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On the MADM solution of coupled wave system with mixed boundary condition

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Abstract: - an analytical solution for a system of parallel vibrating strings is derived. To treat the problem mixed boundary condition (velocity feedback), modified Adomian decomposition method is employed. A close conformity exists between the obtained results and those of numerical ones.

Key-Words: - Modified Adomian Decomposition Method–Mixed Boundary condition– Analytical solution.

1 Introduction

Stability is very desirable for an elastic system. The energy of system should be evaluated, and if rate of energy is negative, the system is stable.

In this paper, we will investigate the stabilization properties of vibrating strings in parallel whose energy will be damped out by boundary velocity feedback via MADM. The governing equation of such a system is described by the following system of wave equations (mixed initial – boundary value problem);

$$\left.\begin{aligned} u_{tt}-\sigma_1{}^2u_{xx}&=\alpha(v-u)\\ v_{tt}-\sigma_2{}^2v_{xx}&=\alpha(u-v)\end{aligned}\right\}\text{in}\begin{cases}t\in(0,\infty)\\ x\in(0,1)\end{cases} \tag{1}$$

where $\sigma>0$ and $\alpha>0$ .The initial conditions are

$$\begin{aligned} &u(x,0)=u_0,\ \ u_t(x,0)=u_1\\ &v(x,0)=v_0,\ \ v_t(x,0)=v_1\end{aligned} \tag{2}$$

with the prescribed boundary conditions

$$\begin{aligned} &u(0,t)=0,\ \ u_x(1,t)=-\beta_1u_t(1,t)\\ &v(0,t)=0,\ \ v_x(1,t)=-\beta_2v_t(1,t),\quad t\ge 0\end{aligned} \tag{3}$$

Here t, x, σ_1 and σ_2 are the time, space variable and wave propagation speeds, respectively. the variables u, v are the deflections of the strings from their equilibrium positions. The wave speed, σ, and the spring constant, α, are the system parameters. The damping coefficients $\beta_i>0\,(i=1,2)$ depend on the control devices. These parameters play an important role in the physical behavior of the system. Generally, this boundary control corresponds to a control mechanism which monitors u_t, v_t at x=1 or at x=0. This phenomenon takes place if the system is exposed to external forces or by velocity feedback boundary conditions (Eq. 3).

This problem is motivated by an analogous problem in ordinary differential equations for coupled oscillators, and has a potential application in oscillation of objects from outside disturbances. Associated with each solution of (1) is its total natural energy at time t [1]:

$$\begin{aligned} E(t)=\frac{1}{2}\int_0^1(&|u_t|^2+c^2|u_x|^2+|v_t|^2+c^2|v_x|^2+\\ &\alpha|u-v|^2)dx\end{aligned} \tag{4}$$

Adomian decomposition method (ADM) is very powerful method to find analytical solutions for ODE and PDE [2-7]. ADM solution to mixed boundary condition problem (for example Eq. 1) is not so straight forward and easily obtainable.

In this study, modified ADM is utilized to solve the system of Eqs. (1).

2 Analysis of MADM to linear PDE

Here, the following linear equation [2] is considered as

$$L_tu+L_xu+Ru=g\,, \tag{5}$$

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Where

$$L_t = \partial^2(.)/\partial t^2,\ L_x = \partial^2(.)/\partial x^2,\quad R = \rho(x,t).$$

Let:

$$\begin{aligned} u &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} a_{n,k}x^n t^k = \sum_{n=0}^{\infty} x^n\left[\sum_{k=0}^{\infty} a_{n,k}t^k\right] \\ &= \sum_{n=0}^{\infty} a_n(t)x^n, \\ R &= \rho(x,t) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \rho_{n,k}x^n t^k \\ &= \sum_{n=0}^{\infty} \rho_n(t)x^n \\ g &= g(x,t) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} g_{n,k}x^n t^k \\ &= \sum_{n=0}^{\infty} g_n(t)x^n. \end{aligned} \tag{6}$$

By the ADM, the inverse operator of L_x is applied on both sides of (5), hence,

$$\begin{aligned} u &= \phi_x + L_x^{-1}g - L_x^{-1}L_t u - L_x^{-1}Ru, \\ \phi_x &= u(x=0,t) + x\,\partial u/\partial x(x=0,t) \end{aligned} \tag{7}$$

where $L_x^{-1} = \int_0^x\int_0^x (.)dxdx$. Substituting for u, g and ρ from (6) into (7), and carry out integrations. Finally one can derive at

$$\begin{aligned} \sum_{n=0}^{\infty} a_n(t)x^n &= \phi_x + \sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+1)(n+2)}g_n(t) - \\ &\sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+1)(n+2)}\frac{\partial^2}{\partial t^2}a_n(t) - \\ &\sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+1)(n+2)}\sum_{v=0}^{n}\rho_v(t)a_{n-v}(t) \end{aligned} \tag{8}$$

