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On the application of ADM and VIM for solving a system of coupled Euler-Bernoulli beams

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Abstract: - In this paper, Adomian decomposition method is applied to solve a system of parallel Euler-Bernoulli beams with distributed springs and dampers. The method does not need linearization, weak nonlinearity or perturbation considerations.

Key-Words: - Adomian Decomposition Method, Partial Differential Equation, coupled Euler-Bernoulli Beams

1 Introduction

Mathematical modeling of many physical systems leads to nonlinear differential equations. An effective method is required to analyze the mathematical model witch provides solutions confirming to physical reality, i.e., the real world of physics. Therefore, we must be able to solve nonlinear differential equations, in space and time, which may be strongly nonlinear. Common analytic procedures linearize the system or assume that nonlinearities are relatively insignificant. Such these assumptions, sometimes strongly, affect the solution with respect to the real physics of the phenomenon. Generally, the numerical methods witch discritize the equation in space and time such as Runge-Kutta can permit us to calculate some values of time and space variables and we should care of chaos and bifurcation. Also, the numerical methods require long computation time. So, solving the problem with considering its nonlinearity and not using perturbation or linearization, … is necessary in our new world.

The Adomian decomposition method [1-10], proposed by Adomian initially with the aims to solve frontier physical problems, has been applied to wide class of deterministic and stochastic problems, linear and nonlinear, in physics, biology, and chemical engineering, …. For nonlinear models, the methods have shown reliable results in analytical approximation that converges very rapidly.

2 Adomian Decomposition Method

Consider the general equation, $Fu = g(t)$, where F represent a general nonlinear operator, which could be decomposed to linear and nonlinear terms. One can decompose the linear term to $Lu+Ru$, where L is the highest order derivative operator and R is the reminder of linear operators. Thus the equation may be rewritten in the form,

$$Lu + Ru + Nu = g \ . \tag{1}$$

N is the nonlinear operator. Solving Lu, we have,

$$Lu = g - Ru - Nu \ . \tag{2}$$

If one applies inverted of L on both sides, the equation can be written as,

$$L^{-1}Lu = L^{-1}g - L^{-1}Ru - L^{-1}Nu \tag{3}$$

where L^{-1} represent the inverted highest order operator L. For example, L is the second order derivative, so L^{-1} is a twofold integration operator. Then Eq. (3) yields,

$$u = u(0) + tu'(0) + L^{-1}g - L^{-1}Ru - L^{-1}Nu \tag{4}$$

According to Adomian decomposition method [1], we decompose u as,

$$u = \sum_{n=0}^{\infty} u_n \tag{5}$$

and the nonlinear term Nu are decomposed using Adomian polynomials.

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$$Nu = \sum_{n=0}^{\infty} A_n(u_0, u_1, \dots, u_n) \quad (6)$$

The Adomian polynomials, A_n, can be calculated […] in the form,

$$A_n = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n} N\left(v\left(\lambda\right)\right)\right]_{\lambda=0} \quad , n = 0,1,2,\dots \quad (7)$$

Now substituting (5) and (6) in (4), we obtain

$$u = \sum_{n=0}^{\infty} u_n = u(0) + tu'(0) + L^{-1}g - L^{-1}R\sum_{n=0}^{\infty} u_n - L^{-1}\sum_{n=0}^{\infty} A_n \quad (8)$$

which u_o identified as $u(0) + tu'(0) + L^{-1}g$. According to Adomian decomposition method, well described in [1], the equation is transformed to a set of recursive relations given by,

$$\begin{aligned} u_0 &= u(0) + tu'(0) + L^{-1}g \\ u_1 &= -L^{-1}Ru_0 - L^{-1}A_0 \\ u_2 &= -L^{-1}Ru_1 - L^{-1}A_1 \\ &\dots \\ u_{n+1} &= -L^{-1}Ru_n - L^{-1}A_n \qquad n \ge 1 \end{aligned} \quad (9)$$

u_0 can be obtained using initial conditions and consequently all of u_n are calculable. Since the series converges very rapidly, the k-term approximation can be use as a practical solution.

$$\varphi_k = \sum_{i=0}^{k-1} u_i, \qquad \lim_{k\to\infty} \varphi_k = u \quad (10)$$

Convergence has been rigorously established by Yves Cherrualt [1] and readers can find more in [2-5].

3 Variational iteration method

To illustrate its basic concepts of the variational iteration method, we consider the following differential equation:

$$Lu + Nu = g(t) \quad (11)$$

where L is a linear operator, N a nonlinear operator, and $g(t)$ a known analytic function. According to He's variational iteration method [11-17], we can construct a correction functional as follows.

