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An Approximated Solution to the Two-dimensional Lid-driven Cavity Flow, Using Adomian Decomposition Method and the Vorticity-Stream Function Formulation

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Abstract:- In this work, we present a reliable algorithm to solve the two-dimensional lid-driven cavity flow, using the Adomian Decomposition Method (ADM). The vorticity-stream function formulation is used for the incompressible flow considered here. The solution is calculated in the form of a series with easily computable coefficients. Also, numerical simulation, using finite difference method (FDM), is performed for comparison purposes. This comparison shows considerably close agreements.

*Key-Words:-*Adomian Decomposition Method, Vorticity-Stream Function Formulation, Lid-driven Cavity Flow.

1 Introduction

Over the last two decades, the Adomian decomposition method has been applied to obtain solutions to a wide class of both deterministic and stochastic PDE's. In recent years however, this method has emerged as an alternation to solve a wide range of problems whose mathematical models involve algebraic, differential, integral, integro-differential, higher-order ordinary differential, and partial differential equations [1-8]. It yields rapidly converging series solutions by using only a few iterations for both linear and non-linear deterministic and stochastic equations. The advantage of this method is that, it provides a direct scheme for solving the problem, i.e., without the need for linearization, perturbation, massive computation, and any transformation.

1.1 Governing Equations

The fundamental equations of fluid dynamics are based on three universal laws of conservation: conservation of mass, conservation of momentum, and conservation of energy. Applied to fluid flows, these laws yield continuity, momentum, and energy equations.

For incompressible Newtonian fluid under no external forces, $\nabla \cdot \vec{V} = 0$ and the momentum equation is simplified to the following form:

$$\frac{\partial \vec{V}}{\partial t} + \vec{V} \cdot \nabla \vec{V} = -\frac{\nabla P}{\rho} + \nu \nabla^2 \vec{V}, \tag{1}$$

where, $\vec{V}$ is velocity vector, P is pressure, ρ is density, and ν is kinematic viscosity.

The two-dimensional incompressible Navier-Stokes equations can be written in Cartesian coordinates as:

continuity:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{2}$$

x-momentum:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \tag{3}$$

y-momentum:

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho}\frac{\partial P}{\partial x}+\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right), \tag{4}$$

Recall, for Incompressible flows many applications, temperature changes are either insignificant or unimportant. Therefore it is not always necessary to solve the energy equation[9]. However, instead of solving the above equations, since our flow is two-dimensional we use vorticity-stream function formulation. In a two-dimensional flow, vorticity (ω) is a scalar quantity defined by:

$$\omega=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y} \quad , \tag{5}$$

which is determined from the general definition: $\vec{\omega}=\nabla\times\vec{V}$. Also, taking the curl of equations (3) and (4), as a vector equation, one obtains the vorticity transport equation, as:

$$\frac{\partial \omega}{\partial t}+u\frac{\partial \omega}{\partial x}+v\frac{\partial \omega}{\partial y}=\nu\left(\frac{\partial^2 \omega}{\partial x^2}+\frac{\partial^2 \omega}{\partial y^2}\right) \tag{6}$$

On the other hand, the stream function (ψ) is a scalar-valued function defined by the following relations:

$$u=\frac{\partial \psi}{\partial y}\,, v=-\frac{\partial \psi}{\partial x}. \tag{7}$$

Substituting these relations into Eq'n. (2) leads to the following Poisson equation:

$$\frac{\partial^2 \psi}{\partial x^2}+\frac{\partial^2 \psi}{\partial y^2}=-\omega\,. \tag{8}$$

Finally, the dimensionless governing equations used here are as follows:

$$\frac{\partial \omega}{\partial t}+\frac{\partial \psi}{\partial y}\frac{\partial \omega}{\partial x}-\frac{\partial \psi}{\partial x}\frac{\partial \omega}{\partial y}=\frac{1}{\mathrm{Re}}\left(\frac{\partial^2 \omega}{\partial x^2}+\frac{\partial^2 \omega}{\partial y^2}\right),$$
$$\frac{\partial^2 \psi}{\partial x^2}+\frac{\partial^2 \psi}{\partial y^2}=-\omega. \tag{9}$$

1.2 Lid-driven Cavity Flow

The “lid-driven cavity problem” is a classical problem, which is also widely used as a benchmark in computational fluid dynamics (CFD). It is an excellent test case for comparing numerical methods that solve the incompressible Navier-Stokes equations. In this problem, the incompressible viscous flow is surrounded by wall boundaries on all three sides and is driven by the uniform translation of the upper surface (lid) [9].

