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Quantum Stochastic Calculus and Some Of Its Applications

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Abstract:-We describe the main features of the Hudson-Parthasarathy quantum stochastic calculus and some of its applications to systems control and, recently, to quantum economics.

Key-Words:-Quantum Stochastic Calculus, Quadratic Optimal Control, Algebraic Riccati equation, Option Pricing, Black-Scholes equation.

Proceedings of the 9th WSEAS International Conference on Applied Mathematics, Istanbul, Turkey, May 27-29, 2006 (pp509-517)

1 Quantum Stochastic Calculus

Let $B_t = \{B_t(\omega)/\,\omega \in \Omega\}$, $t \geq 0$, be one-dimensional classical Brownian motion. Integration with respect to B_t was defined by Itô. Stochastic integral equations of the form

$$X_t = X_0 + \int_0^t b(s, X_s)\, ds + \int_0^t \sigma(s, X_s)\, dB_s$$

are thought of as stochastic differential equations of the form

$$dX_t = b(t, X_t)\, dt + \sigma(t, X_t)\, dB_t$$

where differentials are handled with the use of Itô's formula

$$(dB_t)^2 = dt, \quad dB_t\, dt = dt\, dB_t = (dt)^2 = 0$$

In [5], Hudson and Parthasarathy defined a non-commutative analogue of classical Itô stochastic calculus on the Boson Fock space $\Gamma = \Gamma(L^2(\mathbb{R}_+, \mathcal{C}))$ over $L^2(\mathbb{R}_+, \mathcal{C})$ defined as the Hilbert space completion of the linear span of the exponential vectors $\psi(f)$ under the inner product

$$< \psi(f), \psi(g) >:= e^{<f,g>}$$

where $f, g \in L^2(\mathbb{R}_+, \mathcal{C})$ and

$$< f, g >= \int_0^{+\infty} \bar{f}(s)\, g(s)\, ds$$

where, here and in what follows, $\bar{z}$ denotes the complex conjugate of $z \in \mathbb{C}$. The annihilation, creation and conservation operators $A(f)$, $A^\dagger(f)$ and $\Lambda(F)$ respectively, are defined on the exponential vectors $\psi(g)$ of Γ by

$$A_t \psi(g) := \int_0^t g(s)\, ds\ \psi(g)$$

$$A_t^\dagger \psi(g) := \frac{\partial}{\partial \epsilon}|_{\epsilon=0}\, \psi(g + \epsilon \chi_{[0,t]})$$

$$\Lambda_t \psi(g) := \frac{\partial}{\partial \epsilon}|_{\epsilon=0}\, \psi(e^{\epsilon \chi_{[0,t]}} g).$$

The basic quantum stochastic differentials dA_t, $dA_t^\dagger$, and $d\Lambda_t$ are defined by

$$dA_t := A_{t+dt} - A_t$$

$$dA_t^\dagger := A_{t+dt}^\dagger - A_t^\dagger$$

and

$$d\Lambda_t := \Lambda_{t+dt} - \Lambda_t.$$

Hudson and Parthasarathy defined stochastic integration with respect to the noise differentials of Definition 3 and obtained the Itô multiplication table

$$dA_t\, dA_t^\dagger = dt$$

$$dA_t\, d\Lambda_t = dA_t$$

$$d\Lambda_t\, dA_t^\dagger = dA_t^\dagger$$

and

$$d\Lambda_t\, d\Lambda_t = d\Lambda_t$$

while all other products involving dt, dA_t, $dA_t^\dagger$, and $d\Lambda_t$ are equal to zero. The two fundamental theorems of the Hudson-Partasarathy quantum stochastic calculus (see Theorems 4.1 and 4.3 of [5]), give formulas for expressing the matrix elements

$$< u \otimes \psi(f), M(t)\, v \otimes \psi(g) >$$

and

$$< M(t)\,u \otimes \psi(f), M'(t)\,v \otimes \psi(g) >$$

of quantum stochastic integrals

$$M(t) = \int_0^t E(s)\,d\Lambda(s) + F(s)\,dA(s)$$

$$+G(s)\,dA^{\dagger}(s) + H(s)\,ds$$

and

$$M'(t) = \int_0^t E'(s)\,d\Lambda(s) + F'(s)\,dA(s)$$

$$+G'(s)\,dA^{\dagger}(s) + H'(s)\,ds$$

in terms of ordinary Riemann-Lebesgue integrals. Here E, F, G, H, E', F', G', and H' are time dependent (in general) adapted processes, while $u \otimes \psi(f)$ and $v \otimes \psi(g)$ are in the exponential domain of $\mathcal{H} \otimes \Gamma$, where $\mathcal{H}$ is the, so called, system Hilbert space.

