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A New Algorithm for Solving Riccati Equation Using Adomian Decomposition Method

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Abstract— **Riccati equation with matrix variable coefficients, arising in optimal and robust control approach, is considered. An analytical approximation of the solution of nonlinear differential Riccati equation is investigated using the Adomian decomposition method. An application in optimal control is presented. Solution in different order of approximations will be compared respect to accuracy.**

I. Introduction

Riccati equations arise in optimal and robust control theory and it is a nonlinear, time-variant matrix coefficient equation. For solving this equation no analytical method exists. A method for solving this equation numerically is discretization of it in time domain and substitution of derivation operator with discrete approximation and finding solution in each iteration. But this method is very sensitive to sample time ΔT in discretization and may be unstable for some ΔT . In this paper we start with linear, time-invariant system and apply optimal control to this system. With using calculus of variations, we reach to a Riccati equation. Then we apply Adomian decomposition method for analytical solving this equation and compare solution of this method with different order approximations.

Adomian decomposition method is a approximated approach for solving nonlinear differential equations by substitution of nonlinear parts of equation with Adomian polynomials and use a step by step method for finding solutions [1]. This method is a powerful approach in nonlinear differential equations and accuracy of it depends on number of used partial solutions. Also, solution of this method has a fast convergence to exact solution generally. In recent years, some modifications on this method have been presented [5,6]. Modification of method is in quality of computation of Adomian polynomials. These modifications affects on convergence of method. In some papers [2,3], Adomian decomposition method used in nonlinear optimal control and non-quadratic cost functions optimal control.

Structure of paper is as following: In section 2, a brief description of optimal control will be presented. In section 3, Adomian decomposition method for solving differential equations will be described. In section 4, we apply Adomian method to Riccati equation and find solution. Also two examples will be presented and method will be applied to them and solutions will be compared. In section 5, we have conclusion and suggestion of future works.

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II. Linear Optimal Control

In this section we have a brief description of optimal control. First consider following linear time-invariant system in state space realization [4]:

$$\begin{cases} \dot{x}(t) = Ax(t) + B_u u(t) \quad , \quad x(0) = x_0 \\ y(t) = C_y x(t) \end{cases} \tag{1}$$

System has no disturbance input. Suppose that A is a $n \times n$ matrix, $x(t)$ a $n \times 1$ state space vector, $y(t)$ output vector and $u(t)$ is control signal. Our propose is control of above system and finding control signal subject to minimizing the following cost function:

$$J(u,y) = \frac{1}{2} y^T(t_f) H_y y(t_f) + \frac{1}{2}\int_0^{t_f} (y^T(t) Q_y y(t) + u^T(t) R u(t)) dt \tag{2}$$

In this cost function, Q_y, R and H_y are positive definite and symmetric with appropriate dimensions. Now we want to rewrite $J(u,y)$ according to $x(t)$. Substitution of $y(t) = C_y x(t)$ in (2) results:

$$J(u,x) = \frac{1}{2} x^T(t_f) H x(t_f) + \frac{1}{2}\int_0^{t_f} (x^T(t) Q x(t) + u^T(t) R u(t)) dt \tag{3}$$

That in (3) we have:

$$H = C_y^T H_y C_y$$

$$Q = C_y^T Q_y C_y$$

That H and Q are positive semi-definite and symmetric matrices. Therefore a constrained optimization problem is obtained with system dynamic equations constrains. With using Lagrange coefficients method and adding constrained equation to cost function (3), we convert it to an unconstrained problem as following:

$$J_a(x,u,p(t)) = J(x,u) + \int_0^{t_f} p^T(t)(Ax(t) + B_u u(t) - \dot{x}(t)) dt \tag{4}$$

That in (4) $p(t)$ is Lagrange coefficient vector or co-state. With using calculus of variations and simplifying the problem the following equations result:

$$\begin{cases} p^T(t_f) = x^T(t_f) H \\ \dot{p}(t) = -x^T(t) Q - p^T(t) A \\ u(t) = -R^{-1} B_u^T p(t) \\ \dot{x}(t) = Ax(t) + B_u u(t) \end{cases} \tag{5}$$

