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Fin and Slab Heat Transfer and Property Distribution, Using Adomian Decomposition Method

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Abstract: In this work, conduction-convection heat transfer through a straight fin, property distribution due to convection-diffusion, and conduction heat transfer through a slab with temperature dependent thermal conductivity are evaluated using Adomian decomposition method. The results are compared with numerical and exact solutions. It is shown that the numerical simulation has some limitations and may not always produce correct results. The Adomian decomposition method, however, follows the correct trend of the exact solution with applying only a few terms.

Key-Words: - Adomian Decomposition Method, Conduction, Convection, Heat Transfer, Property Distribution, Fin, Slab

1 Introduction

Generally, a physical problem in nature will be governed by nonlinear and stochastic equations. Therefore, the solutions of these governing equations, which will be obtained using linearization, perturbation, closure approximations, or discretization methods, may not be physically realistic. However, in Adomian's decomposition method (ADM), the closed form solution of the non-linear problems could be obtained without applying any non-realistic simplifications and/or approximations [1, 2]. Although this decomposition method is a n-term approximation, however, it does not change the physics of the original problem and it converges to the exact solution by increasing the number of the computed decomposition terms [1, 3].

In this work, conduction-convection heat transfer in a fin, property distribution by means of convection-diffusion through a one-dimensional domain, and conduction heat transfer through a slab with temperature dependent thermal conductivity are determined, using the ADM. Also, the exact and numerical solutions, using finite volume method (FVM) are compared.

2 Adomian Decomposition Method

The Adomian's equation in an operator form is [1, 4]:

$$Lu + Ru + Nu = g, \tag{1}$$

where, L is the highest-order derivative, R is the remainder of the linear operator, Nu represents nonlinear terms, and g is an inhomogeneous or forcing term. Solving for Lu, one has:

$$Lu = g - Ru - Nu. \tag{2}$$

Leting L^{-1} be an integral operator and applying L^{-1} to both sides of Eq. (2), one gets:

$$u = \Phi + L^{-1}g - L^{-1}Ru - L^{-1}Nu, \tag{3}$$

where, Φ satisfies $L\Phi = 0$. Note, the inverse operator L^{-1} is an indefinite integral. The constant of integrations can be found by utilizing boundary/initial conditions which will be absorbed into Φ. The Adomian's method assumes that the solution u can be expanded as an infinite series as:

$$u = \sum_{n=0}^{\infty} u_n . \tag{4}$$

The non-linear operator Nu is represented by an infinite series $\sum_{n=0}^{\infty} A_n$. Therefore, one can find the final solution as:

$$u = u_0 - L^{-1}R\sum_{n=0}^{\infty} u_n - L^{-1}\sum_{n=0}^{\infty} A_n, \tag{5}$$

where, $u_0 = \Phi + L^{-1} g$. The recursive form of u components can be written as:

$$u_{n+1} = -L^{-1} R u_n - L^{-1} A_n . \quad (6)$$

Polynomials A_n are generated for all kinds of non-linearity and depend on initial/boundary conditions (see Section 5). Consequently, all terms of the decomposition are identified and are calculable. Hence, the n-term partial sum $\phi_n = \sum_{i=0}^{n-1} u_i$ will be the approximate solution. Of course, increasing the number of computed components results in higher accuracy, i.e.,

$$\underset{n\to\infty}{Limit}\ \phi_n = \sum_{i=0}^{\infty} u_i = u. \quad (7)$$

3 Conduction-Convection Heat Transfer through a Fin

We first consider the cooling of a cylindrical fin with uniform cross-sectional area (A) by means of convective heat transfer along its length. Figure 1 shows the geometry and the boundary conditions of the problem. The base is at a temperature of $100°C$ (T_B) and the end is insulated. The fin is exposed to an ambient temperature of $20°C$. The governing equation of one-dimensional heat transfer for this situation is:

$$\frac{d}{dx}(kA\frac{dT}{dx}) - hP(T - T_\infty) = 0, \quad (8)$$

where, x is the distance along the fin, k is the thermal conductivity of the material, A is the cross section, h is the convective heat transfer coefficient, P is the perimeter, and T_∞ is the ambient temperature. The boundary conditions are:

$$T(x=0) = T_B = 100\ °C, \qquad \frac{\partial T}{\partial x}\Big|_{x=l} = 0, \quad (9)$$

where, l is the length of the fin and is set to unity. Supposing kA is constant, the exact solution is as follows [5]:

$$\frac{T - T_\infty}{T_B - T_\infty} = \frac{\cosh[n(l-x)]}{\cosh(nl)}, \quad (10)$$

where, $n^2 = hp/(kA)$. Using the variable change of $T = T - T_\infty$ and applying the ADM, i.e. Eq. (2), to Eq. (8) one can obtain:

$$L_x T = RT, \quad (11)$$

where, $L = \partial^2 / \partial x^2$ is a second-order differential operator, $R = \alpha = hp/(ka)$ is a constant, and Nu, and g are zero for this problem.

