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A study of Blasius viscous flow: an ADM analytical solution

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Abstract: - In this study, an analytical solution is obtained for the Blasius viscous flow problem, utilizing Adomian decomposition method (ADM). A very close agreement between ADM results and corresponding numerical ones is apparent.

Key-Words: - Adomian Decomposition Method– Adomian Polynomials – Blasius– Analytical solution.

1 Introduction

Adomian decomposition method [1, 2] is a powerful straightforward method. ADM is apt to be utilized as an alternative approach to current techniques being employed to a wide variety of physical problems. Recently, this method has attracted a wide class of audiences in all fields of science.

In this study, ADM is adopted to study the Blasius flow. Flow over flat plate was first considered by T. Von Kerman in 1921 [3]. He provided an analysis to the problem by assuming a simple parabolic approximation for velocity profiles. In 1908, Blasius, by use of an ingenious coordinate transformation, solved the boundary-layer equations for laminar flow over a flat plate [3-6].

Blasius obtained a single third-order nonlinear ordinary differential equation for *f*, as,

$$f'''(\eta)+\frac{1}{2}f(\eta)f''(\eta)=0.$$

It is worthy to note, accurate solutions for Blasius equation have been computed only by numerical integration [3].

2 Basic Method

Normally, a general equation of the form, $Fu=g(t)$, is decomposed into linear and nonlinear terms. In turn, the linear part can be separated further into highest order derivative and the remainder of the linear terms. Thus the above general equation can be represented by,

$$Lu+Ru+Nu=g. \tag{1}$$

where the operators *F, L, R, N* represent the general nonlinear equation, the highest order derivative, the remainder of linear terms and the nonlinear part, respectively.

The next step in ADM is solving for *Lu* and applying the inverse operator of *L*, that is L^{-1}, on both sides of equation. Hence,

$$Lu=g-Ru-Nu, \tag{2}$$

$$L^{-1}Lu=L^{-1}g-L^{-1}Ru-L^{-1}Nu. \tag{3}$$

Let *L* represents a second order derivative with respect to *t*, so L^{-1} will be a twofold integral operator. Then (3) yields,

$$u=u(0)+tu'(0)+L^{-1}g-L^{-1}Ru-L^{-1}Nu. \tag{4}$$

Of course, $u(0)$ and $tu'(0)$ terms are the constants of the integration.

Pursuing the ADM procedure [1], the decomposed *u* (5) and the decomposed nonlinear terms (6) are introduced into (4).

$$u=\sum_{n=0}^{\infty}u_n, \tag{5}$$

$$Nu=\sum_{n=0}^{\infty}A_n(u_0,u_1,\ldots,u_n). \tag{6}$$

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The Adomian polynomials, A_n, are generated by [3],

$$A_n = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n} N(v(\lambda))\right]_{\lambda=0} \quad, n = 0,1,2,\dots \tag{7}$$

The resulting equation is,

$$\begin{aligned} u &= \sum_{n=0}^{\infty} u_n = u(0) + tu'(0) + L^{-1}g \\ &- L^{-1}R\sum_{n=0}^{\infty} u_n - L^{-1}\sum_{n=0}^{\infty} A_n, \end{aligned} \tag{8}$$

in which the terms $u(0)+tu'(0)+L^{-1}g$ are recognized as u_0 which is to be calculated by use of boundary conditions.

According to Adomian decomposition method, well described in [1], the equation is transformed to a set of recursive relations given by,

$$\begin{aligned} u_0 &= u(0) + tu'(0) + L^{-1}g, \\ u_1 &= -L^{-1}Ru_0 - L^{-1}A_0, \\ u_2 &= -L^{-1}Ru_1 - L^{-1}A_1, \\ &\vdots \\ u_{n+1} &= -L^{-1}Ru_n - L^{-1}A_n. \qquad n \ge 1 \end{aligned} \tag{9}$$

Now, all the components of the solution, u_n, are calculated (9). The complete solution is $u = \lim_{k\to\infty} \varphi_k$. But, since the series converges very rapidly, the k-term approximation can be use as a practical solution.