2 ADM Applied to Vorticity-stream Function Formulation

The principal algorithm of the Adomian decomposition method when applied to a general non-linear equation is in the form:

$$Lu+Ru+Nu=g\,. \tag{10}$$

The linear terms are decomposed into $L+R$, while the non-linear terms are represented by Nu. Note, L is taken as the highest order derivative to avoid difficult integration involving complicated Green’s functions and R is the remainder of the linear operator. Also, L^{-1} is a definite integration, i.e.,

$$L^{-1}[.]=\int_0^{\alpha}\int_0^{\beta}[.]dxdy\,. \tag{11}$$

Obviously, if L is a second-order operator, L^{-1} is a two-fold indefinite integral, as:

$$L^{-1}Lu=u(x,y)-u(x,0)-y\frac{\partial u(x,0)}{\partial y}. \tag{12}$$

Operating on both sides of Eq'n. (10) with L^{-1} yields:

$$L^{-1}Lu = L^{-1}g - L^{-1}Ru - L^{-1}Nu\,, \qquad (13)$$

and gives

$$u(x,y) = u(x,0) + yu_t(x,0) + L^{-1}g - L^{-1}Ru - L^{-1}Nu \qquad (14)$$

The decomposition method represents the solution of Eq'n. (14) as the following series

$$u(x,y) = \sum_{n=0}^{\infty} u_n(x,y)\,. \qquad (15)$$

The non-linear operator, Nu, is decomposed as:

$$Nu = \sum_{n=0}^{\infty} P_n\,. \qquad (16)$$

Substituting Eq'ns. (15) and (16) into Eq'n. (14), we obtain:

$$\sum_{n=0}^{\infty} u_n(x,y) = u_0 - L^{-1}R\sum_{n=0}^{\infty} u_n - L^{-1}\sum_{n=0}^{\infty} P_n\,, \qquad (17)$$

where,

$$u_0 = u(x,y) + yu_y(x,0) + L^{-1}g\,. \qquad (18)$$

Consequently, it can be written as:

$$\begin{aligned} u_1 &= -L^{-1}Ru_0 - L^{-1}P_0, \\ u_2 &= -L^{-1}Ru_1 - L^{-1}P_1, \\ &\vdots \\ u_n &= -L^{-1}Ru_n - L^{-1}P_n, \quad n \geq 0, \end{aligned} \qquad (19)$$

where, P_n are the Adomian's polynomials of $u_0, u_1, \ldots, u_n$ and are obtained from the formula:

$$P_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[F\left(\sum_{i=0}^{\infty}\lambda^i u_i\right)\right]_{\lambda=0}, \quad n = 0,1,2,\ldots\,. \qquad (20)$$

Equation (20) gives:

$$\begin{aligned} P_0 &= f(u_0), \\ P_1 &= u_1\frac{d}{du_0}f(u_0), \\ P_2 &= u_2\frac{d}{du_0}f(u_0) + \frac{u_1^2}{2!}\frac{d^2}{du_0^2}f(u_0), \\ P_3 &= u_3\frac{d}{du_0}f(u_0) + u_1u_2\frac{d^2}{du_0^2}f(u_0) + \frac{u_1^3}{3!}\frac{d^3}{du_0^3}f(u_0), \\ &\vdots\;. \end{aligned} \qquad (21)$$

The accuracy level of the approximation of $u(x,y)$ can be dramatically enhanced by computing coefficients (as many as we would like). The $n-$ terms approximation:

$$\lim \Phi_{n\to\infty} = u(x,y) \quad where, \quad \Phi_n(x,y) = \sum_{k=0}^{n-1} u_k(x,y), \;\; n \geq 0 \qquad (22)$$

can be used to approximate the solution.
By using the above procedures for system of Eq'ns. (9), we obtain:

$$\psi_0 = y\cdot\psi_y(x,0), \qquad (23)$$

$$\omega_0 = \omega(x,0) + y\cdot\omega_y(x,0), \qquad (24)$$

$$A_0 = \psi_y(x,0)\cdot(\partial_x\omega(x,0) + y\cdot\partial_x\omega_y(x,0)), \qquad (25)$$