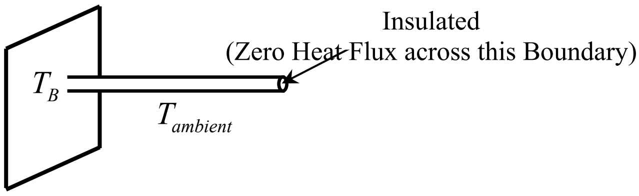

Figure 1: Geometry and boundary conditions of the first problem.

Having considered Eqs. (5), (6), and the inverse operator L_x^{-1}, a two-fold integration represented by $L_x^{-1}(.) = \int_0^x \int_0^x (.)dxdx,$ the decomposition components can be calculated as:

$$\begin{aligned} T_0 &= c_1 + c_2 x \\ T_1 &= L_x^{-1} R T_0 = \alpha(\frac{c_1 x^2}{2!} + \frac{c_2 x^3}{3!}) \\ T_2 &= L_x^{-1} R T_1 = \alpha^2(\frac{c_1 x^4}{4!} + \frac{c_2 x^5}{5!}) \\ T_3 &= L_x^{-1} R T_2 = \alpha^3(\frac{c_1 x^6}{6!} + \frac{c_2 x^7}{7!}) \\ &\vdots \\ T_n &= L_x^{-1} R T_{n-1} = \alpha^n(\frac{c_1 x^{2n}}{(2n)!} + \frac{c_2 x^{2n+1}}{(2n+1)!}). \end{aligned} \quad (12)$$

Consequently, the solution is:

$$\begin{aligned} T = \sum_{n=0}^{\infty} T_n &= c_1(1 + \frac{\alpha x^2}{2!} + \frac{\alpha^2 x^4}{4!} + \frac{\alpha^3 x^6}{6!} + ... \\ &+ \frac{\alpha^n x^{2n}}{(2n)!}) + c_2(x + \frac{\alpha x^3}{3!} + \frac{\alpha^2 x^5}{5!} \\ &+ \frac{\alpha^3 x^7}{7!} + ... + \frac{\alpha^n x^{2n+1}}{(2n+1)!}). \end{aligned} \quad (13)$$

By applying the boundary conditions (Eq. 9) the constants in Eq. (13) can be obtained and the solution becomes:

$$\frac{T-T_\infty}{T_B-T_\infty} = (1+\frac{\alpha x^2}{2!}+\frac{\alpha^2 x^4}{4!}+\frac{\alpha^3 x^6}{6!}+...+\frac{\alpha^n x^{2n}}{(2n)!}) + \frac{(\alpha l+\frac{\alpha^2 l^3}{3!}+\frac{\alpha^3 l^5}{5!}+...+\frac{\alpha^n l^{2n-1}}{(2n-1)!})}{(1+\frac{\alpha l^2}{2!}+\frac{\alpha^2 l^4}{4!}+\frac{\alpha^3 l^6}{6!}+...+\frac{\alpha^n l^{2n}}{(2n)!})} \times (x+\frac{\alpha x^3}{3!}+\frac{\alpha^2 x^5}{5!}+\frac{\alpha^3 x^7}{7!}+...+\frac{\alpha^n l^{2n+1}}{(2n+1)!}). \quad (14)$$

In our numerical simulation using FVM, the central difference approach is used with a uniform grid and the length (which is considered to be $1.0\ m$) is divided into five control volumes, so that $\delta x = 0.2\ m$ [6]. Figure 2 shows the finite volume, exact, and 5-, 6-, 7-, and 8-terms of Adomian solutions. It can be observed that, the Adomian decomposition method has higher accuracy in comparison with the numerical solution. Also, it converges to the analytical solution as the number of its terms increases.