If delete $u(t)$ in (5), we have:

$$\begin{pmatrix}\dot{x}(t)\\ \dot{p}(t)\end{pmatrix}=\begin{pmatrix}A & -B_uR^{-1}B_u^T\\ -Q & -A\end{pmatrix}\begin{pmatrix}x(t)\\ p(t)\end{pmatrix}=Z\begin{pmatrix}x(t)\\ p(t)\end{pmatrix}\quad;\quad\begin{cases}x(0)=x_0\\ p(t_f)=Hx(t_f)\end{cases}\qquad(6)$$

Above system is corresponding Hamiltonian system for (1) and (3). Solution of (6) in $t=t_f$ with using state transient matrix will be:

$$\begin{pmatrix}x(t_f)\\ p(t_f)\end{pmatrix}=e^{Z(t_f-t)}\begin{pmatrix}x(t)\\ p(t)\end{pmatrix}=\begin{pmatrix}\phi_{11}(t_f-t) & \phi_{12}(t_f-t)\\ \phi_{21}(t_f-t) & \phi_{22}(t_f-t)\end{pmatrix}\begin{pmatrix}x(t)\\ p(t)\end{pmatrix}\qquad(7)$$

We have:

$$p(t)=\left[\phi_{22}(t_f-t)-H\phi_{12}(t_f-t)\right]^{-1}\left[H\phi_{11}(t_f-t)-\phi_{21}(t_f-t)\right]x(t)\qquad(8)$$

Or:

$$p(t)=P(t)x(t)\qquad(9)$$

That:

$$P(t)=\left[\phi_{22}(t_f-t)-H\phi_{12}(t_f-t)\right]^{-1}\left[H\phi_{11}(t_f-t)-\phi_{21}(t_f-t)\right]\qquad(10)$$

If we derive from (9) and substitute from (6) then simplify it, we have:

$$\begin{cases}-\dot{P}(t)=P(t)A+A^TP(t)+Q+P(t)B_uR^{-1}B_u^TP(t)\\ P(t_f)=H\end{cases}\qquad(11)$$

This equation called "Riccati Equation" and is a nonlinear time-variant differential equation. Because of $Q,R\geq 0$ and are symmetric, global existence of solutions is guarantee. It has two solutions that positive semi-definite solution ($P(t)\geq 0$) is desirable. Optimal control signal obtained from:

$$\begin{aligned}u_{opt}(t)&=-R^{-1}B_u^TP(t)x(t)=-K(t)x(t)\\ K(t)&=R^{-1}B_u^TP(t)\end{aligned}\qquad(12)$$

Example 1-1: consider the following linear scalar time-invariant system:

$$\dot{x}(t)=x(t)+u(t)$$

We want to find $u(t)$ such that minimize the following cost Function:

$$J=\frac{1}{2}8x^2(10)+\frac{1}{2}\int_0^{10}(3x^2(t)+u^2(t))dt$$

First we have:

$$A=1,B_u=1,t_f=10,H=8,Q=3,R=1$$

Organize Hamiltonian matrix Z:

$$Z=\begin{pmatrix}1 & -1\\ -3 & -1\end{pmatrix}$$

State transient matrix obtained as:

$$e^{Zt}=\begin{pmatrix}\frac{3}{4}e^{2t}+\frac{1}{4}e^{-2t} & -\frac{1}{4}e^{2t}+\frac{1}{4}e^{-2t}\\ -\frac{3}{4}e^{2t}+\frac{3}{4}e^{-2t} & \frac{1}{4}e^{2t}+\frac{3}{4}e^{-2t}\end{pmatrix}$$

Using mentioned equations results:

$$K(t)=\frac{27e^{2(10-t)}+5e^{-2(10-t)}}{9e^{2(10-t)}-5e^{-2(10-t)}}$$

It is clear that optimal feedback law is a nonlinear time-variant vector.

