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Analytical and Computational Study of the Stability of Coupled Wave Equations

M. NAJAFI[1,2], H.R. MASSAH[3,4]
Department of Mathematics
Kent State University, Ashtabulo
3325 west street, OH 44004
USA

Abstract: - Analytical solution as well as numerical computation of coupled wave equations is under consideration. To this end, ADM and spectral methods are utilized, respectively. Also, stability of the system is discussed due to its coupled -distributed velocity feedback controllers.

Key-Words: - Adomian Decomposition Method- Spectral Methods- Adomian Polynomials- coupled wave equations- Analytical solution.

1 Introduction

Many problems in structural dynamics deal with stabilizing the elastic energy of partial differential equations by boundary or internal energy dissipative controllers for wave equations or the Euler-Bernoulli beam equation [1-3]. Exponential stability is a very desirable property for such elastic systems. In this study stability of a system of wave equations coupled in parallel with distributed viscous damping and springs [1] is revisited. What comes new in this work is to find an analytical solution for the system (1), below, via Adomian Decomposition Method (ADM) [4-7].

This paper is organized as follows: In the present section, Section. 1, the equation of coupled wave system is given, and some basic knowledge of spectral method and ADM are presented, respectively. In Section 2, the application of these methods will be utilized to study the stability of the system due to solution of the system (1-3), see Eq. (1) below. In Section 3, the numerical computation of this system will be discussed followed by conclusion.

1.1 Coupled wave system in one dimension

The following governing equation is under consideration:

$$\begin{aligned} & u_{tt} - c_1^2 u_{xx} = l(v-u) + \beta(v_t - u_t), \\ & \quad \text{in } \Omega_1 \times (0,\infty), \\ & v_{tt} - c_2^2 v_{xx} = l(u-v) + \beta(u_t - v_t), \\ & \quad \text{in } \Omega_2 \times (0,\infty), \end{aligned} \tag{1}$$

with initial conditions,

$$\begin{aligned} & u(0) = f_1, \quad u_t(0) = g_1, \quad \text{in } \Omega_1, \\ & v(0) = f_1, \quad v_t(0) = g_2, \quad \text{in } \Omega_2, \end{aligned} \tag{2}$$

along with prescribed Dirichlet boundary conditions,

$$u = v = 0, \quad \text{on } \partial\Omega \times (0,\infty). \tag{3}$$

Here, $\Omega_1 = \Omega_2 = \Omega = (0,1)$ are open sets. Let $\partial\Omega_1, \partial\Omega_2$ be the boundaries of Ω_1 and Ω_2, respectively. The coupling constants $\beta > 0$ and $l > 0$ are damping and spring coefficients, respectively. We assume that the projection of Ω_1 into Ω_2, denotes as Ω. Also, $u(x,t)$ and $v(x,t)$ are the displacements of two vibrating strings measured from their equilibrium position, and c_1, c_2 are wave propagation speeds. The distributed springs and dampers linking two vibrating springs are the coupling terms; that is, $l(u-v)$ and $\beta(u_t - v_t)$. Energy can flow from one object to another through this parameter (l) and damp via shock absorber (β). Also $u(x,t)$ and $v(x,t)$ are the

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[1] Associate Professor, Mathematics Department, Kent State University, USA
[2] Khayyam Research Institute, Tehran, Iran
[3] Physics Department, Amirkabir University, Tehran, Iran
[4] Acoustics Research Center, Institute of Applied Physics, Tehran, Iran

displacement of two vibrating strings measured from their equilibrium positions.

