_l(j)\in\mathbb{R}^1$ and these coefficients are not depend on the parameter λ, and furthermore

$$\sum_{i=1}^{j-1}\xi_i=\max_{l=1,N(j)}\tau_l(j)$$

and

$$-\sum_{i=1}^{j-1}\xi_i=\min_{l=1,N(j)}\tau_l(j).$$

Now we will take into account these representations of constants $B_j^1(\lambda)$ and $B_j^2(\lambda)$ from (35) in the expression for the function $V(x,p)$, $x\in[x_{n-1},x_n]$ we get

$$V(x,\lambda)=\frac{1}{2}\sum_{l=1}^{N(n)}\theta_l(n)e^{\sqrt{\frac{\lambda}{k_n}}(x-x_{n-1})+i\sqrt{\lambda}\tau_l(n)}+$$

$$+\frac{1}{2}\sum_{l=1}^{N(n)}\theta_l(n)e^{-\sqrt{\frac{\lambda}{k_n}}(x-x_{n-1})-i\sqrt{\lambda}\tau_l(n)}.$$

Now we can calculate the expression $V(x_n,\lambda)+C\cdot k_n\cdot V_x'(x_n,\lambda)$:

$$V(x_n,\lambda)+Ck_n V_x'(x_n,\lambda)=\sum_{l=1}^{N(n)}\theta_l(n)\cos\left[\sqrt{\lambda}(\xi_n+\tau_l(n))\right]- \tag{36}$$

$$-\sqrt{k_n\lambda}C\sum_{l=1}^{N(n)}\theta_l(n)\sin\left[\sqrt{\lambda}(\xi_n+\tau_l(n))\right].$$

Absolutely analogously computing the expression $\tilde{V}(\tilde{x}_m,\lambda)+\tilde{C}\cdot\tilde{k}_m\cdot\tilde{V}_x'(\tilde{x}_m,\lambda)$, where $\tilde{k}_m=k_n$, we get

$$\tilde{V}(\tilde{x}_m,\lambda)+\tilde{C}\tilde{k}_m\tilde{V}_x'(\tilde{x}_m,\lambda)=\sum_{l=1}^{\tilde{N}(m)}\tilde{\theta}_l(m)\cos\left[\sqrt{\lambda}(\tilde{\xi}_m+\tilde{\tau}_l(m))\right]-$$

$$-\sqrt{\tilde{k}_m\cdot\lambda}\cdot\tilde{C}\cdot\sum_{l=1}^{\tilde{N}(m)}\tilde{\theta}_l(m)\cdot\sin\left[\sqrt{\lambda}\cdot(\tilde{\xi}_m+\tilde{\tau}_l(m))\right]. \tag{37}$$

Here the numbers $\tilde{\theta}_l(m)$, $\tilde{\tau}_l(m)$, $\tilde{\xi}_m$ and $\tilde{N}(m)$ have a similar to the numbers $\theta_l(n)$, $\tau_l(n)$, ξ_n and $N(n)$ interpretations.

Now let us return to the equality (32):

$V(x_n,p)+Ck_nV_x'(x_n,p)=\tilde{V}(\tilde{x}_m,p)+\tilde{C}\tilde{k}_m\tilde{V}_x'(\tilde{x}_m,p)$, $\tilde{x}_m=x_n$.

In this equality that is a relation between the integer functions $V(x,p)$, $V_x'(x,p)$ and corresponding functions $\tilde{V}(x,p)$, $\tilde{V}_x'(x,p)$ at the fixed points $x=x_n$ and $x=\tilde{x}_m$, considering the formulae (36) and (37) we get finally

$$\sum_{l=1}^{N(n)}\theta_l(n)\cos\left[\sqrt{\lambda}(\xi_n+\tau_l(n))\right]-\sqrt{\lambda k_n}C\sum_{l=1}^{N(n)}\theta_l(n)\times$$

$$\times\sin\left[\sqrt{\lambda}(\xi_n+\tau_l(n))\right]=\sum_{l=1}^{\tilde{N}(m)}\tilde{\theta}_l(m)\cdot\cos\left[\sqrt{\lambda}\cdot(\tilde{\xi}_m+\tilde{\tau}_l(m))\right]-$$

$$-\sqrt{\lambda\cdot\tilde{k}_m}\cdot\tilde{C}\cdot\sum_{l=1}^{\tilde{N}(m)}\tilde{\theta}_l(m)\cdot\sin\left[\sqrt{\lambda}\cdot(\tilde{\xi}_m+\tilde{\tau}_l(m))\right]. \tag{38}$$