III. Adomian Decomposition Method

In this section we have a brief description of Adomian method. Suppose that we have a nonlinear differential equation in he form of [1]:

$$Lu+Ru+Nu=g(x)\qquad(13)$$

Where L is the highest order derivative which assumed to be easily invertible, R the linear differential operator of less order than L, Nu represents the nonlinear parts and g is the input part. Using inverse operator L^{-1} to both side of (13), we obtain:

$$u=f(x)-L^{-1}(Ru)-L^{-1}(Nu)\qquad(14)$$

That $f(x)$ produced after integration from $g(x)$ and using given initial conditions. In this regard, the nonlinear operator $N(u)=F(u)$ is usually represented by an infinite series of the so-called Adomian polynomials as following:

$$F(u)=\sum_{j=0}^{\infty}A_j\qquad(15)$$

The polynomials A_j are produced for all of nonlinearities so that A_0 depends only on u_0, A_1 depends on u_0 and u_1, and so on. The modified decomposition method defines the solution $u(x)$ by the series $u=\sum_{n=0}^{\infty}u_n$, that components $u_0,u_1,u_2,...$ are usually determined recursively from following equations:

$$\begin{cases}u_0=f_0(x)\\ u_{i+1}=f_{i+1}(x)-L^{-1}(Ru)-L^{-1}(A_i)\end{cases}\qquad(16)$$

And $f(x)$ can be expressed in Taylor series $f(x)=\sum_{k=0}^{\infty}f_k(x)$. We have:

$$\begin{aligned}A_0&=f(u_0)\\ A_1&=u_1(\frac{d}{du_0})f(u_0)\\ A_2&=u_2\frac{d}{du_0}f(u_0)+(\frac{u_1^2}{2!})(\frac{d^2}{du_0^2})f(u_0)\\ A_3&=u_3(\frac{d}{du_0})f(u_0)+u_1u_2(\frac{d^2}{du_0^2})f(u_0)+(\frac{u_1^3}{3!})(\frac{d^3}{du_0^3})f(u_0)\end{aligned}\qquad(17)$$

And so on.

Example 3-1: consider the following nonlinear differential equation [1]:

$$\frac{du}{dt}-u^2=0\quad,\quad u(0)=1$$

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We use Adomian method for this problem. It is obtained that:

$$u=\sum_{n=0}^{\infty}u_n=u(0)+L^{-1}\sum_{n=0}^{\infty}A_n$$
$$u_0=u(0)=1$$
$$u_1=L^{-1}(1)=t$$
$$u_2=t^2$$
$$u_3=t^3$$

And $u=\sum_{n=0}^{\infty}t^n=\frac{1}{1-t}$ is the exact solution.

IV. Description of Method

In this section we describe the application of Adomian method for solving Riccati equation. According to (11), we have a nonlinear matrix equation with two solutions. The positive definite solution is acceptable. First, we introduce variables $\tau=t_f-t$ and $B=B_uR^{-1}B_u^T$. Then we have:

$$\begin{cases}\dot{P}(\tau)=P(\tau)A+A^TP(\tau)+Q+P(\tau)BP(\tau)\\ P(0)=H\end{cases} \quad (18)$$

Let now $L=\frac{d}{dx}$, so we have $LP=\dot{P}$ and $NP=PBP$, that N is nonlinear operator. With substitution of mentioned parameters, (18) becomes:

$$\dot{P}=PA+A^TP+Q+NP \quad (19)$$

And in terms of inverse operator $L^{-1}=\int_0^{\tau}[.]da$:

$$P=L^{-1}(PA+A^TP)+L^{-1}Q+L^{-1}NP \quad (20)$$

Now, we can apply the Adomian decomposition method mentioned in previous section to (20) and find solution. Suppose that solution is $P=\sum_{n=0}^{\infty}P_n$ and writing nonlinear part in the form of below Adomian polynomials:

$$NP=\sum_{n=0}^{\infty}A_n \quad (21)$$

Therefore (20) becomes:

$$\sum_{n=0}^{\infty}P_n=H+L^{-1}(\sum_{n=0}^{\infty}P_nA+A^T\sum_{n=0}^{\infty}P_n)+L^{-1}Q+L^{-1}\sum_{n=0}^{\infty}A_n \quad (22)$$