1.2 Spectral Method

Separation of variables is a valuable tool when there are time derivatives. It is rather direct method for our analysis since the part involving time is only an exponential. For the heat equation, it is a decay $e^{-\lambda t}$, and for the wave equation it is an oscillation e^{-iwt}. The key is to find the eigenvectors since they solve the time–dependent problem by combining with $e^{-\lambda t}$, or e^{-iwt} into pure exponential solutions. For partial differential equations, they are eigenfunctions. The term $\frac{\partial^2(.)}{\partial x^2}$ has negative eigenvalues, $\frac{\partial^2}{\partial x^2}(e^{2\pi ikx}) = -(2\pi k)^2 e^{2\pi ikx}$ for a periodic case, and $\frac{\partial^2}{\partial x^2}\sin(\pi kx) = -(\pi k)^2$ $\sin(\pi kx)$, for zero boundary conditions. The separated solutions, $\phi(t)\Phi(x)$, can be written down immediately. The heat equation has decaying solutions $e^{-\lambda t}\Phi$, the wave equation has oscillating solutions $e^{iwt}\Phi$ and $e^{-iwt}\Phi$. The eigenvalues $-\lambda = -w^2$ have eigenfunctions $\Phi-$, one for each frequency k. The solutions to wave equations are combinations of these exponential solutions. For example,

$u\sum(c_{k^e}{}^{iw}k^t + d_{k^e}{}^{-iw}k^t)\Phi_k(x).$

What is unique about this approach is that it may be generalized so that any infinite series of smooth, and preferably, orthogonal functions may be used to eliminate the physical space variable from the problem and reduce the solutions of the partial differential equations to the solution of a set of ordinary differential equations in the other independent variable, time. Because of their close association with the Fourier series, the expansion coefficients are referred to as spectra and this approach is called the spectral method.

1.3 Basics of ADM

The ADM consists of splitting the given equation into linear and nonlinear parts. Then the inverse of the highest-order derivative operator, usually, contained in the linear operator, is applied to the both sides of the given equation. The process is followed by decomposing the unknown function into a series whose components are to be determined. Decomposing of the nonlinear part in terms of the so-called Adomian polynomials is the essential part of ADM. Recurrent relation using Adomian's polynomials finds the successive terms of the series solution. This method, usually, starts with a general equation $Fu = g(t)$. Where F represents a general nonlinear operator, which could be decomposed into linear and nonlinear operators. Further decomposition of the linear term leads to $Lu + Ru$, where L is the highest order derivative operator and R is the reminder of the linear operator. Thus, the equation may be rewritten in the form,

$$Lu + Ru + Nu = g, \tag{4}$$

where N is a nonlinear operator. Solving for Lu, gives,

$$Lu = g - Ru - Nu. \tag{5}$$

Applying inverse of L on both sides, the equation (5) can be written as,

$$L^{-1}Lu = L^{-1}g - L^{-1}Ru - L^{-1}Nu. \tag{6}$$

Where L^{-1} indicates the inverse of the highest order operator L. For example if L is considered to be a second order derivative operator in t, then L^{-1} is a twofold linear integral operator. Hence (6) becomes,

$$u = u_0 + L^{-1}g - L^{-1}Ru - L^{-1}Nu, \tag{7}$$

where u_0 is yet to be calculated. By ADM, u and the nonlinear term, Nu, are decomposed to:

$$u = \sum_{n=0}^{\infty} u_n, \tag{8}$$

and

$$Nu = \sum_{n=0}^{\infty} A_n(u_0, u_1, \ldots, u_n), \tag{9}$$

respectively, where Adomian polynomials, A_n, are used to compute nonlinear terms and are calculated by the following relation [4], By substituting (8) and (9) into (7), one can get,

$$\begin{aligned} u = \sum_{n=0}^{\infty} u_n &= u(0) + tu'(0) + L^{-1}g \\ &- L^{-1}R\sum_{n=0}^{\infty} u_n - L^{-1}\sum_{n=0}^{\infty} A_n. \end{aligned} \tag{10}$$

The above equation is rewritten, according to ADM [4], in the form of a set of following recursive relations,

$$\begin{aligned} u_0 &= u(0) + tu'(0) + L^{-1}g, \\ u_{n+1} &= -L^{-1}Ru_n - L^{-1}A_n, \qquad n \geq 0. \end{aligned} \tag{11}$$