The fundamental result which connects classical with quantum stochastics is that the processes $B_t := A_t + A_t^{\dagger}$ and $P_t := \Lambda_t + \sqrt{\lambda}(A_t + A_t^{\dagger}) + \lambda t$ are identified through their vacuum characteristic functionals

$$< \psi(0), e^{i\,s\,B_t}\,\psi(0) >= e^{-\frac{s^2}{2}\,t}$$

and

$$< \psi(0), e^{i\,s\,P_t}\,\psi(0) >= e^{\lambda\,(e^{i\,s}-1)\,t}$$

with classical Brownian motion and the Poisson process of intensity λ respectively.

Within the framework of Quantum Stochastic Calculus, classical quantum mechanical evolution equations take the form

$$dU_t = -((iH + \frac{1}{2}\,L^* L)\,dt + L^*\,W\,dA_t$$

$$-L\,dA_t^{\dagger} + (1-W)\,d\Lambda_t)U_t$$

with $U_0 = 1$, where, for each $t \geq 0$, U_t is a unitary operator defined on the tensor product $\mathcal{H} \otimes \Gamma(L^2(\mathbb{R}_+, \mathcal{C}))$ of the system Hilbert space $\mathcal{H}$ and the noise (or reservoir) Fock space Γ. Here H, L, W are in $\mathcal{B}(\mathcal{H})$, the space of bounded linear operators on $\mathcal{H}$, with W unitary and H self-adjoint. We identify time-independent, bounded, system space operators X with their ampliation $X \otimes 1$ to $\mathcal{H} \otimes \Gamma(L^2(\mathbb{R}_+, \mathcal{C}))$.

The quantum stochastic differential equation (analogue of the Heisenberg equation for quantum mechanical observables) satisfied by the quantum flow $j_t(X) = U_t^*\,X\,U_t$, where X is a bounded system space operator, is

$$dj_t(X) = j_t(i\,[H, X]$$

$$-\frac{1}{2}(L^*LX + XL^*L - 2L^*XL))\,dt$$

$$+j_t([L^*, X]\,W)\,dA_t + j_t(W^*\,[X, L])\,dA_t^{\dagger}$$

$$+j_t(W^*\,X\,W - X)\,d\Lambda_t$$

with

$$j_0(X) = X, \quad t \in [0, T]$$

2. Quantum Control

The quantum stochastic analogue of the problem of minimizing a quadratic performance criterion associated with a stochastic differential equation has been considered in [1], [2], and [4].

Proceedings of the 9th WSEAS International Conference on Applied Mathematics, Istanbul, Turkey, May 27-29, 2006 (pp509-517)

Let $\{U_t \,/\, t \geq 0\}$ be an adapted process satisfying the quantum stochastic differential equation

$$dU_t = (F_t U_t + u_t)\, dt + \Psi_t\, U_t\, dA_t + \Phi_t\, U_t\, dA_t^\dagger$$

with $U_0 = I$, $t \in [0, T]$, where $T > 0$ is a fixed finite horizon, and the coefficient processes are adapted, bounded, strongly continuous and square integrable processes living on the exponential vectors domain $\mathcal{E} = span\{h = v \otimes \psi(f)\}$. Treating $u = \{u_t \,/\, t \geq 0\}$ as a control process, the quadratic performance functional

$$J_{h,T}(u) = \int_0^T [< U_t h, X^* X\, U_t h >$$

$$+ < u_t h, u_t h >]\, dt + < U_T h, M\, U_T h >$$

where X, M are bounded system space operators, with $M \geq 0$, is minimized by the feedback control $u_t = -P_t U_t$, where the bounded, positive, self-adjoint process $\{P_t \,/\, t \in [0, T]\}$, with $P_T = M$, is the solution of the quantum stochastic Riccati equation

$$dP_t + (P_t F_t + F_t^* P_t + \Phi_t^* P_t \Phi_t - P_t^2 + X^* X)\, dt$$

$$+ (P_t \Psi_t + \Phi_t^* P_t)\, dA_t + (P_t \Phi_t + \Psi_t^* P_t)\, dA_t^\dagger = 0$$

and the minimum value is $< h, P_0 h >$.