Let $n \to n-2$ on the right side of Eq. (8), then Eq. (8) becomes:

$$\begin{aligned} \sum_{n=0}^{\infty} a_n(t)x^n &= \phi_x + \sum_{n=2}^{\infty}\frac{x^n}{(n)(n-1)}g_{n-2}(t) - \\ &\sum_{n=2}^{\infty}\frac{x^n}{(n)(n-1)}\frac{\partial^2}{\partial t^2}a_{n-2}(t) - \\ &\sum_{n=2}^{\infty}\frac{x^n}{(n)(n-1)}\sum_{v=0}^{n-2}\rho_v(t)a_{n-2-v}(t) \end{aligned} \tag{9}$$

Finally, equating coefficients of like power of x in Eq. (9), we derive the recursion formula for the coefficient a_n

$$\begin{aligned} a_0 &= u(x=0,t) = c_0, \\ a_1 &= \partial u/\partial x(x=0,t) = c_1 \end{aligned} \tag{10}$$

and for $n \ge 2$,

$$a_n = \frac{g_{n-2}(t) - \left(\partial^2/\partial t^2\right)a_{n-2}(t)}{n(n-1)} - \frac{\sum_{v=0}^{n-2}\rho_v(t)a_{n-2-v}(t)}{n(n-1)} \tag{11}$$

The final solution is given by

$$u(x,t) = \sum_{n=0}^{\infty} a_n(t)x^n. \tag{12}$$

The "modified decomposition" series solutions have been found for initial-value problems by incorporating and adapting ideas of the decomposition method. The procedure can be further generalized by using the double decomposition technique [2]. This enables one to treat initial-value and boundary-value problems in a similar fashion and computationally efficient formulation.For this end, using double decomposition,

$$\begin{aligned} c_0 &= \sum_{m=0}^{\infty} c_0^m, \\ c_1 &= \sum_{m=0}^{\infty} c_1^m, \\ a_n &= \sum_{m=0}^{\infty} a_n^m \end{aligned} \tag{13}$$

By manipulating Eqs. (10, 11) to fulfill the boundary conditions,

$$a_0^m = c_0^m,\ a_1^m = c_1^m \tag{14}$$

and for $n \ge 2$,

$$a_n^m = \frac{g_{n-2}^m(t) - \left(\partial^2/\partial t^2\right)a_{n-2}^m(t)}{n(n-1)} - \frac{\sum_{v=0}^{n-2}\rho_v(t)a_{n-2-v}^m(t)}{n(n-1)} \tag{15}$$

Now, we have from Eq'ns. (12) and (13):

$$u=\sum_{n=0}^{\infty}a_n x^n=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}a_n^{(m)}x^n \qquad (16)$$

The next task is to determine the components of c_0 and c_1 using Eq. (13). Since the solution u is unknown, generally at $x=\alpha$ and $x=\beta$, the approximant φ_{m+1} for solution u is used (see Eq. 17). for this end, we stagger the series:

$$\begin{aligned}
u_0&=a_0^{(0)}+a_1^{(0)}x\\
u_1&=a_0^{(1)}+a_1^{(1)}x+a_2^{(0)}x^2+a_3^{(0)}x^3\\
u_2&=a_0^{(2)}+a_1^{(2)}x+a_2^{(1)}x^2+a_3^{(1)}x^3+a_4^{(0)}x^4+a_5^{(0)}x^5\\
&.\\
&.\\
&.\\
u_m&=a_0^{(m)}+a_1^{(m)}x+\ldots+a_{2m}^{(0)}x^{2m}+a_{2m+1}^{(0)}x^{2m+1}.
\end{aligned}$$

This can be written as:

$$u_m=\sum_{n=0}^{2m+1}a_n^{(m-\{n/2\})}x^n,$$

Where, {n/2} is the first integer greater than n/2.
We now have a different decomposition of u, which is suitable for boundary-value problems as:

$$u=\sum_{n=0}^{\infty}u_m=\sum_{m=0}^{\infty}\sum_{n=0}^{2m+1}a_n^{(m-\{n/2\})}x^n.$$