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Where u_0 can be obtained by prescribed initial/boundary conditions and consequently all of the u_n will be calculable. The k-term approximation can be used as a practical solution, as:

$$u \cong \varphi_k = \sum_{i=0}^{k-1} u_i , \tag{12}$$

and if more accuracy is desirable, more terms of the approximation should be utilized. So,

$$u = \sum_{i=0}^{\infty} u_i . \tag{13}$$

Hereafter, ADM is applied to the following system of wave equations; see Eqs. (1-3),

$$\begin{aligned} &u_{tt} - u_{xx} + R_1(u,v) + N_1(u,v) = g_1(x), \\ &v_{tt} - v_{xx} + R_2(u,v) + N_2(u,v) = g_2(x), \end{aligned} \tag{14}$$

with initial conditions,

$$\begin{aligned} &u(x,0) = f_1(x), \quad v(x,0) = \sigma_1(x), \\ &u_t(x,0) = f_2(x), \quad v_t(x,0) = \sigma_2(x), \end{aligned} \tag{15}$$

Rewriting the system in the operator form, as in (4), yields

$$\begin{aligned} &L_t u - L_x u + R_1(u,v) + N_1(u,v) = g_1(x), \\ &L_t v - L_x v + R_2(u,v) + N_2(u,v) = g_2(x). \end{aligned} \tag{16}$$

Here L_t and L_x are operators in t and x, respectively. The inverse operator L_t^{-1} is a two-fold integration represented by $L_t^{-1}(.) = \int_0^t \int_0^t (.) dt\, dt$. Applying the operator L_t^{-1} on (16), analogous Eqs. (4-7), one gets

$$\begin{aligned} &u(x,t) = f_1(x) + tf_2(x) + L_t^{-1} L_x u - \\ &L_t^{-1} R_1(u,v) - L_t^{-1} N_1(u,v) + L_t^{-1} g_1(x), \\ &v(x,t) = \sigma_1(x) + t\sigma_2(x) + L_t^{-1} L_x v \\ &- L_t^{-1} R_2(u,v) - L_t^{-1} N_2(u,v) + L_t^{-1} g_2(x). \end{aligned} \tag{17}$$

The first two terms in (17) are constants of integration, which can be calculated by (15). By (13), $u(x,t)$ and $v(x,t)$ are decomposed to,

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t), \quad v(x,t) = \sum_{n=0}^{\infty} v_n(x,t). \tag{18}$$

Also, the nonlinear terms $N_1(u,v)$ and $N_2(u,v)$ are decomposed into infinite series by use of Adomian polynomials,

$$N_1(u,v) = \sum_{n=0}^{\infty} A_n , \quad N_2(u,v) = \sum_{n=0}^{\infty} B_n . \tag{19}$$

The Adomian polynomials A_n and B_n can be calculated by [2], Introducing (18) and (19) into (17); gives

$$\begin{aligned} &\sum_{n=0}^{\infty} u_n(x,t) = f_1(x) + tf_2(x) \\ &+ L_t^{-1}\left(L_x \left(\sum_{n=0}^{\infty} u_n \right)\right) - L_t^{-1} R_1\left(\sum_{n=0}^{\infty} u_n , \sum_{n=0}^{\infty} v_n \right) \\ &- L_t^{-1}\left(\sum_{n=0}^{\infty} A_n \right) + L_t^{-1} g_1(x), \\ &\sum_{n=0}^{\infty} v_n(x,t) = \sigma_1(x) + t\sigma_2(x) \\ &+ L_t^{-1}\left(L_x \left(\sum_{n=0}^{\infty} v_n \right)\right) - L_t^{-1} R_2\left(\sum_{n=0}^{\infty} u_n , \sum_{n=0}^{\infty} v_n \right) \\ &- L_t^{-1}\left(\sum_{n=0}^{\infty} B_n \right) + L_t^{-1} g_2(x). \end{aligned} \tag{20}$$