In the case of quantum flows $\{j_t(X) \,/\, t \in [0, T]\}$ defined by $j_t(X) = U_t^*\, X\, U_t$, if U_t is for each t a unitary operator, then the equation for U_t takes the form

$$dU_t = -((iH + \frac{1}{2}\, L^* L)\, dt$$

$$+ L^* dA_t - L\, dA_t^\dagger)\, U_t$$

with $U_0 = I$, $t \in [0, T]$. The adjoint equation is

$$dU_t^* = -U_t^*((-iH + \frac{1}{2}\, L^* L)\, dt$$

$$-L^* dA_t + L\, dA_t^\dagger)$$

with $U_0^* = I$, $t \in [0, T]$, where H, L are bounded system space operators, with H self-adjoint. Using quantum Itô's formula, we see that $\{j_t(X) \,/\, t \in [0, T]\}$ satisfies the quantum stochastic differential equation

$$dj_t(X) = j_t(i[H, X]$$

$$-\frac{1}{2}(L^* L X + X L^* L - 2L^* X L))\, dt$$

$$+ j_t([L^*, X])\, dA_t + j_t([X, L])\, dA_t^\dagger$$

with $j_0(X) = X$, $t \in [0, T]$. Letting $u_t = -\frac{1}{2} L^* L U_t$ and taking $M = \frac{1}{2} L^* L$, the quadratic performance functional becomes

$$J_{h,T}(L) = \int_0^T [\, \|j_t(X) h\|^2 + \frac{1}{4} \|j_t(L^* L) h\|^2 \,]\, dt$$

$$+ \frac{1}{2} \|j_T(L) h\|^2$$

Thinking of L as a control, we interpret the first term of $J_{h,T}(L)$ as a measure of the size of the flow over $[0, T]$, the second as a measure of the control effort over $[0, T]$ and the third as a "penalty" for allowing the evolution to go on for a long time. In order for L to be optimal it must satisfy $\frac{1}{2} L^* L = P_t$ where P_t is the solution of the Riccati equation for $F_t = -iH$, $\Phi_t = L$ and $\Psi_t = -L^*$. For these choices, the Riccati equation reduces, by the time independence of P_t and the linear independence of dt, dA_t and $dA_t^\dagger$, to the equations

$$[L, L^*] = 0 \text{ (i.e } L \text{ is normal)}$$

and the Algebraic Riccati Equation (ARE) of [6]

$$\frac{i}{2}[H, P_\infty] + \frac{1}{4} P_\infty^2 + X^* X = 0$$

Proceedings of the 9th WSEAS International Conference on Applied Mathematics, Istanbul, Turkey, May 27-29, 2006 (pp509-517)

where $P_\infty = \frac{1}{2} L^* L$. If there exists a bounded system space operator K such that $\frac{i}{2} H + K X^*$ is the generator of an asymptotically stable semigroup (i.e if the pair $(\frac{i}{2} H,\, X^*)$ is stabilizable) then Algebraic Riccati Equation has a positive self-adjoint solution P_∞. We may summarize as follows:

Let $h \in \mathcal{E}$, $0 < T < +\infty$, and let H, L, X be bounded system space operators, such that H is self-adjoint and the pair $(\frac{i}{2} H,\, X^*)$ is stabilizable. The quadratic performance criterion $J_{h,T}(L)$ associated with the quantum flow $j_t(X)$, is minimized by $L = \sqrt{2}\, P_\infty^{1/2}\, W$ where P_∞ is a positive self-adjoint solution of the Algebraic Riccati Equation and W is any bounded, unitary, system space operator commuting with P_∞. Moreover $\min_L \, J_{h,T}(L) =< h, P_\infty h >$ independent of T.