$$\varphi_k = \sum_{i=0}^{k-1} u_i. \tag{10}$$

3 Blasius' viscous flow equation

A two dimensional laminar viscous flow past a semi-infinite flat plate, is governed by,

$$f'''(\eta) + \frac{1}{2} f(\eta) f''(\eta) = 0, \tag{11}$$

with the boundary conditions,

$$f(0) = f'(0) = 0, \quad f'(\infty) = 1, \tag{12}$$

where

$$f'(\eta) = \frac{df(\eta)}{d\eta}. \tag{13}$$

And η is a similarity variable,

$$\eta = y\sqrt{U_\infty /(\upsilon x)}. \tag{14}$$

The dimensionless function $f(\eta)$ is related to the stream function $\psi(x,y)$ by

$$f(\eta) = \psi / \sqrt{\upsilon x U_\infty}, \tag{15}$$

where U_∞ is the constant velocity of mainstream at infinity, υ is the kinematic viscosity coefficient and x, y are two independent variables. For details see [6].

In 1908, Blasius [8] provided a solution in power series, as follows,

$$f(\eta) = \sum_{k=0}^{+\infty}\left(-\frac{1}{2}\right)^k \frac{A_k \sigma^{k+1}}{(3k+2)!}\eta^{3k+2}, \tag{16}$$

where $\sigma = f''(0)$,

$$\begin{aligned} A_0 &= A_1 = 1, \\ A_k &= \sum_{r=0}^{k-1}\binom{3k-1}{3r} A_r A_{k-r-1}, \ (k \ge 2). \end{aligned} \tag{17}$$

If the Eq. (16) is expanded, we have,

$$\begin{aligned} f(\eta) &= \frac{1}{2}\sigma\eta^2 - \frac{1}{240}\sigma^2\eta^5 \\ &+ \frac{11}{161280}\sigma^3\eta^8 - \frac{5}{4257792}\sigma^4\eta^{11} + \cdots. \end{aligned} \tag{18}$$

Blasius evaluated σ by demonstrating another approximation of $f(\eta)$ at large η. Then, by means of matching two different approximations at a proper point, he obtained the numerical result $\sigma = 0.332$. And in 1938, by means of a numerical technique, Howarth [9] gained a more accurate value $\sigma = 0.332057$ utilized to solve Belasius Eq. (11).

4 Applying ADM to Blasius equation

The Blasius equation, (11), is rewritten in the operator form,

$$Lf + \frac{1}{2} ff'' = 0. \tag{19}$$

Where L is a third order derivative with respect to η. Solving for Lf,

$$Lf = -\frac{1}{2} ff''. \tag{20}$$

Following the rationale of ADM as in (9), one can find,

$$L^{-1}Lf = L^{-1}(-\frac{1}{2} ff''). \tag{21}$$

Where L^{-1} is three fold integral operator,

$$L^{-1}(.) = \int_0^\eta\int_0^\eta\int_0^\eta (.)d\eta\, d\eta\, d\eta. \tag{22}$$

So, (21) leads to,

$$f(\eta) = c_1 + c_2\eta + c_3\frac{\eta^2}{2} + L^{-1}(-\frac{1}{2} ff''), \tag{23}$$

where the first three terms are constants of integration, yet to be determined by employing (12). By ADM, f is decomposed into,

$$f = \sum_{n=0}^{\infty} f_n \tag{24}$$

The nonlinear term in (23) is decomposed to Adomian polynomials,

$$Nf = \sum_{n=0}^{\infty} A_n\left(f_0, f_1, \dots, f_n\right). \tag{25}$$

For the nonlinear term in (23), $\left(-\frac{1}{2} ff''\right)$, the corresponding Adomian polynomials are evaluated by (7),

$$\begin{aligned} A_0 &= -\frac{1}{4}\sigma^2\eta^2, \\ A_1 &= \frac{11}{480}\sigma^3\eta^5, \\ A_2 &= -\frac{25}{21504}\sigma^4\eta^8, \\ A_3 &= \frac{9299}{212889600}\sigma^5\eta^{11}, \\ &\vdots \end{aligned} \tag{26}$$

Introducing (26) into (23) yields,

$$\sum_{n=0}^{\infty} f_n = c_1 + c_2\eta + c_3\frac{\eta^2}{2} + L^{-1}\sum_{n=0}^{\infty} A_n. \tag{27}$$

Eq. (27) can be rewritten as,

$$f_0 + \sum_{n=0}^{\infty} f_{n+1} = c_1 + c_2\eta + c_3\frac{\eta^2}{2} + L^{-1}\sum_{n=0}^{\infty} A_n. \tag{28}$$

The required recursive relations take the form of,

$$\begin{cases} f_0 = c_1 + c_2\eta + c_3\dfrac{\eta^2}{2}, \\ \displaystyle\sum_{n=0}^{\infty} f_{n+1} = L^{-1}A_n. \end{cases} \tag{29}$$

The approximant solution for f can be found by,

$$\varphi_{k+1} = \sum_{n=0}^{k} f_n(\eta). \tag{30}$$

If more accuracy is needed, more terms should be calculated.