It is clear that without loss of generality we can suppose the summation index in formula (38) is taken ascending order for $\tau_l(n)$ and $\tilde{\tau}_l(m)$. Then from (38) it follows that $N(n)=\tilde{N}(m)$ and also

$$C\cdot\sqrt{k_n}=\tilde{C}\cdot\sqrt{\tilde{k}_m}, \tag{39}$$

$$\tau_l(n)+\xi_n=\tilde{\tau}_l(m)+\tilde{\xi}_m, \tag{40}$$

$$\theta_l(n)=\tilde{\theta}_l(m). \tag{41}$$

Let $l=1$. Then from (40) we have $\tau_1(n)+\xi_n=\tilde{\tau}_1(m)+\tilde{\xi}_m$.

Now let $l=N(n)=\tilde{N}(m)$. Then from (40) we have $\tau_{N(n)}(n)+\xi_n=\tilde{\tau}_{N(n)}(m)+\tilde{\xi}_m$.

From these two equalities we get that $\xi_n=\tilde{\xi}_m$. Taking into account this statement in (40) we obtain

$$\tau_l(n)=\tilde{\tau}_l(m),\qquad l=\overline{1,N(n)}. \tag{42}$$

Now we will designate as $\tilde{C}_i^1(\lambda)$ and $\tilde{C}_i^2(\lambda)$ the constants determining the function $\tilde{V}(x,\lambda)$ at the segment $[\tilde{x}_{i-1},\tilde{x}_i]$. Then from (41) and (42) we get that

$C_n^1(\lambda)=\tilde{C}_m^1(\lambda)$ and $C_n^2(\lambda)=\tilde{C}_m^2(\lambda)$.

From the conjugate conditions (29), (30) for the function $\tilde{V}(x,\lambda)$ and also from (39) we receive that

$$\begin{cases}V(x_{n-1}-0,\lambda)=C_n^1(\lambda)+C_n^2(\lambda)=\tilde{C}_m^1(\lambda)+\tilde{C}_m^2(\lambda)=\tilde{V}(\tilde{x}_{m-1}-0,\lambda);\\ C\cdot k_{n-1}\cdot V_x'(x_{n-1}-0,\lambda)=C\cdot\sqrt{k_n}\cdot i\cdot\sqrt{\lambda}\cdot\left[C_n^1(\lambda)-C_n^2(\lambda)\right]=\\ \quad=\tilde{C}\cdot\sqrt{\tilde{k}_m}\cdot i\cdot\sqrt{\lambda}\cdot\left[\tilde{C}_m^1(\lambda)-\tilde{C}_m^2(\lambda)\right]=\tilde{C}\cdot\tilde{k}_{m-1}\cdot\tilde{V}_x'(\tilde{x}_{m-1}-0,\lambda).\end{cases}$$

From here

$$V(x_{n-1}-0,\lambda)+C\cdot k_{n-1}\cdot V_x'(x_{n-1}-0,\lambda)=$$

$$=\tilde{V}(\tilde{x}_{m-1}-0,\lambda)+\tilde{C}\cdot\tilde{k}_{m-1}\cdot\tilde{V}_x'(\tilde{x}_{m-1}-0,\lambda).$$

We observe that this equality is absolutely identical with the equality (32). Then repeating previous calculations we can write

$C\cdot\sqrt{k_{n-1}}=\tilde{C}\cdot\sqrt{\tilde{k}_{m-1}}$, $\xi_{n-1}=\tilde{\xi}_{m-1}$, $C_{n-1}^1(\lambda)=\tilde{C}_{m-1}^1(\lambda)$, $C_{n-1}^2(\lambda)=\tilde{C}_{m-1}^2(\lambda)$.