$$B_0 = y\cdot(\partial_x\psi_y(x,0))\cdot\omega_y(x,0), \qquad (27)$$

$$A_{j+1} = (\partial_y\psi_j)(\partial_x\omega_{i-j}) + (\partial_y\psi_{i-j})(\partial_x\omega_j) + A_j$$
$$i = 0,\cdots,n \quad j = 0,\cdots,\frac{i-1}{2}, \qquad (28a)$$

$$A_i = (\partial_y \psi_{\frac{i}{2}})(\partial_x \omega_{\frac{i}{2}}) + A_j \quad \textit{if} \quad i \quad \textit{is even}, \tag{28b}$$

$$B_{k+1} = (\partial_x \psi_k)(\partial_y \omega_{i-k}) + (\partial_x \psi_{i-k})(\partial_y \omega_k) + B_k$$
$$i = 0,\cdots,n \quad k = 0,\cdots,\frac{i-1}{2}, \tag{29a}$$

$$B_i = (\partial_x \psi_{\frac{i}{2}})(\partial_y \omega_{\frac{i}{2}}) + B_k \quad \textit{if} \quad i \quad \textit{is even}, \tag{29b}$$

$$\omega_{i+1} = -\iint \left(\frac{\partial^2}{\partial x^2} \omega_i \right) dydy + \mathrm{Re} \iint (A_i - B_i) dydy$$
$$i = 0,\cdots,n, \tag{30}$$
$$\psi_{i+1} = -\iint \left(\frac{\partial^2}{\partial x^2} \psi_i + \omega_i \right) dydy$$
$$i = 0,\cdots,n. \tag{31}$$

3 Computational Methodology-An Alternative Approach

To computationally solve the vorticity transport equation (Eq'n. 6) on a discrete grid, we used the forward-time and centered-space (FTCS) scheme of FDM. This was done via replacing the time derivatives by one-sided forward and the spatial derivatives by centered differences. Rewriting Eq'n. (6) by using its discrete approximation [9, 10], yields:

$$\frac{\omega_{i,j}(t+\Delta t) - \omega_{i,j}(t)}{\Delta t} + u_{i,j}\frac{\omega_{i+1,j} - \omega_{i-1,j}}{2\Delta x} + v_{i,j}\frac{\omega_{i,j+1} - \omega_{i,j-1}}{2\Delta y}$$
$$= \frac{1}{\mathrm{Re}}\left[\frac{\omega_{i+1,j} - 2\omega_{i,j} + \omega_{i-1,j}}{\Delta x^2} + \frac{\omega_{i,j+1} - 2\omega_{i,j} + \omega_{i,j-1}}{\Delta y^2}\right], \tag{32}$$

where, the velocities $u_{i,j}$ and $v_{i.j}$ are given by:

$$u_{i,j} = \frac{\psi_{i+1,j} - \psi_{i-1,j}}{2\Delta x},$$
$$v_{i,j} = \frac{\psi_{i,j+1} - \psi_{i,j-1}}{2\Delta y}. \tag{33}$$

Note, by substituting Eq'ns. (32) and (33) directly into Eq'n. (6), the velocity vectors do not need to be explicitly computed. In order to ensure the stability and convergence of the algorithm, Δt must be set small enough for a given viscosity ν and grid spacing. Rather than manually setting Δt, one can compute it automatically using the following equations, where Re is the flow Reynolds number:

$$\mathrm{Re} = \frac{U_{wall}}{\nu},$$
$$\Delta t \le \frac{[\max(\Delta x, \Delta y)]^2}{|U_{\max}| \cdot \max(\Delta x, \Delta y) + \frac{4}{\mathrm{Re}}}. \tag{34}$$

3.1 Stream Function Poisson Equation

The stream function Poisson equation (Eq'n. 8) can be rewritten as:

$$\frac{\psi_{i+1,j} - 2\psi_{i,j} + \psi_{i-1,j}}{\Delta x^2} + \frac{\psi_{i,j+1} - 2\psi_{i,j} + \psi_{i,j-1}}{\Delta y^2} = -\omega_{i,j}. \tag{35}$$