To complete the solution, the integration constants should be evaluated via boundary conditions (12).

In this step, let $\sigma = f''(0)$, where σ was obtained numerically [9].

One-term approximation is evaluated via boundary conditions, as follows,

$$\begin{aligned} &\varphi_1(\eta) = f_0(\eta) = c_1 + c_2\eta + c_3\frac{\eta^2}{2}, \\ &f''(0) = \sigma \to c_3 = \sigma, \\ &f'(0) = 0 \to c_2 = 0, \\ &f(0) = 0 \to c_1 = 0. \end{aligned} \tag{31}$$

Then,

$$\varphi_1(\eta) = f_0(\eta) = \sigma\frac{\eta^2}{2}. \tag{32}$$

By using (29), proceeding terms for f_n are calculated, as,

$$\begin{aligned} f_1(\eta) &= -\frac{1}{240}\sigma^2\eta^5, \\ f_2(\eta) &= \frac{11}{161280}\sigma^3\eta^8, \\ f_3(\eta) &= -\frac{5}{4257792}\sigma^4\eta^{11}, \\ &\vdots \end{aligned} \tag{33}$$

So, the approximation solution of $f(\eta)$ is,

$$\begin{aligned} f(\eta) &= \frac{1}{2}\sigma\eta^2 - \frac{1}{240}\sigma^2\eta^5 \\ &+ \frac{11}{161280}\sigma^3\eta^8 - \frac{5}{4257792}\sigma^4\eta^{11} + \cdots. \end{aligned} \tag{34}$$

According to Howarth calculation [9], inserting $\sigma = 0.332057$ in above equation leads to,

$$\begin{aligned} f(\eta) &= 0.16603\eta^2 - 4.6\times10^{-4}\eta^5 \\ &+ 2.5\times10^{-6}\eta^8 - 1.4\times10^{-8}\eta^{11} + \cdots, \end{aligned} \tag{35}$$

Eq. (35) is an approximant solution of Blasius problem (11) by ADM.

The results of 5-, 10-, 15-, 20-term ADM approximate solutions for Blasius equation and also, its velocity profile and their Runge-Kutta counterpart are presented in tables 1 and 2, respectively.

The results show good agreement with each other, as the number of terms in ADM solution increases so does the accuracy between the ADM solution and the corresponding numerical results. A high degree of accuracy between the 20-term solution and the relevant Runge-Kutta results is evident for different values of η.

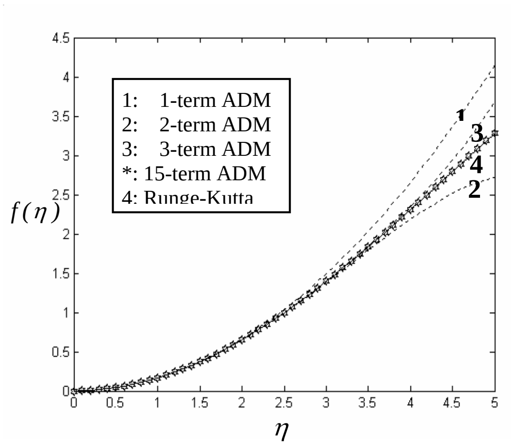

Fig.1- comparison between 1-, 2-, 3-, 15-term ADM solutions and Runge-Kutta 45 for Blasius profile.

The curves of 1-, 2-, 3-, and 5-term ADM solutions for Blasius equation and its velocity profile are shown in figures 1 and 2. As the number of ADM terms increases, more conformity to the relevant Runge-Kutta is observed. The complete conformity of 15-term ADM solution to the numerical curve is illustrated in Fig. 1 and 2.