Thus we can find the components of solution (P_n) as:

$$\begin{aligned}&P_0=H+L^{-1}Q\\ &P_1=L^{-1}(P_0A+A^TP_0)+L^{-1}A_0\\ &P_2=L^{-1}(P_1A+A^TP_1)+L^{-1}A_1\\ &.\\ &.\\ &P_n=L^{-1}(P_{n-1}A+A^TP_{n-1})+L^{-1}A_{n-1}\end{aligned} \quad (23)$$

Now, we shall produce A_n polynomials for completion of method. There is a step by step method for finding A_n as:

$$\begin{aligned}&A_0=P_0BP_0\\ &A_1=P_1BP_0+P_0BP_1\\ &A_2=P_2BP_0+P_1BP_1+P_0BP_2\\ &.\\ &.\\ &A_n=\sum_{i=0}^{n}P_iBP_{n-i}, \qquad n\geq 0\end{aligned} \quad (24)$$

Thus with substitution (24) in (23) we have the following step by step equations:

$$\begin{aligned}&P_0=H+L^{-1}Q\\ &P_1=L^{-1}(P_0A+A^TP_0)+L^{-1}P_0BP_0\\ &P_2=L^{-1}(P_1A+A^TP_1)+L^{-1}\{P_0BP_1+P_1BP_0\}\\ &.\\ &.\\ &P_n=L^{-1}(P_{n-1}A+A^TP_{n-1})+L^{-1}\left\{\sum_{i=0}^{n-1}P_iBP_{n-i}\right\}, \quad n\geq 1\end{aligned} \quad (25)$$

Therefore with computing partial solution P_n and calculation of sum of them, we can find approximated response with a desirable accuracy. It is clear that when we use a great number of partial solution, obtained response is more accurate.

Now, we use the mentioned algorithm for a typical example.

Example 4-1: consider the following differential equation:

$$\begin{cases}\dot{P}(t)=-P^2(t)+1\\ P(0)=0\end{cases}$$

In this equation we have:

$$A=0, B=-1, Q=1, H=0$$

Exact solution of this equation is:

$$P(t)=\frac{e^{2t}-1}{e^{2t}+1}$$

If we use the mentioned method introduced in this section, we have:

$$P_0=H+L^{-1}(Q)=t$$
$$P_1=L^{-1}(P_0A+A^TP_0)+L^{-1}P_0BP_0=L^{-1}(-t^2)=-\frac{1}{3}t^3$$
$$P_2=L^{-1}(P_1A+A^TP_1)+L^{-1}(P_1BP_0+P_0BP_1)=\frac{2}{15}t^5$$

And so on.

Therefore, we consider $\Phi_n=\sum_{i=0}^{n-1}P_i, \quad n\geq 1$ as partial solution of Riccati equation. So:

$$\Phi_1=t$$
$$\Phi_2=t-\frac{1}{3}t^3$$
$$\Phi_3=t-\frac{1}{3}t^3+\frac{2}{15}t^5$$

We plot exact and approximated solutions of Riccati equation in figure 1.

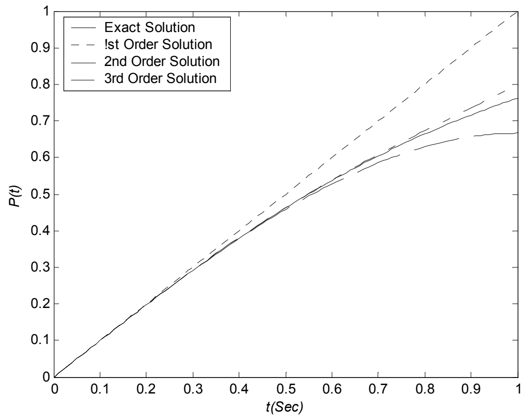

Fig. 1. Exact and approximated solutions of Riccati equation mentioned in example 4-1.