Now, according to ADM, the system of equations in (20) is transformed into a set of recursive relations given by,

$$\begin{aligned} &u_0(x,t) = f_1(x) + tf_2(x) + L_t^{-1} g_1(x), \\ &u_{n+1}(x,t) = L_t^{-1} L_x u_n \\ &\qquad - L_t^{-1} R_1(u_n, v_n) - L_t^{-1} A_n , n \geq 0, \end{aligned} \tag{21}$$

and similarly,

$$\begin{aligned} &v_0(x,t) = \sigma_1(x) + t\sigma_2(x) + L_t^{-1} g_2(x), \\ &v_{n+1}(x,t) = L_t^{-1} L_x v_n - L_t^{-1} R_2(u_n, v_n) \\ &- L_t^{-1} B_n , n \geq 0. \end{aligned} \tag{22}$$

The (k+1)-term approximant solutions for u and v can be determined respectively by,

$$\varphi_{k+1} = \sum_{n=0}^{k} u_n(x,t), \quad \psi_{k+1} = \sum_{n=0}^{k} v_n(x,t). \tag{23}$$

Finally, there appears a power series solution, in some cases a closed form solution is at hand by summing up the power series,

$$\begin{aligned} &u(x,t) = \sum_{n=0}^{\infty} u_n(x,t), \\ &v(x,t) = \sum_{n=0}^{\infty} v_n(x,t). \end{aligned} \tag{24}$$

2 Application of Spectral Method and ADM

Having considered the above two methods, we are now in a position to pose the following theorem, which is necessary in studying the rate of convergence of the system:

Theorem. System (1-3);

(a) Decays strongly if $c_1 \neq c_2$; that is, the system is stable.

(b) The stability of the system fails if $c_1 = c_2$ with identical initial conditions.

Proof of (a) (stability via spectral method)**:** let $\Omega\{x \in \Re^n | 0 < x_i < \pi, i = 1,...,n\}$. Then the following solutions satisfy the system (1-3):

$$u(x,t) = \phi(t)\prod_1^n \sin(\lambda_i x_i),$$
$$v(x,t) = \psi(t)\prod_1^n \sin(\lambda_i x_i). \quad (25)$$

Introducing (25) into (1), one obtains the system of differential equations in ϕ and ψ with respect to time t as follows:

$$\ddot{\phi} = -w_1^2\phi(l(\psi - \phi) + \beta(\dot{\psi} - \dot{\phi}),$$
$$\ddot{\psi} = -w_1^2\psi + l(\psi - \phi) + \beta(\dot{\phi} - \dot{\psi}). \quad (26)$$

System (26) can be written as a system of first – order ordinary differential equations:

$$\dot{\overline{X}} = \overline{A}_w \overline{X}, \quad (27)$$

where

$$\overline{X} = [\phi, \psi, \dot{\phi}, \dot{\psi}]^T, \overline{A}_w = \begin{bmatrix} \overline{0} & I \\ \overline{A}_w & \overline{B} \end{bmatrix},$$

and

$$\overline{A}_w = \begin{bmatrix} -w_1^2 & l \\ l & -w_2^2 \end{bmatrix}, \overline{B} = \begin{bmatrix} -\beta & \beta \\ \beta & -\beta \end{bmatrix}.$$

The characteristic polynomial of the matrix $\overline{A}_w$ is

$$P(s) = det(\overline{A}_w - sI)$$
$$= s^4 + 2\beta s^3 + (w_2^2 + w_1^2 + 2l)s^2\beta$$
$$+ (w_2^2 + w_1^2)s + lw_1^2 + w_1^2 w_2^2 + lw_2^2, \quad (28)$$

where

$$w_1^2 = c_1^2\lambda^2, w_2^2 = c_2^2\lambda^2, \lambda^2 = \sum_{i=1}^n \lambda_i^2.$$

Completion of Proof of (a) (asymptotic behavior of the solutions) $c_1 \neq c_2$**:**

Consider the characteristic equation in the following factored form:

$$P(s) = (s^2 + xs + (w_1^2 - l - z)) \times (s^2 + ys + (w_2^2 - l - z)). \quad (29)$$