To computationally solve this equation, we used the successive over-relaxation (SOR) method [10, 11], as:

$$\psi_{i,j}^{n+1} = \frac{\beta}{2}\left\{\begin{aligned} &\frac{\Delta y^2}{\Delta x^2 + \Delta y^2}\left[\psi_{i+1,j}^{n} + \psi_{i-1,j}^{n+1}\right] \\ &+ \frac{\Delta x^2}{\Delta x^2 + \Delta y^2}\left[\psi_{i,j+1}^{n} + \psi_{i,j-1}^{n+1}\right] \\ &+ \frac{\Delta x^2 \Delta y^2}{\Delta x^2 + \Delta y^2}\,\omega_{i,j} \end{aligned}\right\} + (1-\beta)\psi_{i,j}^{n}, \tag{36}$$

where, β is the relaxation parameter which needs to be greater than one for extrapolation and less than two for stability purposes: $1 < \beta < 2$ [9, 10].

3.2 Boundary Conditions

The boundary conditions at the four walls are given as follows [9]:

Left and Right Walls:

$$u = 0,\ v = 0,\ \psi = 0,\ \text{and}\ \omega = -\frac{\partial^2 \psi}{\partial x^2}, \tag{37}$$

Bottom wall:

$u = +\dfrac{\partial \psi}{\partial y} = U_{wall}$, $v = 0$, $\psi = 0$, and

$$\omega = -\frac{\partial^2 \psi}{\partial y^2}, \tag{38}$$

Top wall:

$$u = 0,\ v = 0,\ \psi = 0, \text{ and } \omega = -\frac{\partial^2 \psi}{\partial y^2}. \tag{39}$$

In order to integrate Eq'n (9), the boundary vorticity (ω_{wall}) needs to be yet computed. We applied a first order approximation by expanding a Taylor series of the stream function around the boundary points as:

$$\psi(\Delta x) = \psi(0) + \frac{\partial \psi(0)}{\partial x}\Delta x + \frac{\partial^2 \psi(0)}{\partial x^2}\frac{\Delta x}{2} + O(\Delta x^3) \tag{40}$$

Substituting the boundary conditions into Eq'n. (15) and solving for ω_{wall}, we get:

$$\omega_{wall} = (\psi(0) - \psi(\Delta x))\frac{2}{\Delta x^2} + U_{wall}\frac{2}{\Delta x} + O(\Delta x) \tag{41}$$

3.3 Computational Algorithm

A general outline of the algorithm to solve the lid-driven cavity problem using the vorticity-stream function formulation is given as Algorithm 1 here. Note, the Poisson equation is solved first, instead of the vorticity transport equation, because initially for all interior points $\omega_{i,j}$ is guessed.

4 Concluding Remarks

In this work, we considered an approximated solution of vorticity-stream function formulation using ADM. From the theoretical analysis and the numerical results, we may come to the following concluding remarks:

1-We can claim that ADM method is useful successfully for solving the considered highly non-linear system of PDEs.
2-From the outlined theoretical analysis, we can conclude that the proposed method is applicable for similar physical equations.
3-The obtained numerical results compared with the analytical approximated solution show that the method needs to assign further terms in series. But our MATHEMATICA/MAPLE program can not run on a high speed PC Pentium IV for more than 3 terms in series.

Conclusions

In this work, we obtained an approximated solution of the vorticity-stream function formulation applied to the lid-driven cavity flow, by using the Adomian decomposition method. We demonstrated that, the decomposition procedure is quite efficient to determine approximated solutions when using numerical boundary conditions.

References:

[1]. S. Guellal and Y. Cherruault. Application of the Decomposition Method to Identify the Disturbed Parameters of an Elliptical Equation, *Math. Comput. Modelling*, Vol. 21, No., 4, pp. 51-55, 1995.

[2]. G. Adomian, Y. Cherruault, and K. Abbaoui. A Non-perturbative Analytical Solution of Immune Response with Time-delays and Possible Generalization, *Math. Comput. Modelling*, Vol. 20, No. 10, pp.89-96, 1996.

[3]. P. Laffez and K. Abbaoui. Modelling of the Thermic Exchanges During a Drilling Resolution with Adomian's Decomposition Method, *Math. Comput. Modelling*, Vol. 23, No. 10, pp. 11-14, 1996.

[4]. M. Ndour, K. Abbaoui, H. Ammar, and Y. Cherruault. An Example of an Interaction Model Between Two Species, *Kybernetes*, Vol. 25, No. 4, pp. 106-118, 1996.