From fig.1, it is clear that by increasing the number of Adomian partial solutions, accuracy of solution increases. Now we calculate error of solutions in $t = 1Sec$ for comparing them. We have absolute error in different cases as following:

- Case1: 0.2384
- Case 2: 0.0949
- Case 3: 0.0384

This confirms that by increasing partial sum of solution, error reduces. Now we consider another example for this purpose.

Example 4-2: consider the following Ricatti equation:

$$\begin{cases} \dot{P}(t) = 2P(t) - P^2(t) + 1 \\ P(0) = 0 \end{cases}$$

In this case we have:

$A = 1, B = -1, Q = 1, H = 0$

Exact solution of this equation is:

$$P(t) = 1 + \sqrt{2}\tanh(\sqrt{2}t + \frac{1}{2}\log(\frac{\sqrt{2}-1}{\sqrt{2}+1}))$$

If we use Adomian decomposition method for this equation, we have:

$$P_0 = H + L^{-1}(Q) = t$$

$$P_1 = L^{-1}(P_0 A + A^T P_0) + L^{-1} P_0 B P_0 = L^{-1}(2t) + L^{-1}(-t^2) = t^2 - \frac{1}{3}t^3$$

$$P_2 = L^{-1}(P_1 A + A^T P_1) + L^{-1}(P_1 B P_0 + P_0 B P_1) = \frac{2}{3}t^3 - \frac{2}{3}t^4 + \frac{2}{15}t^5$$

Therefore we consider $\Phi_n = \sum_{i=0}^{n-1} P_i, \quad n \geq 1$ as partial solution of Riccati equation. So:

$$\Phi_1 = t$$

$$\Phi_2 = t + t^2 - \frac{1}{3}t^3$$

$$\Phi_3 = t + t^2 + \frac{1}{3}t^3 - \frac{2}{3}t^4 + \frac{2}{15}t^5$$

We plot exact and approximated solutions of Riccati equation in figure 2.

From figure 2 it is clear that accuracy of solution increases with using more number of Adomian partial solutions in our response. Now, we calculate absolute error of solution in $t = 0.8Sec$ for comparing them. We have absolute error in different cases as following:

- Case1: 0.5464
- Case 2: 0.0771
- Case 3: 0.0301

This confirms that by increasing partial sum of solution, error reduces.

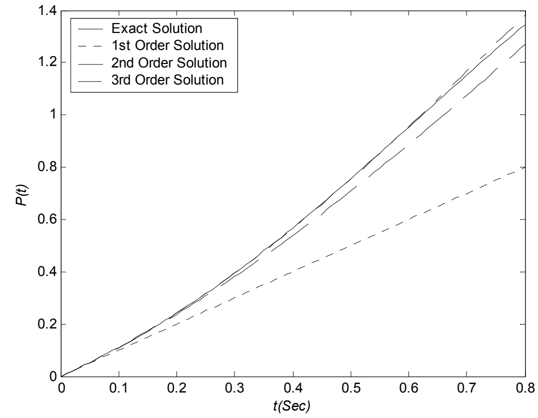

Fig. 2. Exact and approximated solutions of Riccati equation mentioned in example 4-2.

Therefore, we used introduced method for solving Riccati equation and considered application of it. Also effect of number of partial solutions considered in accuracy of solution.

V. CONCLUSION

We introduced a new method for solving Riccati equation and it was considered that increasing in number of partial solutions causes decreasing in error of approximated solution.

For future works, we can use this method with some modification for solving HJB and HJI equations. Also we can use this method for analyzing singular perturbation systems.

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