Then, we derive the solution approximation as:

$$\varphi_{m+1}\{u\}=\sum_{n=0}^{m}u_n=\sum_{n=0}^{m}a_0^{(n)}+x\sum_{n=0}^{m}a_1^{(n)}+x^2\sum_{n=0}^{m-1}a_2^{(n)}+$$
$$x^3\sum_{n=0}^{m-1}a_3^{(n)}+\ldots+x^{2m}a_{2m}^{(0)}+x^{2m+1}a_{2m-1}^{(0)},$$

and hence,

$$\varphi_{m+1}\{u\}=\varphi_{m+1}\{a_0\}+x\varphi_{m+1}\{a_1\}+$$
$$x^2\varphi_m\{a_2\}+x^3\varphi_m\{a_3\}+\ldots+$$
$$x^{2m}\varphi_1\{a_{2m}\}+x^{2m+1}\varphi_1\{a_{2m-1}\}$$

or,

$$\varphi_{m+1}\{u\}=\sum_{n=0}^{2m+1}\varphi_{m+1-[n/2]}\{a_n\}x^n \qquad (17)$$

Where, $\varphi_m\{a_n\}=\sum_{v=0}^{m-1}a_n^v$, and [n/2] is the first integer smaller than n/2.
Obviously, as $m\to\infty$, we have:

$\lim\varphi_{m+1}\{u\}\to u.$

Therefore, using the approximate φ_{m+1} with boundary conditions, we can compute the constants $c_0^{(m)}$, $c_1^{(m)}$, $a_0^{(m)}$, and $a_1^{(m)}$. Having found the coefficients of the Maclaurin series, the final approximated solution is obtained (Eq. 17).

3 Application MADM to system (1)

Let us consider the system of coupled wave equation (1) with following initial conditions

$$\left.\begin{aligned}u_{tt}-\sigma_1{}^2u_{xx}&=\alpha(v-u)\\ v_{tt}-\sigma_2{}^2v_{xx}&=\alpha(u-v)\end{aligned}\right\}\text{in}\begin{cases}t\in(0,\infty)\\ x\in(0,1)\end{cases} \qquad (18)$$

$$\begin{aligned}&u(x,0)=\sin(\pi x),\quad u_t(x,0)=0,\\ &v(x,0)=\sin(\pi x),\quad v_t(x,0)=0,\end{aligned} \qquad (19)$$

and prescribed boundary conditions

$$\begin{aligned}&u(0,t)=0,\quad u_x(1,t)=-\beta_1u_t(1,t)\\ &v(0,t)=0,\quad v_x(1,t)=-\beta_2v_t(1,t),\ \mathrm{t}\geq 0\end{aligned} \qquad (20)$$

Without loss of generally, to compare solutions of MADM with finite difference method in [1], we assume:

$\sigma_1=\sigma_2=1, \alpha=1, \beta_1=1$ and $\beta_2=2$

The MADM solution using the x partial solution (in this system that boundary is mixed, t partial solution gives incorrect results because the solution doesn't confirm boundary condition) is given by:

$$\begin{aligned}u&=c_0+c_1x+L_x^{-1}u_{tt}-L_x^{-1}(v-u),\\ v&=d_0+d_1x+L_x^{-1}v_{tt}-L_x^{-1}(u-v),\end{aligned} \qquad (21)$$

where c_0, c_1, d_0 and d_1 are the constant of integrations. In order to find c_0 and d_0 in Eq. (21), we apply boundary conditions, Eq. (20).

$$\begin{aligned}&u(0,t)=0\to c_0=0\\ &v(0,t)=0\to d_0=0\end{aligned}$$

Having considered Eq. (6), one can find that:

$$
\begin{aligned}
u &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} a_{n,k}x^n t^k = \sum_{n=0}^{\infty} x^n\left[\sum_{k=0}^{\infty} a_{n,k}t^k\right] \\
&= \sum_{n=0}^{\infty} a_n(t)x^n \\
v &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} b_{n,k}x^n t^k = \sum_{n=0}^{\infty} x^n\left[\sum_{k=0}^{\infty} b_{n,k}t^k\right] \\
&= \sum_{n=0}^{\infty} b_n(t)x^n
\end{aligned} \tag{22}
$$

Substituting Eq. (22) into Eq. (20), we have:

$$
\begin{aligned}
&\sum_{n=0}^{\infty} a_n(t)x^n = c_1 x + \int_0^x\int_0^x \sum_{n=0}^{\infty} \ddot{a}_n(t)x^n dxdx - \\
&\int_0^x\int_0^x \sum_{n=0}^{\infty} (b_n(t)-a_n(t))dxdx, \\
&\sum_{n=0}^{\infty} b_n(t)x^n = d_1 x + \int_0^x\int_0^x \sum_{n=0}^{\infty} \ddot{b}_n(t)x^n dxdx - \\
&\int_0^x\int_0^x \sum_{n=0}^{\infty} (a_n(t)-b_n(t))dxdx.
\end{aligned} \tag{23}
$$