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda[Lu_n(\tau) + N\overline{u_n(\tau)} - g(\tau)]d\tau \quad (12)$$

where λ is a general Lagrangian multiplier [11-17], which can be identified optimally via the variational theory, the subscript n denotes the nth order approximation, $\overline{u_n}$ is considered as a restricted variation, i.e. $\overline{u_n} = 0$.

Eq. (12) is called a correction functional. The variational iteration method proposed by He has been shown to solve effectively, easily and accurately a large class of nonlinear problems with approximations converging rapidly to accurate solutions. For linear problems, its exact solution can be obtained by only one iteration step due to the fact that Lagrange multiplier can be exactly identified.

4 Application of Adomian decomposition for a linear PDE system

The problem considered in this paper is motivated by an analogous problem in ordinary differential equations for coupled oscillator, and has potential application in isolating a vibrating a vibratory object from the outside disturbances. For example, rubber or rubberlike materials can be used to either absorb or shield a structure from vibration. As an approximation, these materials can be modeled as distributed springs. Interested readers are referred to Najafi (1996) [18] for further application of such a configuration.

Dynamics of the system under consideration are governed by the following set of partial differential equations.

$$\begin{cases} u_{tt} + a_1^4 u_{xxxx} = k(v - u) - \beta_1 u_t & 0 \le x \le 1, \quad t > 0 \\ v_{tt} + a_2^4 v_{xxxx} = k(u - v) - \beta_2 v_t & 0 \le x \le 1, \quad t > 0 \end{cases} \quad (13)$$

with initial conditions,

$$\begin{aligned} &u(x,0) = p_1(x), \quad v(x,0) = q_1(x) \quad 0 \le x \le 1 \\ &u_t(x,0) = p_2(x), \quad v_t(x,0) = q_2(x) \end{aligned} \quad (14)$$

and boundary conditions,

$$\begin{aligned} &u(0,t) = u_{xx}(0,t) = 0, v(0,t) = v_{xx}(0,t) = 0, t > 0 \\ &u(1,t) = u_{xx}(1,t) = 0, \quad v(1,t) = v_{xx}(1,t) = 0 \end{aligned} \quad (15)$$

where t and x represent the time and space variables respectively, and $u = u(x,t)$ and $v = v(x,t)$ are the vertical displacements of the beams measured from the horizontal equilibrium positions. The system parameters, $a_i \ge 0, i = 1,2$, are described in terms of flexural rigidity coefficient, $E_i I_i$, and mass density, m_i, as $a_i^4 = E_i I_i / m_i$, where E_i denotes Young modulus of elasticity and I_i denotes the cross-sectional area. Uniform beam properties are assumed; that is, m_i, E_i and I_i are assumed to be constants. The terms $\pm k(u - v)$ represent the coupling between the two beams and k denotes the elastic coupling constant. The terms $-\beta_1 u_t$ and $-\beta_2 v_t$ represent the velocity feedback control. Rewriting the system in the operator form

$$\begin{cases} L_t(u) + a_1^4 L_x(u) + R_1(u,v) = 0, \\ L_t(v) + a_2^4 L_x(v) + R_2(u,v) = 0. \end{cases} \tag{16}$$

The linear term is decomposed to L and R, where L is the highest order derivative and R is the reminder of the linear operator. L_t and L_x are considered second order and forth partial differential operator, respectively. L_t^{-1} is twofold integration with respect to t from 0 to t. Applying the L_t^{-1} to system (16) and using initial condition (14) yields,

$$\begin{cases} u(x,t) = p_1(x) + tp_2(x) \\ -a_1^4 L_t^{-1} L_x(u) - L_t^{-1} R_1(u,v), \\ v(x,t) = q_1(x) + tq_2(x) \\ -a_2^4 L_t^{-1} L_x(v) - L_t^{-1} R_2(u,v). \end{cases} \tag{17}$$

where the first two terms of equation (17) are integration constants. According to ADM [1] $u(x,t)$ and $v(x,t)$ can be decomposed as,

$$\begin{aligned} u(x,t) &= \sum_{n=0}^{\infty} u_n(x,t), \\ v(x,t) &= \sum_{n=0}^{\infty} v_n(x,t). \end{aligned} \tag{18}$$

Substituting (18) to (17) gives,

$$\begin{cases} \sum_{n=0}^{\infty} u_n(x,t) = p_1(x) + tp_2(x) \\ -a_1^4 L_t^{-1} L_x(\sum_{n=0}^{\infty} u_n(x,t)) \\ -L_t^{-1} R_1(\sum_{n=0}^{\infty} u_n(x,t), \sum_{n=0}^{\infty} v_n(x,t)) \\ \sum_{n=0}^{\infty} v_n(x,t) = q_1(x) + tq_2(x) \\ -a_2^4 L_t^{-1} L_x(\sum_{n=0}^{\infty} v_n(x,t)) \\ -L_t^{-1} R_2(\sum_{n=0}^{\infty} u_n(x,t), \sum_{n=0}^{\infty} v_n(x,t)) \end{cases} \tag{19}$$