[5]. S. Guellal, P. Grimalt, and Y. Cherruault. Numerical Study of Lorentz's Equation by the Adomian Model, *Comput. Math. Appl.*, Vol. 33, No. 3, pp. 25-29, 1997.

[6]. K. Abbaoui and Y. Cherruault. The Decomposition Method Applied to the Cauchy Problem, *Kybernetes*, Vol. 28, No. 1, pp. 68-74, 1999.

[7]. B. Adjedj. Application of the Decomposition Method to the Understanding of HIV Immune Dynamics, *Kybernetes*, Vol. 28, No. 3, pp. 271-283, 1999.

[8]. F. Sanchez, K. Abbaoui, and Y. Cherruault. Beyond the Thin-Sheet Approximation: Adomian's Decomposition, *Optics Commun.*, Vol. 173, pp. 397-401, 2000.

[9]. J.C. Tannehill, D.A. Anderson, and R.H. Pletcher. *Computational Fluid Mechanics and Heat Transfer*, Taylor & Francis, 2nd edition, 1997.

[10]. Z.U.A. Warsi. *Fluid Dynamics: Theoretical and Computational Approaches*, CRC Press, 1993.

[11]. J.U. Ghia, K.N. Ghia, and C.T. Shin, "*High-Re Solutions for Incompressible Flow Using the Navier-Stokes Equations and a Multi-grid Method*", J. Comp. Phys. Vol. 48, pp. 387-411, 1982.

Figures and Graphs

Algorithm 1: Vorticity-stream function formulation

$\omega_{i,j} = 0$ **for all interior points**

For i=1 to N do

Iteratively solve the Poisson's equation for $\psi_{i,j}$ at t.

Compute the boundary vorticity, ω_{wall}.

Solve the vorticity transport equation for $\omega_{i,j}$ at $t+\Delta t$.

Update $\omega_{i,j}$ and t.

End For

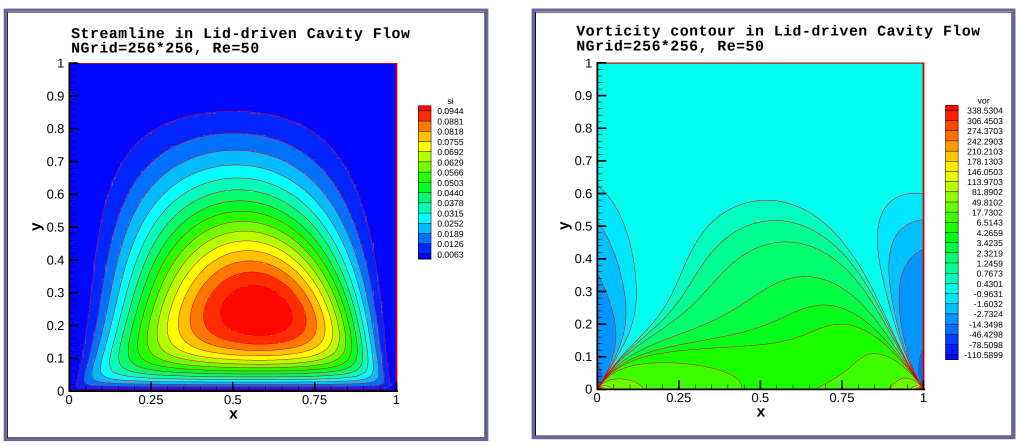

Fig. 1: Streamline and Vorticiy contour in Lid-driven cavity flow computational method.

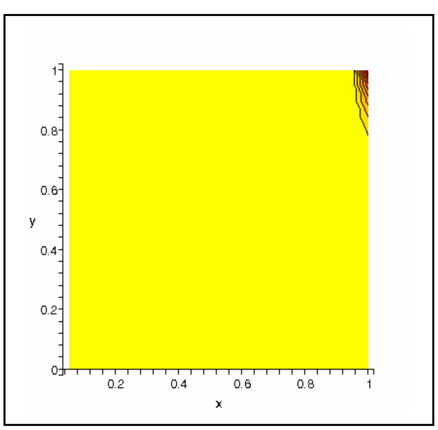

Fig 2. Vorticity contours using ADM for n=3 terms in Re=50.