By carrying out integrations and let $n \to n-2$, we have

$$
\begin{aligned}
&\sum_{n=0}^{\infty} a_n(t)x^n = c_1 x + \sum_{n=2}^{\infty} \ddot{a}_{n-2}(t)x^n - \\
&\sum_{n=2}^{\infty} (b_{n-2}(t)-a_{n-2}(t))x^n, \\
&\sum_{n=0}^{\infty} b_n(t)x^n = d_1 x + \sum_{n=2}^{\infty} \ddot{b}_{n-2}(t)x^n - \\
&\sum_{n=2}^{\infty} (a_{n-2}(t)-b_{n-2}(t))x^n.
\end{aligned} \tag{24}
$$

Following the procedure in preceding section, one can derive

$$
\begin{aligned}
&a_0 = u(x=0,t) = c_0 = 0, \\
&a_1 = \partial u/\partial x(x=0,t) = c_1 \\
&b_0 = v(x=0,t) = d_0 = 0, \\
&b_1 = \partial v/\partial x(x=0,t) = d_1
\end{aligned}
$$

and for $n \ge 2$

$$
\begin{aligned}
a_n &= \frac{1}{n(n-1)}(\ddot{a}_{n-2} - (b_{n-2} - a_{n-2})) \\
b_n &= \frac{1}{n(n-1)}(\ddot{b}_{n-2} - (a_{n-2} - b_{n-2}))
\end{aligned}
$$

Then considering double decomposition, we get

$$
\begin{aligned}
&a_n = \sum_{m=0}^{\infty} a_n^m, b_n = \sum_{m=0}^{\infty} b_n^m \\
&c_i = \sum_{m=0}^{\infty} d_i^m, d_i = \sum_{m=0}^{\infty} d_i^m \\
&(i=0,1) \\
&\begin{cases} c_0 = 0 \\ d_0 = 0 \end{cases} \to \begin{cases} a_0^m = 0, \\ b_0^m = 0, \end{cases} \\
&\mathit{and} \\
&\begin{cases} a_1^m = c_1^m \\ b_1^m = d_1^m \end{cases}
\end{aligned} \tag{25}
$$

and for $n \ge 2$

$$
\begin{aligned}
a_n^m &= \frac{1}{n(n-1)}(\ddot{a}_{n-2}^m - (b_{n-2}^m - a_{n-2}^m)) \\
b_n^m &= \frac{1}{n(n-1)}(\ddot{b}_{n-2}^m - (a_{n-2}^m - b_{n-2}^m))
\end{aligned} \tag{26}
$$

Now following the preceding section, we use boundary condition, Eq. (20), and Eq. (17) to compute the constants c_1^m and d_1^m :

$$
\begin{aligned}
&\varphi_{m+1}\{u\}(t,x=0) = 0, \\
&\frac{d}{dx}(\varphi_{m+1}\{u\}(t,x))\Big|_{x=1} = \\
&-\frac{d}{dt}(\varphi_{m+1}\{u\}(t,x))\Big|_{x=1} \\
&\varphi_{m+1}\{v\}(t,x=0) = 0, \\
&\frac{d}{dx}(\varphi_{m+1}\{v\}(t,x))\Big|_{x=1} = \\
&-2\frac{d}{dt}(\varphi_{m+1}\{v\}(t,x))\Big|_{x=1}
\end{aligned} \tag{27}
$$

Consequently, for m=0 we have

$$
\begin{aligned}
&\varphi_1\{u\}(t,x=0) = \sum_{n=0}^{1} \varphi_{1-[n/2]}\{a_n\}x^n \\
&\to a_0^0 + a_1^0(0) = 0 \to 0+0=0 \\
&\frac{d}{dx}(\varphi_1\{u\}(t,x))\Big|_{x=1} = \\
&-\frac{d}{dt}(\varphi_1\{u\}(t,x))\Big|_{x=1}
\end{aligned} \tag{28}
$$