3 Quantum Economics

In Economics, an option is a ticket which is bought at time $t = 0$ and which allows the buyer at (in the case of European call options) or until (in the case of American call options) time $t = T$ (the time of maturity of the option) to buy a share of stock at a fixed exercise price K. In what follows we restrict to European call options. The question is: how much should one be willing to pay to buy such an option. Let X_T be a reasonable price. The answer given by Black and Scholes (cf. [7]) is that an investment of this reasonable price in a mixed portfolio (i.e part is invested in stock and part in bond) at time $t = 0$, should allow the investor, through a self-financing strategy (i.e one where the only change in the investor's wealth comes from changes of the prices of the stock and bond), to end up at time $t = T$ with an amount of $(X_T - K)^+ := \max(0, X_T - K)$ which is the same as the payoff, had the option been purchased. If $(a_t, b_t), t \in [0, T]$ is a self -financing trading strategy (i.e an amount a_t is invested in stock at time t and an amount b_t is invested in bond at the same time) then the value of the portfolio at time t is given by $V_t = a_t\, X_t + b_t\, \beta_t$ where, by the self-financing assumption, $dV_t = a_t\, dX_t + b_t\, d\beta_t$. Here X_t and β_t denote, respectively, the price of the stock and bond at time t. We assume that $dX_t = c\, X_t\, dt + \sigma\, X_t\, dB_t$ and $d\beta_t = \beta_t\, r\, dt$ where B_t is classical Brownian motion, $r > 0$ is the constant interest rate of the bond, $c > 0$ is the mean rate of return, and $\sigma > 0$ is the volatility of the stock. The assets a_t and b_t are in general stochastic processes. Letting $V_t = u(T - t, X_t)$ where $V_T = u(0, X_T) = (X_T - K)^+$ it can be shown (cf. [7]) that $u(t, x)$ is the solution of the Black-Scholes equation

$$\frac{\partial}{\partial t} u(t,x) = (0.5\, \sigma^2\, x^2\, \frac{\partial^2}{\partial x^2}$$

$$+ r\, x\, \frac{\partial}{\partial x} - r)\, u(t,x)$$

with $u(0, x) = (X_T - K)^+$, where $x > 0$, $t \in [0, T]$, and it is explicitly given by

$$u(t,x) = x\, \Phi(g(t,x)) - K\, e^{-r\,t}\, \Phi(h(t,x))$$

where

$$g(t,x) = \sigma^{-1}\, t^{-1/2}\, (\ln(x/K) + (r + 0.5\, \sigma^2)\, t)$$

$$h(t,x) = g(t,x) - \sigma \sqrt{t}$$

and

$$\Phi(x) = (2\,\pi)^{-1/2} \int_{-\infty}^{x} e^{-y^2/2}\, dy.$$

Thus a rational price for a European call option is

$$V_0 = u(T, X_0)$$

$$= X_0\, \Phi(g(T, X_0)) - K\, e^{-r\,T}\, \Phi(h(T, X_0))$$

and the self-financing strategy $(a_t, b_t), t \in [0, T]$ is given by

Proceedings of the 9th WSEAS International Conference on Applied Mathematics, Istanbul, Turkey, May 27-29, 2006 (pp509-517)

$$a_t = \frac{\partial}{\partial x} u(T-t, X_t)$$
$$b_t = (u(T-t, X_t) - a_t\, X_t)\, \beta_t^{-1}.$$

In recent years the fields of Quantum Economics and Quantum Finance have appeared in order to interpret erratic stock market behavior with the use of quantum mechanical concepts as in [8]. The Black-Schole model has recently been extended in [3] to the quantum setup, within the framework of Hudson-Parthasarathy quantum stochastic calculus. The stock process X_t of the classical Black-Scholes theory is replaced by the quantum mechanical process $j_t(X) = U_t^*\, X \otimes 1\, U_t$ where , for each $t \geq 0$, U_t is a unitary operator defined on the tensor product $\mathcal{H} \otimes \Gamma(L^2(\mathbb{R}_+, \mathcal{C}))$ of a system Hilbert space $\mathcal{H}$ and the noise Boson Fock space $\Gamma = \Gamma(L^2(\mathbb{R}_+, \mathcal{C}))$, satisfying the quantum stochastic differential equation

$$dU_t = -((iH + \frac{1}{2}\, L^* L)\, dt + L^*\, dA_t - L\, dA_t^\dagger) U_t$$

with $U_0 = 1$, where $X > 0$, H, L, are in $\mathcal{B}(\mathcal{H})$, the space of bounded linear operators on $\mathcal{H}$, with X and H self-adjoint. The value process V_t is defined for $t \in [0, T]$ by

$$V_t = a_t\, j_t(X) + b_t\, \beta_t$$

with terminal condition

$$V_T = (j_T(X) - K)^+ = \max(0, j_T(X) - K)$$

where $K > 0$ is a bounded self-adjoint system operator corresponding to the strike price of the quantum option, a_t is a real-valued function, b_t is in general an observable quantum stochastic processes (i.e b_t is a self-adjoint operator for each $t \geq 0$) and