4 Property Distribution Due to Convection-Diffusion

In this section, the distribution of a property ϕ by means of convection and diffusion through the one-dimensional domain is considered, as our second problem (figure 3 shows the problem schematically). In the absence of sources, the steady state governing equation is:

$$\frac{d}{dx}(\rho u \phi) = \frac{d}{dx}(\Gamma \frac{d\phi}{dx}), \quad (15)$$

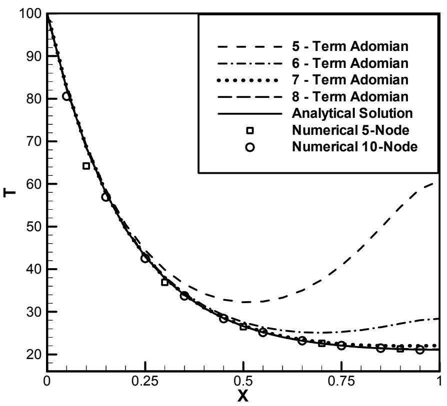

Figure 2: Exact, numerical, and Adomian solutions of temperature distribution in a fin.

where, ρ, u, and Γ are density, velocity, and diffusion coefficient, respectively, and are assumed to be constant. The boundary conditions are:

$$\phi(x=0) = \phi_0 = 1, \quad \phi(x=l) = \phi_L = 0. \quad (16)$$

The exact solution of this problem is:

$$\frac{\phi-\phi_0}{\phi_L-\phi_0} = \frac{\exp(\rho u x/\Gamma)-1}{\exp(\rho u l/\Gamma)-1}. \quad (17)$$

Applying the Adomian decomposition method to Eq. (15), one can obtain:

$$L_x\phi - R\phi = 0, \quad (18)$$

where, $L = \partial^2/\partial x^2$ is a second-order differential operator, $R = \beta(\partial/\partial x)$, Nu, and g are zero, and $\beta = \rho u/\Gamma$ (a constant).

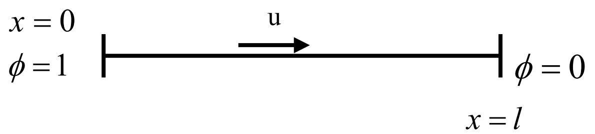

Figure 3: Distribution of a property, ϕ, by means of convection and diffusion through the one-dimensional domain.

Similarly, as in Section 2, the decomposition components are as follows:

$$\begin{aligned} \phi_0 &= c_1 + c_2 x, \\ \phi_1 &= L_x^{-1}R\phi_0 = \frac{c_2\beta x^2}{2!}, \\ \phi_2 &= L_x^{-1}R\phi_1 = \frac{c_2\beta^2 x^3}{3!}, \\ \phi_3 &= L_x^{-1}R\phi_2 = \frac{c_2\beta^3 x^4}{4!}, \\ &\vdots \\ \phi_n &= L_x^{-1}R\phi_{n-1} = \frac{c_2\beta^n x^{n+1}}{(n+1)!}. \end{aligned} \quad (19)$$

Consequently, the solution is:

$$\phi = \sum_{n=0}^{\infty}\phi_n = c_1 + c_2(x+\frac{\beta x^2}{2!}+\frac{\beta^2 x^3}{3!}+\frac{\beta^3 x^4}{4!} +...+\frac{\beta^n x^{n+1}}{(n+1)!}), \quad (20)$$

Applying the boundary conditions (Eq. 16), the constants in Eq. (20) are obtained and the solution becomes:

$$\frac{\phi-\phi_0}{\phi-\phi_L}=\frac{\sum_{n=0}^{\infty}\frac{\beta^n x^{n+1}}{(n+1)!}}{\sum_{n=0}^{\infty}\frac{\beta^n l^{n+1}}{(n+1)!}}. \tag{21}$$

In FVM simulation, central difference approach is used with a uniform grid and the length, which is considered to be $1.0\ m$, is divided into five control volumes, so that $\delta x=0.2\ m$ [6]. Also, ρ and Γ are assumed to be $1.0\,kg/m^3$ and $0.1\,kg/m/s$. However, two different cases are considered for the velocity, i.e., for case (i) $u=0.1\,m/s$ and for case (ii) $u=2.5\,m/s$. Figures 4 and 5 show the exact, numerical and Adomian decomposition solutions for the two cases.

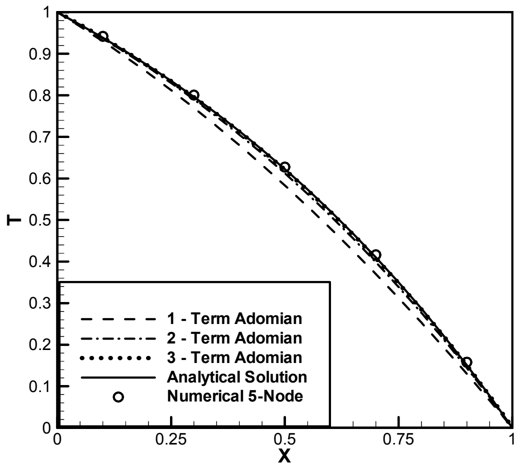

Figure 4: Exact, numerical, and Adomian solutions of property distribution due to convection-diffusion for case (i).