To find values for x,y, and z, we should equate the coefficients of the like powers in s of (28) and (29), so that the following system of equations can be solved in terms of x, y, and z:

$$x + y = 2\beta$$
$$(w_1^2 + l - z)x + (w_2^2 + l + z)y = \beta(w_1^2 + w_2^2)$$
$$(w_1^2 + l - z)(w_2^2 + l + z)y = lw_1^2 + w_1^2 w_2^2 + lw_2^2. \quad (30)$$

Solution to the system (30) is as follows:

$$x = \beta - \frac{2l\beta}{(w_1^2 - w_2^2) - 2z},$$
$$y = 2\beta - x, \quad (31)$$
$$z = \frac{(w_1^2 - w_2^2) \pm \sqrt{(w_1^2 - w_2^2)^2 + 4l^2}}{2}.$$

Eq. (31) can be expressed as

$$P(s) = \left(s^2 + (\beta - \frac{2l\beta}{(w_1^2 - w_2^2) - 2z} s + (w_1^2 + l - \frac{(w_1^2 - w_2^2) \pm \sqrt{(w_1^2 - w_2^2)^2 + 4l^2}}{2}) \right)$$
$$\times \left(s^2 + (2\beta - (\beta \frac{2l\beta}{(w_1^2 - w_2^2) - 2z}))s + (w_2^2 + l + \frac{(w_1^2 - w_2^2) \pm \sqrt{(w_1^2 - w_2^2)^2 + 4l^2}}{2}) \right). \quad (32)$$

We will notice from (32) that every eigenvalue of $\overline{A}_w$ has $-\frac{\beta}{2}$ as a negative real part, since

$$\lim_{\lambda \to \infty} \chi(\lambda) = \beta. \quad (33)$$

Eq. (33) shows that a sequence of solutions to the system (1) can be found, which go to equilibrium state; that is, as $t \to \infty$, the energy of the system $E(t) \to 0$, see Eq. (51) in Sec. 3. Hence the system is strongly stabilized, and that furnishes the proof of part (a) of the theorem.

Proof of (b) (instability via application of ADM): without loss of generality let $l=\beta=1,$ and $c_1=c_2=c$, then system (1) becomes

$$\begin{aligned} u_{tt}-c^2u_{xx}&=(v-u)+(v_t-u_t),\\ v_{tt}-c^2v_{xx}&=(u-v)+(u_t-v_t), \end{aligned} \tag{34}$$

with identical initial conditions,

$$\begin{aligned} &u(x,0)=\sin\pi x, && v(x,0)=\sin\pi x,\\ &u_t(x,0)=0, && v_t(x,0)=0, \end{aligned} \tag{35}$$

and boundary conditions,

$$\begin{aligned} &u(0,t)=0, && v(0,t)=0,\\ &u(1,t)=0, && v(1,t)=0. \end{aligned} \tag{36}$$

Rewriting Eq. (34) in operator form, as discussed in Sec. 1:

$$\begin{aligned} L_tu&=c^2L_xu+(v-u)+(v_t-u_t),\\ L_tv&=c^2L_xv+(u-v)+(u_t-v_t), \end{aligned} \tag{37}$$

where L_t and L_x are second order partial differential operators in respect to t and x, respectively. Now, applying the inversed operator L_t to system (37), yields,

$$\begin{aligned} u(x,t)&=f_1(x)+tf_2(x)\\ &+c^2L_t^{-1}L_xu+L_t^{-1}(v-u)+L_t^{-1}(v_t-u_t),\\ v(x,t)&=\sigma_1(x)+t\sigma_2(x)\\ &+c^2L_t^{-1}L_xv+L_t^{-1}(u-v)+L_t^{-1}(u_t-v_t), \end{aligned} \tag{38}$$

where L_t^{-1} is a two-fold integration in respect to t from 0 to t, and first two terms are generated due to integrations. After decomposing $u(x,t)$ and $v(x,t)$, Eq. (38) is rewritten as:

$$\begin{aligned} \sum_{n=0}^{\infty}u_n(x,t)&=f_1(x)+tf_2(x)\\ &+c^2L_t^{-1}\left(L_x\left(\sum_{n=0}^{\infty}u_n\right)\right)+L_t^{-1}\left(\sum_{n=0}^{\infty}v_n-\sum_{n=0}^{\infty}u_n\right)\\ &+L_t^{-1}\left(\frac{\partial}{\partial t}\left(\sum_{n=0}^{\infty}v_n-\sum_{n=0}^{\infty}u_n\right)\right),\\ \sum_{n=0}^{\infty}v_n(x,t)&=\sigma_1(x)+t\sigma_2(x)\\ &+c^2L_t^{-1}\left(L_x\left(\sum_{n=0}^{\infty}v_n\right)\right)+L_t^{-1}\left(\sum_{n=0}^{\infty}u_n-\sum_{n=0}^{\infty}v_n\right)\\ &+L_t^{-1}\left(\frac{\partial}{\partial t}\left(\sum_{n=0}^{\infty}u_n-\sum_{n=0}^{\infty}v_n\right)\right). \end{aligned} \tag{39}$$

Unlike Equations (16) and (17), Eq. (39) is linear, so the nonlinear terms, $N_i(u,v)$, are not appeared, therefore Eq. (39) becomes:

$$\begin{aligned} u_0+\sum_{n=0}^{\infty}u_{n+1}(x,t)&=f_1(x)+tf_2(x)\\ &+c^2L_t^{-1}\left(L_x\left(\sum_{n=0}^{\infty}u_n\right)\right)+L_t^{-1}\left(\sum_{n=0}^{\infty}v_n-\sum_{n=0}^{\infty}u_n\right)\\ &+L_t^{-1}\left(\frac{\partial}{\partial t}\left(\sum_{n=0}^{\infty}v_n-\sum_{n=0}^{\infty}u_n\right)\right),\\ v_0+\sum_{n=0}^{\infty}v_{n+1}(x,t)&=\sigma_1(x)+t\sigma_2(x)\\ &+c^2L_t^{-1}\left(L_x\left(\sum_{n=0}^{\infty}v_n\right)\right)+L_t^{-1}\left(\sum_{n=0}^{\infty}u_n-\sum_{n=0}^{\infty}v_n\right)\\ &+L_t^{-1}\left(\frac{\partial}{\partial t}\left(\sum_{n=0}^{\infty}u_n-\sum_{n=0}^{\infty}v_n\right)\right), \end{aligned} \tag{40}$$

each of the equations in (40) can be rewritten in a set of recursive relations, as follows:

$$\begin{aligned} u_0(x,t)&=f_1(x)+tf_2(x),\\ u_{n+1}(x,t)&=c^2L_t^{-1}\left(L_x\left(\sum_{n=0}^{\infty}u_n\right)\right)\\ &+L_t^{-1}\left(\sum_{n=0}^{\infty}v_n-\sum_{n=0}^{\infty}u_n\right)\\ &+L_t^{-1}\left(\frac{\partial}{\partial t}\left(\sum_{n=0}^{\infty}v_n-\sum_{n=0}^{\infty}u_n\right)\right) \quad n\geq 0. \end{aligned} \tag{41}$$

Similarly,

$$\begin{aligned} v_0(x,t)&=\sigma_1(x)+t\sigma_2(x),\\ v_{n+1}(x,t)&=c^2L_t^{-1}\left(L_x\left(\sum_{n=0}^{\infty}v_n\right)\right)\\ &+L_t^{-1}\left(\sum_{n=0}^{\infty}u_n-\sum_{n=0}^{\infty}v_n\right)\\ &+L_t^{-1}\left(\frac{\partial}{\partial t}\left(\sum_{n=0}^{\infty}u_n-\sum_{n=0}^{\infty}v_n\right)\right) \quad n\geq 0. \end{aligned} \tag{42}$$

The ($k+1$)-term approximant results for u and v can be found, respectively as

$$\begin{aligned} \varphi_{k+1}&=\sum_{n=0}^{k}u_n(x,t),\\ \psi_{k+1}&=\sum_{n=0}^{k}v_n(x,t). \end{aligned} \tag{43}$$