According to ADM, the system of equations (16) is transformed to a set of recursive relations given by,

$$\begin{cases} u_0(x,t) = p_1(x) + tp_2(x) \\ u_{n+1}(x,t) = -a_1^4 L_t^{-1} L_x(u_n) - L_t^{-1} R_1(u_n, v_n) \quad ,n \ge 0 \end{cases} \tag{20}$$

and

$$\begin{cases} v_0(x,t) = q_1(x) + tq_2(x) \\ v_{n+1}(x,t) = -a_1^4 L_t^{-1} L_x(v_n) - L_t^{-1} R_1(u_n, v_n) \quad ,n \ge 0 \end{cases} \tag{21}$$

u_0 and v_0 can be obtained using initial conditions. The terms u_{n+1} and v_{n+1} are calculated using preceding terms. Consequently the summation of u_n and v_n terms is the desired solution which converges rapidly. In real world, we can calculated k terms of the summation, so the approximate solution is,

$$\begin{cases} \varphi_k = \sum_{n=0}^{k-1} u_n(x,t) \\ \psi_k = \sum_{n=0}^{k-1} v_n(x,t) \end{cases} \tag{22}$$

5 The specific case of a parallel system of Euler-Bernoulli Beams

The system which we are willing to deal with is a parallel system of Euler-Bernoulli beams with distributed springs and dampers.

$$\begin{cases} u_{tt} + u_{xxxx} = v - u - u_t \qquad 0 < x < 1, \quad t > 0 \\ v_{tt} + v_{xxxx} = u - v - v_t \qquad 0 < x < 1, \quad t > 0 \end{cases} \tag{23}$$

and initial conditions are

$$\begin{aligned} &u(x,0) = \sin(\pi x), \qquad v(x,0) = -\sin(\pi x) \\ &u_t(x,0) = 0, \qquad v_t(x,0) = 0 \end{aligned} \tag{24}$$

and boundary conditions

$$\begin{aligned} &u(0,t) = u_{xx}(0,t) = 0, \qquad v(0,t) = v_{xx}(0,t) = 0 \\ &u(1,t) = u_{xx}(1,t) = 0, \qquad v(1,t) = v_{xx}(1,t) = 0 \end{aligned} \tag{25}$$

After decomposition $u(x,t)$ and $v(x,t)$ according to (19), the system of equations (23) can be rewritten as,

$$\begin{cases} \sum_{n=0}^{\infty} u_n(x,t) = \sin(\pi x) + \iint [(\sum_{n=0}^{\infty} u_n)_{xxxx} \\ + \sum_{n=0}^{\infty} v_n - \sum_{n=0}^{\infty} u_n - (\sum_{n=0}^{\infty} u_n)_t] dtdt \\ \sum_{n=0}^{\infty} v_n(x,t) = -\sin(\pi x) + \iint [(\sum_{n=0}^{\infty} v_n)_{xxxx} \\ + \sum_{n=0}^{\infty} u_n - \sum_{n=0}^{\infty} v_n - (\sum_{n=0}^{\infty} v_n)_t] dtdt \end{cases} \tag{26}$$

Each of equations in (26) can be rewritten in a set of recursive relations.

$$\begin{cases} u_0(x,t) = \sin(\pi x) \\ u_{n+1}(x,t) = \iint [(u_n)_{xxxx} \\ + v_n - u_n - (u_n)_t] dtdt \qquad n \ge 0 \end{cases} \tag{27}$$

and

$$\begin{cases} u_0(x,t) = -\sin(\pi x) \\ u_{n+1}(x,t) = \iint [(v_n)_{xxxx} \\ + u_n - v_n - (v_n)_t] dtdt \qquad n \ge 0 \end{cases} \tag{28}$$

The procedure is clear and forward. These calculations can be programmed using symbolic packages such as Maple V. We computed a 2-tem solution for u and v.

$$
\begin{aligned}
u_2 &= \sum_{n=0}^{2} u_n(t) = (-1 + \frac{1}{2}(-2\pi^4 + 2)t^2 \\
&- \frac{1}{6}(-2\pi^4 + 2)t^3 + \frac{1}{12}(-2\pi^8 + 2\pi^4)t^4 \\
&+ \frac{1}{24}(2\pi^4 - 2)t^4 - \frac{1}{24}(-2\pi^4 + 2)t^4)\sin \pi x, \\
v_2 &= \sum_{n=0}^{2} v_n(t) = (1 + \frac{1}{2}(2\pi^4 - 2)t^2 \\
&- \frac{1}{6}(2\pi^4 - 2)t^3 + \frac{1}{12}(2\pi^8 - 2\pi^4)t^4 \\
&+ \frac{1}{24}(-2\pi^4 + 2)t^4 - \frac{1}{24}(2\pi^4 + 2)t^4)\sin \pi x.
\end{aligned} \tag{29}
$$