$$\beta_t = \beta_0\, e^{t\, r}$$

where β_0 and r are positive real numbers. Therefore

$$b_t = (V_t - a_t\, j_t(X))\, \beta_t^{-1}.$$

We interpret the above in the sense of expectation i.e given $u \otimes \psi(f)$ in the exponential domain of $\mathcal{H} \otimes \Gamma$, where we will always assume $u \neq 0$ so that $\|u \otimes \psi(f)\| \neq 0$,

$$< u \otimes \psi(f), V_t\, u \otimes \psi(f) >=$$

$$a_t\ < u \otimes \psi(f), j_t(X)\, u \otimes \psi(f) >$$

$$+ < u \otimes \psi(f), b_t\, u \otimes \psi(f) >\ \beta_t$$

i.e the value process is always in reference to a particular quantum mechanical state and

$$< u \otimes \psi(f), V_T\, u \otimes \psi(f) >=$$

$$\max(0, < u \otimes \psi(f), (j_T(X) - K)\, u \otimes \psi(f) >).$$

As in the classical case we assume that the portfolio $(a_t, b_t), t \in [0, T]$ is self -financing i.e

$$dV_t = a_t\, dj_t(X) + b_t\, d\beta_t$$

By the Quantum Itô table of Section 1, and the homomorhism property $j_t(x\, y) = j_t(x)\, j_t(y)$, we obtain

$$dj_t(X) = j_t(\alpha^\dagger)\, dA_t^\dagger + j_t(\alpha)\, dA_t + j_t(\theta)\, dt$$

and

$$(dj_t(X))^2 = j_t(\alpha\, \alpha^\dagger)\, dt$$

while for $k \geq 2$, $(dj_t(X))^k = 0$. Here, and in what follows,

$$\alpha = [L^*, X]$$

$$\alpha^\dagger = [X, L]$$

and

$$\theta = i\,[H, X] - \frac{1}{2}\,\{L^*\,L\,X + X\,L^*\,L - 2\,L^*\,X\,L\}.$$

In the above framework, let $V_t := F(t, j_t(X))$ where $F : [0, T] \times \mathcal{B}(\mathcal{H} \otimes \Gamma) \longrightarrow \mathcal{B}(\mathcal{H} \otimes \Gamma)$ is the extension to self-adjoint operators $x = j_t(X)$ of the analytic function

$$F(t, x) = \sum_{n,k=0}^{+\infty} a_{n,k}(t_0, x_0)\,(t - t_0)^n\,(x - x_0)^k$$

where x and $a_{n,k}(t_0, x_0)$ are in $\mathbb{C}$, and for $\lambda, \mu \in \{0, 1, ...\}$

$$F_{\lambda\,\mu}(t, x) = \frac{\partial^{\lambda+\mu} F}{\partial t^\lambda\,\partial x^\mu}(t, x).$$

If 1 denotes the identity operator then

$$a_{n,k}(t_0, x_0) = a_{n,k}(t_0, x_0)\,1 = \frac{1}{n!\,k!}\,F_{n\,k}(t_0, x_0).$$

Moreover for $(t_0, x_0) = (0, 0)$ we have

$$V_t = \sum_{n,k=0}^{+\infty} a_{n,k}(0, 0)\,t^n\,j_t(X)^k.$$

By the Quantum Itô table

$$dV_t = (a_{1,0}(t, j_t(X)) + a_{0,1}(t, j_t(X))\,j_t(\theta)$$

$$+a_{0,2}(t, j_t(X))\,j_t(\alpha\,\alpha^\dagger))\,dt$$

$$+a_{0,1}(t, j_t(X))\,j_t(\alpha^\dagger)\,dA_t^\dagger + a_{0,1}(t, j_t(X))\,j_t(\alpha)\,dA_t.$$

while, by the self-financing property,

$$dV_t = (a_t\,j_t(\theta) + V_t\,r - a_t\,j_t(X)\,r)\,dt$$

$$+a_t\,j_t(\alpha^\dagger)\,dA_t^\dagger + a_t\,j_t(\alpha)\,dA_t.$$

Equating the coefficients of dt and the quantum stochastic differentials in the two expressions for dV_t and combining the resulting two equations, after simplifying, we obtain