Repeating this process still $(n-1)$ times we get finally the following three statements:

$$n=m, \tag{43}$$

$$C\cdot\sqrt{k_j}=\tilde{C}\cdot\sqrt{\tilde{k}_j},\quad \forall j=\overline{1,n}, \tag{44}$$

$$\xi_j=\tilde{\xi}_j,\quad \forall j=\overline{1,n}. \tag{45}$$

Further we will proceed on the following way: for each fixed $j=\overline{1,n}$ we will multiply the equality (44) by the equality (45). Then we will summarize obtained equality on $j=\overline{1,n}$: $\sum_{j=1}^{n}C\cdot\sqrt{k_j}\cdot\xi_j=\sum_{j=1}^{n}\tilde{C}\cdot\sqrt{\tilde{k}_j}\cdot\tilde{\xi}_j$.

Let us rewrite this obtained expression with regard to our designations

$$\xi_j=\frac{x_j-x_{j-1}}{\sqrt{k_j}}\ \left(j=\overline{1,n}\right);\ \tilde{\xi}_j=\frac{\tilde{x}_j-\tilde{x}_{j-1}}{\sqrt{\tilde{k}_j}}\ \left(j=\overline{1,m}\right);\ m=n.$$

Then we have $C\cdot x_n=\tilde{C}\cdot\tilde{x}_m$. Since $x_n=\tilde{x}_m$ consequently $C=\tilde{C}$. Then from (44) we get $k_j=\tilde{k}_j$, $j=\overline{1,n}$. Therefore from

$\xi_j=\frac{x_j-x_{j-1}}{\sqrt{k_j}}$ and $\tilde{\xi}_j=\frac{\tilde{x}_j-\tilde{x}_{j-1}}{\sqrt{\tilde{k}_j}}$ it follows that $x_j=\tilde{x}_j$ for each $\forall j=\overline{1,n}$.

Thus we have obtained $n = m$; $C = \tilde{C}$; $k_j = \tilde{k}_j \left(j = \overline{1,n} \right)$; $x_j = \tilde{x}_j$, $\left(j = \overline{1,n} \right)$.

From here $u(x,t) = \tilde{u}(x,t)$, $x \in \left[0, x_n\right]$ and $t \geq 0$.

The Theorem 1 is proved.

6 Some important notes

Remark. The condition $x_n = \tilde{x}_m$ in the Theorem 1 is very important, and we have not to ignore this information. The following example illustrates this fact.

Really, if we assume $n = m = 1$ (i.e. we have no any discontinuity points of heat conduction coefficient) we receive

$$V(x_1, p) + k_1 \cdot C \cdot V_x'(x_1, p) = \frac{1}{2} \cdot \left[e^{\sqrt{\frac{p}{k_1}} \cdot x_1} + e^{-\sqrt{\frac{p}{k_1}} \cdot x_1} \right] +$$

$$+ \frac{C}{2} \cdot \sqrt{p \cdot k_1} \cdot \left[e^{\sqrt{\frac{p}{k_1}} \cdot x_1} - e^{-\sqrt{\frac{p}{k_1}} \cdot x_1} \right].$$

From here it is obvious that the equality $\mathrm{V}(0, p) = \tilde{\mathrm{V}}(0, p)$ will be correct if we will suppose $x_1 = 1,\ k_1 = 1,\ C_1 = 1$ and $\tilde{x}_1 = 2,\ \tilde{k}_1 = 4,\ \tilde{C} = \frac{1}{2}$.

Theorem 2. The statement of the Theorem 1 is valid if we have the condition $u(x,t)\big|_{x=0} = 0, \quad t > 0$ instead of the condition (11), and also if we have another additional information $k_1 \cdot u_x(x,t)\big|_{x=0} = \nu(t), \quad t \geq 0$ instead of the condition (15).

The course and idea of the proofs of the Theorem 1 and the previous theorem are not differing essentially.

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