6 system of parallel Euler-Bernoulli Beams via VIM

Following the variational iteration method, its correction variational functional in t-direction can be expressed as follows.

$$
\begin{aligned}
u_{n+1}(x,t) &= u_n(x,t) \\
&+ \int_0^t \lambda_1 \left[\frac{d^2 u_n}{d\tau^2} + \frac{d^4 u_n}{d\tau^4} + \overline{u_n} - \overline{v_n} + \overline{\frac{du_n}{d\tau}} \right] d\tau \\
v_{n+1}(x,t) &= v_n(x,t) \\
&+ \int_0^t \lambda_2 \left[\frac{d^2 v_n}{d\tau^2} + \frac{d^4 v_n}{d\tau^4} + \overline{v_n} - \overline{u_n} + \overline{\frac{dv_n}{d\tau}} \right] d\tau
\end{aligned} \tag{30}
$$

$\delta\overline{u_n}$ is considered as a restricted variation, i.e. $\delta\overline{u_n} = 0$. Making the correction functional, Eq. (30), stationary, noticing that $\delta\overline{u_n} = 0$,

$$
\begin{aligned}
\delta u_{n+1}(x,t) &= \delta u_n(x,t) \\
&+ \int_0^t \lambda_1 \left[\frac{d^2 u_n}{d\tau^2} + \frac{d^4 u_n}{d\tau^4} + \overline{u_n} - \overline{v_n} + \overline{\frac{du_n}{d\tau}} \right] d\tau
\end{aligned} \tag{31}
$$

$$
\delta u_{n+1}(x,t) = \delta u_n(x,t) + \delta \int_0^t \lambda_1 \left[\frac{d^2 u_n}{d\tau^2} \right] d\tau \tag{32}
$$

$$
\begin{aligned}
\delta u_{n+1}(x,t) &= \delta u_n(x,t)\left(1 - \lambda_1'(\tau)\right) \\
&+ \delta \frac{d^2 u_n}{d\tau^2} \lambda_1(\tau) + \int_0^t \delta u_n(x,\tau) \lambda_1''(\tau) d\tau = 0
\end{aligned} \tag{33}
$$

yields the following stationary conditions:

$$
\begin{aligned}
\delta u_n &: \quad 1 - \lambda_1'(\tau) = 0 \\
\delta \frac{du_n}{d\tau} &: \quad \lambda_1(\tau) = 0 \\
\delta u_n &: \quad \lambda_1''(\tau) = 0
\end{aligned} \tag{34}
$$

The Lagrange multiplier, therefore, can be identified.

$$
\lambda_1(\tau) = \tau - t \tag{35}
$$

As a result, we obtain the following iteration formulae in t-direction.

$$
\begin{aligned}
u_{n+1}(x,t) &= u_n(x,t) \\
&+ \int_0^t (\tau - t) \left[\frac{d^2 u_n}{d\tau^2} + \frac{d^4 u_n}{x^4} \right. \\
&\left. + u_n - v_n + \frac{du_n}{dt} \right] d\tau
\end{aligned} \tag{36}
$$

We start with the initial condition given by Eq. (24). Therefore, we have u_0 as follows.

$$
\begin{aligned}
u_0(x,t) &= \sin(\pi x) \\
v_0(x,t) &= -\sin(\pi x)
\end{aligned} \tag{37}
$$

By the iteration formulae, we can obtain the following results:

$$
\begin{aligned}
u_0(x,t) &= \sin(\pi x) \\
u_1(x,t) &= \sin(\pi x) \\
&- \frac{1}{2}(\sin(\pi x)\pi^4 + 2\sin(\pi x))t^2 \\
u_2(x,t) &= \ldots \\
v_0(x,t) &= -\sin(\pi x) \\
v_1(x,t) &= -\sin(\pi x) \\
&+ \frac{1}{2}(\sin(\pi x)\pi^4 + 2\sin(\pi x))t^2 \\
v_2(x,t) &= \ldots
\end{aligned} \tag{38}
$$

and so on, in the same manner the rest of components of the iteration formulae (36) can be obtained using symbolic packages such as Maple.

7 Conclusion

In this paper, Adomian decomposition method is used to solve a system of coupled Euler-Bernoulli beams. In comparison with perturbation or linearization methods, this method gives analytical solution in series form which converges rapidly. The reliability and the reduction in the size of computational work is certainly a sign of a wider applicability of the method.

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