It is observed that for case (i), only three-terms of the Adomian's decomposition shows a very high accuracy. However, for case (ii), the numerical solution produces a result that appears to oscillate about the exact solution and does not follow the exact solution trend. Although the truncation error of central differencing scheme is second order, it may not produce approximately the correct solution of the problem (at least within the approximation used). The Adomian's decomposition method, however, follows the correct trend of the exact solution even applying 5-terms. Of course, it converges to the analytical solution by increasing the number of computed components.

5 Heat Transfer through a Slab with Temperature Dependent Thermal Conductivity

Conduction heat transfer through a slab in $0\le x\le l$ with heat generation at a constant rate of g_0 is investigated in this section. The thermal conductivity depends on temperature in the form of $k(T)=k_0(1+\beta T)$. The governing equation is as follows:

$$\frac{d}{dx}\left[k_0(1+\beta T)\frac{dT}{dx}\right]+g_0=0, \tag{22}$$

where, k_0 is the thermal conductivity of the slab at the ambient temperature and β is the parameter describing the variation of the thermal conductivity. The boundary conditions are:

$$\frac{\partial T}{\partial x}\Big|_{x=0}=0, \qquad T(x=l)=0. \tag{23}$$

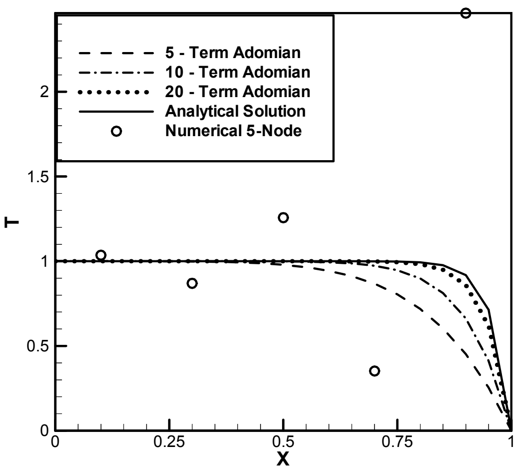

Figure 5: Exact, Finite volume, and Adomian solutions of property distribution due to convection-diffusion for case (ii).

The exact solution to Eq. (22) is:

$$T(x)=\frac{1}{\beta}\left[\sqrt{1+\beta\frac{g_0}{k_0}(l^2-x^2)}-1\right]. \tag{24}$$

Let, $\alpha=g_0/k_0$, then Eq. (22) reduces to:

$$\frac{d^2T}{dx^2}=-\alpha-\beta\left(T\frac{d^2T}{dx^2}\right)-\beta\left(\frac{dT}{dx}\right)^2. \tag{25}$$

Applying the Adomian decomposition method to Eq. (25), one finds:

$$L_x\, T = -\alpha - \beta\, N\, A - \beta\, N\, B, \tag{26}$$

where $L = \partial^2 / \partial x^2$ is a second-order differential operator. The non-linear terms in Eq. (26) can be defined as:

$$N\,A = T\frac{d^2T}{dx^2} = \sum_{m=0}^{\infty} A_m, \qquad N\,B = \left(\frac{dT}{dx}\right)^2 = \sum_{m=0}^{\infty} B_m, \tag{27}$$

which are introduced as

$$\begin{aligned} A_0 &= T_0\frac{d^2T_0}{dx^2}, \\ A_1 &= T_1\frac{d^2T_0}{dx^2} + T_0\frac{d^2T_1}{dx^2}, \\ A_2 &= T_2\frac{d^2T_0}{dx^2} + T_1\frac{d^2T_1}{dx^2} + T_0\frac{d^2T_2}{dx^2}, \\ A_3 &= T_3\frac{d^2T_0}{dx^2} + T_2\frac{d^2T_1}{dx^2} + T_1\frac{d^2T_2}{dx^2} + T_0\frac{d^2T_3}{dx^2}, \\ &\vdots \end{aligned} \tag{28}$$

also,

$$\begin{aligned} B_0 &= \left(\frac{dT_0}{dx}\right)^2, \\ B_1 &= 2\frac{dT_0}{dx}\frac{dT_1}{dx}, \\ B_2 &= \left(\frac{dT_1}{dx}\right)^2 + 2\frac{dT_0}{dx}\frac{dT_2}{dx}, \\ B_3 &= 2\frac{dT_1}{dx}\frac{dT_2}{dx} + 2\frac{dT_0}{dx}\frac{dT_3}{dx}, \\ &\vdots \end{aligned} \tag{29}$$