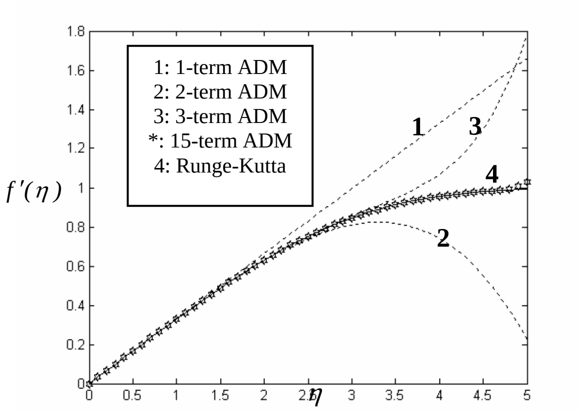

Fig.2- comparison between 1-, 2-, 3-, 15-term ADM solutions and Runge-Kutta 45 for Blasius velocity profile.

5 Conclusion

In view of analytical solution, the standard ADM adopted here, bears a good potential in dealing with the inherent nonlinearities of real physical problems. The results obtained in this study justify the need for more trails with the application of ADM to nonlinear phenomena.

References:

[1] G. Adomian, A review of the decomposition method and some recent results for nonlinear equations, Computers. Math. Application, Vol.21, No.5, 1991, p.p. 101-127.

[2] G. Adomian, Solving Frontier Problems of Physics: The Decomposition Method, Kluwer Academic, 1994.

[3] F. M. White, Fluid mechanics, second ed., McGraw Hill, 1986.

[4] H. Schlichting, Boundary Layer theory, seventh ed., McGraw Hill, 1979.

[5] L. Rosenhead, Laminar Boundary Layer, Oxford University Press, 1963.

[6] F. M. White, Viscous Fluid Flow, McGraw-Hill, New York, 1991.

[7] Yonggui Zhu, Qianshun Chang, Shengchang Wu, A new algorithm for calculating Adomian polynomials, Applied Mathematics and Computation, In Press, 2004.

[8] H. Blasius, Grenzschichten in Füssigkeiten mit kleiner Reilbung. Z. Math. Phys., 56:1-37, 1908.

[9] L. Howarth, On the solution of the laminar boundary layer equations. Proc. Roy. Soc. London A, 164:547-579,1938.

Table 1 -Comparison between numerical and ADM results for BLASIUS equation with different values of η.

η	Numerical (Runge-kutta)	5 terms of ADM	10 terms of ADM	15 terms of ADM	20 terms of ADM	(20-t ADM)-Numerical
0	0	0	0	0	0	0
0.4	0.02655980	0.02655986	0.02655986	0.02655986	0.02655986	-6E-8
0.8	0.10610804	0.10610811	0.10610811	0.10610811	0.10610811	-7E-8
1.2	0.23794841	0.23794848	0.23794848	0.23794848	0.23794848	-7E-8
1.6	0.42032030	0.42032035	0.42032034	0.42032034	0.42032034	-5E-8
2	0.65002371	0.65002378	0.65002373	0.65002373	0.65002373	-1E-8
2.4	0.92228926	0.92229044	0.92228922	0.92228922	0.92228922	4E-8
2.8	1.23097621	1.23099215	1.23097612	1.23097612	1.23097612	9E-8
3.2	1.56909362	1.56924080	1.56909346	1.56909349	1.56909349	14E-8
3.6	1.92952360	1.93054788	1.92952230	1.92952340	1.92952340	20E-8
4	2.30574473	2.31145241	2.30571452	2.30574450	2.30574435	38E-8

Table 2 -Comparison between numerical and ADM results for BLASIUS velocity with different values of η.

η	Numerical (Runge-kutta)	5 terms of ADM	10 terms of ADM	15 terms of ADM	20 terms of ADM	(20-t ADM)-Numerical
0	0	0	0	0	0	0
0.4	0.13276402	0.13276403	0.13276403	0.13276403	0.13276403	0
0.8	0.26470887	0.26470887	0.26470887	0.26470887	0.26470887	-1E-8
1.2	0.39377571	0.39377571	0.39377571	0.39377571	0.39377571	0
1.6	0.51675628	0.51675629	0.51675628	0.51675628	0.51675628	0
2	0.62976515	0.62976561	0.62976513	0.62976513	0.62976513	2E-8
2.4	0.72898131	0.72898976	0.72898126	0.72898126	0.72898126	5E-8
2.8	0.81150900	0.81160436	0.81150890	0.81150890	0.81150890	10E-8
3.2	0.87608087	0.87684215	0.87608045	0.87608071	0.87608071	15E-8
3.6	0.92332913	0.92799183	0.92331932	0.92332894	0.92332892	21E-8
4	0.95551758	0.97865776	0.95528459	0.95551924	0.95551749	9E-8