$\rightarrow a_1^0(1) = -\dot{a}_1^0(1)$

$\rightarrow a_1^0 = k_1 \exp(-t),\ k_1 = \text{constant}$

Similarly,

$$\varphi_1\{v\}(t, x=0) = \sum_{n=0}^{1} \varphi_{1-[n/2]}\{b_n\}x^n$$

$\rightarrow b_0^0 + b_1^0(0) = 0 \rightarrow 0 + 0 = 0$

$$\frac{d}{dx}(\varphi_1\{v\}(t,x))\bigg|_{x=1} =$$

$$-2\frac{d}{dt}(\varphi_1\{v\}(t,x))\bigg|_{x=1}$$

$\rightarrow b_1^0(1) = -2\dot{b}_1^0(1)$

$\rightarrow b_1^0 = l_1 \exp(-\frac{1}{2}t),\ l_1 = \text{constant}$

Now, using Eq. (26), we can evaluate $a_m^0,\ b_m^0$. To obtain $\varphi_2\{u\}, \varphi_2\{v\}$, substituting the above solution into Eq. (27). Similarly, we have

$$a_1^1 = -\frac{2}{3}k_1 t \exp(-t) + \frac{5}{6}l_1 \exp(-\frac{1}{2}t) +$$
$$k_2 \exp(-t)$$
$$b_1^1 = -\frac{5}{24}l_1 t \exp(-\frac{1}{2}t) - \frac{1}{6}k_1 \exp(-t) +$$
$$l_2 \exp(-\frac{1}{2}t)$$

where k_2 and l_2 are constants. Using Eq. (26), we have $a_m^l,\ b_m^l$. Then we have $\varphi_2\{u\}, \varphi_2\{v\}$:

$$\varphi_2\{u\} = [(k_1 + k_2)\exp(-t) -$$
$$\frac{2}{3}k_1 t \exp(-t) - \frac{5}{6}k_1 \exp(-\frac{1}{2}t)]x +$$
$$\frac{1}{6}[2k_1 \exp(-t) - l_1 \exp(-\frac{1}{2}t)]x^3$$
$$\varphi_2\{v\} = [(l_1 + l_2)\exp(-\frac{1}{2}t) -$$
$$\frac{5}{24}l_1 t \exp(-\frac{1}{2}t) - \frac{1}{6}k_1 \exp(-t)]x +$$
$$\frac{1}{6}[\frac{5}{4}l_1 \exp(-\frac{1}{2}t) - k_1 \exp(-t)]x^3$$

One can continue this procedure to find $\varphi_3, \varphi_4, \ldots$. In order to find constants φ_{m+1}, k_i and l_i (i=1-2,…, m) we apply initial condition, Eq. (19).

For example, when we compute $\varphi_7\{u\}, \varphi_7\{v\}$, with Maple 10, we have 14 constants (k_i and l_i (i=1-7). Due to initial conditions in Eq. (19), these constants can be found by the expansion of the $\sin(\pi x)$, and equating the coefficients of like powers of x (4 equations for u and 4 equations for v), and also, similarly, equating $\dot{\phi}_7\{u\}(x,0)$ and $\dot{\phi}_7\{v\}(x,0)$ coefficients to zero (3 equations for u and 3 equations for v).

Now we use the solution in the energy of the system, Eq. (4), for solutions, $\varphi_7\{u\}$ and $\varphi_7\{v\}$ to investigate stability of velocity feedback controllers. The final results are shown in Fig. 1, which is very close to the finite difference solution in [1]. Obviously with more terms in $\varphi\,(\varphi_8, \varphi_9, \ldots)$ one can obtain better confirmation.

This shows that MADM is strong method to find approximate solution analytically without using numerical computation. We also conclude from Fig. 1 that system (1) is stable.

4 Conclusions

We found the approximate solution analytically to the coupled wave equations with velocity feedback boundary conditions (mixed conditions), using MADM. We used Maple 10 to evaluate this solution. We then computed the energy of the system to investigate the stability of parallel strings with velocity feedback. The results are reasonably close to the benchmark finite difference data. Thus, MADM is a strong approach to solve such problems without using numerical techniques.

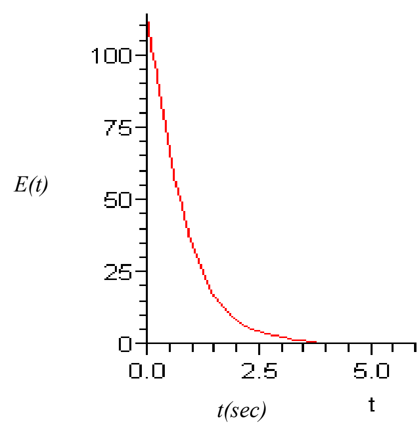

Figure 1. Curve energy (E) VS. t (E solved with $\varphi_7\{u\}$ and $\varphi_7\{v\}$).

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