Applying the inverse operator, L_x^{-1} to both sides of Eq. (26), one obtains:

$$T = T_0 - \beta\, L_x^{-1} N\, A - \beta\, L_x^{-1} N\, B, \tag{30}$$

where, $T_0 = c_1 x + c_2 + \alpha \frac{x^2}{2!}$. The other terms can be obtained using the following recursive form:

$$T_{m+1} = -\beta\, L_x^{-1} A_m - \beta\, L_x^{-1} B_m. \tag{31}$$

Applying the boundary condition (Eq. 23) at $x = 0$ and letting $T(x=0) = C$, one can obtain from Eq. (31) that:

$$\begin{aligned} T_0 &= C - \alpha\frac{x^2}{2!}, \\ T_1 &= \beta\,\alpha\, C\frac{x^2}{2!} - 3\,\beta\,\alpha^2\frac{x^4}{4!}, \\ T_2 &= -\beta^2\,\alpha\, C^2\frac{x^2}{2!} + 9\,\beta^2\,\alpha^2 C\frac{x^4}{4!} \\ &\quad - 45\,\beta^2\,\alpha^3\frac{x^6}{6!}, \\ T_3 &= \beta^3\,\alpha\, C^3\frac{x^2}{2!} - 18\,\beta^3\,\alpha^2 C^2\frac{x^4}{4!} \\ &\quad + 225\,\beta^3\,\alpha^3 C\frac{x^6}{6!} - 1575\,\beta^3\,\alpha^4\frac{x^8}{8!}, \\ T_4 &= -\beta^4\,\alpha\, C^4\frac{x^2}{2!} + 30\,\beta^4\,\alpha^2 C^3\frac{x^4}{4!} \\ &\quad - 675\,\beta^4\,\alpha^3 C^2\frac{x^6}{6!} + 11025\,\beta^4\,\alpha^4 C\frac{x^8}{8!} \\ &\quad - 99225\,\beta^4\,\alpha^5\frac{x^{10}}{10!}, \\ &\vdots \end{aligned} \tag{32}$$

Consequently, the final solution in terms of α and β is:

$$\begin{aligned} T &= \sum_{m=0}^{\infty} T_m = T_0 + T_1 + T_2 + T_3 + T_4 + \ldots + T_n, \\ T &= C - \alpha(1 - \beta\, C + \beta^2\, C^2 - \beta^3\, C^3 + \beta^4\, C^4 - \ldots)\frac{x^2}{2!} \\ &\quad - 3\beta\,\alpha^2(1 - 3\beta\, C + 6\beta^2\, C^2 - 10\beta^3\, C^3 + \ldots)\frac{x^4}{4!} \\ &\quad - 45\beta^2\,\alpha^3(1 - 5\beta\, C + 15\beta^2\, C^2 - \ldots)\frac{x^6}{6!} \\ &\quad - 1575\beta^3\,\alpha^4(1 - 7\beta\, C + \ldots)\frac{x^8}{8!} \\ &\quad - 99225\beta^4\,\alpha^5(1 - \ldots)\frac{x^{10}}{10!}, \end{aligned} \tag{33}$$

where, C is yet to be obtained using Newton-Raphson method for given values of $\alpha = 10$ and $\beta = 0.5$.

Figure 6 shows the exact and the Adomian decomposition solutions. Here, it is worth to mention that, as the terms of Adomian's series increase, the solution converges to the exact one.

6 Conclusions

The Adomian decomposition method is applied to attain the analytical solution to conduction-convection heat transfer through a straight fin, property distribution due to convection-diffusion, and conduction heat transfer through a slab with temperature dependent thermal conductivity. Also, the problems are solved numerically applying finite volume method and the results are compared with the exact solutions. It is observed that, the ADM is a reliable method which converges to the exact solution, unlike the numerical results.

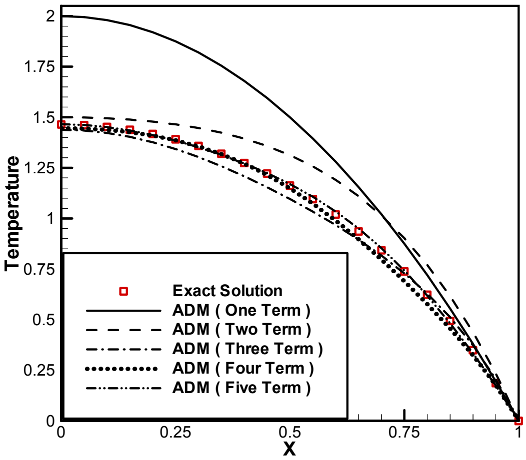

Figure 6: The exact and the Adomian solutions of heat transfer through a slab.

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