$$a_{1,0}(t, j_t(X)) + a_{0,2}(t, j_t(X))\,j_t([L^*, X]\,[X, L])$$

$$+a_{0,1}(t, j_t(X))\,j_t(X)\,r - V_t\,r = 0$$

which can be written as

$$F_{1\,0}(t, j_t(X)) + \frac{1}{2}\,F_{0\,2}(t, j_t(X))\,j_t([L^*, X]\,[X, L])$$

$$+F_{0\,1}(t, j_t(X))\,j_t(X)\,r = F(t, j_t(X))\,r$$

with $F(T, j_T(X)) = (j_T(X) - K)^+$. Letting $x = j_t(X)$, $y = j_t(L)$ be arbitrary elements in $\mathcal{B}(\mathcal{H} \otimes \Gamma)$ and $g(x) = [y^*, x]\,[x, y]$, $h(x) = x\,r$, we obtain

$$F_{1\,0}(t, x) + \frac{1}{2}\,F_{0\,2}(t, x)\,g(x)$$

$$+F_{0\,1}(t, x)\,h(x) = F(t, x)\,r.$$

Letting

$$u(t, x) = F(T - t, x)$$

we obtain the Quantum Black-Scholes Equation

$$u_{1\,0}(t, x) = \frac{1}{2}\,u_{0\,2}(t, x)\,g(x)$$

$$+u_{0\,1}(t, x)\,h(x) - u(t, x)\,r$$

with

$$u(0, j_T(X)) = (j_T(X) - K)^+.$$

To solve the Quantum Black-Scholes Equation we assume that $j_t(X^2) = j_t([L^*, X]\,[X, L])$ which implies that $[X, L] = W\,X$ and $[L^*, X] = X\,W^*$

where W is an arbitrary unitary operator acting on the system space. The Quantum Black-Scholes Equation then takes the form

$$u_{1\,0}(t,x) = \frac{1}{2}\,u_{0\,2}(t,x)\,x^2$$

$$+u_{0\,1}(t,x)\,x\,r - u(t,x)\,r$$

where we may assume that x is a bounded self-adjoint operator. At $(0,0)$,

$$u(t,x) = \sum_{n,k=0}^{+\infty} a_{n,k}(0,0)\,(T-t)^n\,x^k$$

and , since $x = j_t(X) > 0$ and K are invertible, we may let $x = K\,e^z$ where z is a bounded self-adjoint operator commuting with K. Letting

$$\omega(t,z) := u(t, K\,e^z)$$

we obtain

$$\omega_{1\,0}(t,z) = \frac{1}{2}\,\omega_{0\,2}(t,z)$$

$$+\omega_{0\,1}(t,z)\,(r-\frac{1}{2}) - \omega(t,z)\,r$$

with $\omega(0,z_T) = (j_T(X)-K)^+$, where z_T is defined by $K\,e^{z_T} = j_T(X)$. The quantum analogue of the classical Black-Scholes option pricing theorem can now be formulated as follows:

The solution $\omega(t,z)$ of the Quantum Black-Scholes Equation is given by

$$\omega(t,z) = K\,e^z\;\Phi(g(t,K\,e^z))$$

$$-K\,\Phi(h(t,K\,e^z))\,e^{-r\,t}$$

where

$$g(t,K\,e^z) = z\,t^{-1/2} + (r+0.5)\,t^{1/2},$$

$$h(t,K\,e^z) = z\,t^{-1/2} + (r-0.5)\,t^{1/2},$$

and

$$\Phi(x) = \frac{1}{2} + \frac{1}{\sqrt{2\,\pi}}\sum_{n=0}^{+\infty}\frac{(-1)^n}{2^n\,n!}\,\frac{x^{2\,n+1}}{2\,n+1}.$$

Moreover, a reasonable price for a quantum option is $\omega(T,z_0)$ where z_0 is defined by $X = K\,e^{z_0}$. The associated quantum portfolio (a_t,b_t) is given by

$$a_t = \omega_{0\,1}(t-T,z_t)$$

and

$$b_t = (\omega(T-t,z_t) - a_t\,j_t(X))\,e^{-t\,r}\,{\beta_0}^{-1}$$

where z_t is defined by $j_t(X) = K\,e^{z_t}$.

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Proceedings of the 9th WSEAS International Conference on Applied Mathematics, Istanbul, Turkey, May 27-29, 2006 (pp509-517)

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