So, one can find the first term approximation for u and v from Equations (41) and (42), respectively as:

$$\begin{aligned} \varphi_1(x,t)&=u_0=f_1(x)+tf_2(x),\\ \psi_1(x,t)&=v_0=\sigma_1(x)+t\sigma_2(x). \end{aligned} \tag{44}$$

The integration constants in (44) are evaluated by (35) as follows:

$$\begin{aligned} &\varphi_1(x,0) = \sin \pi x \rightarrow f_1(x) = \sin \pi x, \\ &\psi_2(x,0) = \sin \pi x \rightarrow \sigma_1(x) = \sin \pi x, \\ &\frac{\partial}{\partial t}\varphi_1(x,0) = 0 \rightarrow f_2(x) = 0 \\ &\frac{\partial}{\partial t}\psi_2(x,0) = 0 \rightarrow \sigma_2(x) = 0, \end{aligned} \tag{45}$$

Eq. (45) leads to:

$$\begin{aligned} &\varphi_1(x,t) = u_0 = \sin \pi x, \\ &\psi_1(x,t) = v_0 = \sin \pi x. \end{aligned} \tag{46}$$

By using (41), proceeding terms for u_n are evaluated as:

$$\begin{aligned} &u_1 = -\frac{(\pi ct)^2}{2!}\sin \pi x, \qquad u_2 = \frac{(\pi ct)^4}{4!}\sin \pi x, \\ &u_3 = -\frac{(\pi ct)^6}{6!}\sin \pi x, \qquad u_4 = \frac{(\pi ct)^8}{8!}\sin \pi x, \\ &u_5 = -\frac{(\pi ct)^{10}}{10!}\sin \pi x, \qquad \ldots \end{aligned} \tag{47}$$

Now, considering Eq. (24), Eq. (47) leads to

$$u(x,t) = \sin \pi x \begin{pmatrix} 1 - \frac{(\pi ct)^2}{2!} + \frac{(\pi ct)^4}{4!} \\ -\frac{(\pi ct)^6}{6!} + \frac{(\pi ct)^8}{8!} - \ldots \end{pmatrix}. \tag{48}$$

Hence, the closed form of the series solution to the system (1), Eq. (48), is as follows:

$$u(x,t) = \sin \pi x \sum_{n=0}^{\infty} \frac{(-1)^n (\pi ct)^{2n}}{(2n)!}. \tag{49}$$

Where the summation is the Maclaurin series of $\cos(\pi ct)$. Hence Eq. (49) leads to the following exact solution:

$$u(x,t) = \sin(\pi x)\cos(\pi ct). \tag{50}$$

Similarly, one can get the exact solution for $v(x, t)$ as:

$$v(x,t) = \sin(\pi x)\cos(\pi ct). \tag{51}$$

Equations (50) and (51) represent an oscillatory system; that is, as $t \rightarrow \infty$, the above system will never get to rest; that is, $E(t) \neq 0$, and hence the system is unstable, and that ends the proof of part (b) of the theorem.

3 Numerical Computation and Conclusion

Another evidence to support our claims in above theorem is to deal with the energy of the system as t increases. To this end, according to [1], the energy of the system is defined by

$$E(t) = \frac{1}{2}\int_0^1 \begin{Bmatrix} |u_t|^2 + c_1^2|u_x|^2 + |v_t|^2 \\ + c_2^2|v_x|^2 + \alpha|u-v|^2 \end{Bmatrix} dx. \tag{52}$$

Now, by introducing (50) and (51) into (52), yields

$$E(t) = \frac{\pi^2}{2}. \tag{53}$$

Eq. (53) implies that the energy of the system is conserved, and consequently when the system possesses identical wave speeds as well as identical initial conditions, the system is unstable, and that confirms the proof of part (b) of the theorem in Sec. 2. When the system excites with different wave propagation speeds, then as time increases, the energy of the system (1) gets smaller and eventually approaches zero, which satisfies the stability of the system and that also confirms the proof of part (a) of the theorem. Here in this paper, the analytical solution of the system of wave equations, Eq. (1) was found by the application of ADM, which distinguishes from other regular methods.

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