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ON THE APPLICATION OF ADOMIAN DECOMPOSITION METHOD AND OSCILLATION EQUATIONS

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Abstract: - In this paper, Adomian Decomposition Method (ADM) is applied to typical oscillation equations (Duffing and Van der Pol equations). The method does not need linearization, weak nonlinearity or perturbation considerations. The applicability of the method is confirmed by a good agreement obtained between the results of ADM solutions and the corresponding numerical ones.

Key-Words: - ADM – Nonlinear Oscillation – Adomian Polynomials – Duffing Equation - Van der Pol Equation.

1 Introduction

Mathematical modeling of many physical systems leads to nonlinear ordinary differential equations as well as PDE. An effective method is required to analyze the mathematical model which provides solutions conforming to physical reality. Therefore, it is much desirable to solve nonlinear differential equations, in space or time, which may be strongly nonlinear. Common analytic procedures linearize the system or assume that nonlinearities are relatively insignificant. Such assumptions, sometimes strongly, affect the solution with respect to the real physics of the phenomenon. Generally, the numerical methods which discretize the equation in space and time, such as finite difference, can permit us to calculate some values of time and space variables and care must be taken for chaos and bifurcation. Also, the numerical methods require long computation time.

Many oscillating systems in engineering and science are represented by $m\ddot{u} + f(u,\dot{u},c,k) = 0$. By employing an especial approximation for restoring force, *f(u)*, Duffing's differential equation results, which models a large number of dynamic systems. Duffing double-well oscillator was first developed to model forced vibrations of industrial machinery.

Even though, Van der Pol's equation was originally developed to describe the dynamics of a triode electronic oscillation, but it demonstrates many of the basic properties of a nonlinear system representing a mechanical self-exited mechanism.

The Adomian decomposition method [1-5], proposed initially by Adomian with the aim to solve frontier physical problems, has been applied to a wide class of deterministic and stochastic problems, linear and nonlinear, in physics, biology, and engineering, … . For nonlinear models, the methods have shown reliable results in analytical approximation that converges very rapidly [1, 2, 4, 5].

2 Basic Method

Consider the general equation, $Fu = g(t)$, where F represent a general nonlinear operator, which could be decomposed into linear and nonlinear terms. One could decompose the linear term into $Lu + Ru$, where L is the highest order derivative operator and R is the reminder of the linear operators. Thus the equation may be rewritten in the form,

$$Lu + Ru + Nu = g \ , \qquad (1)$$

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where N is a nonlinear operator. Solving for Lu, we have,

$$Lu = g - Ru - Nu\,. \tag{2}$$

If one applies the inverse operator of L on both sides, the equation can be written as,

$$L^{-1}Lu = L^{-1}g - L^{-1}Ru - L^{-1}Nu\,, \tag{3}$$

where L^{-1} represents the inverse of the highest order operator L. For example, L is a second order derivative operator, so L^{-1} is a twofold integral operator. Then Eq. (3) yields,

$$u = u(0) + tu'(0) + L^{-1}g - L^{-1}Ru - L^{-1}Nu\,, \tag{4}$$

The first two terms, on the right hand side, are constants of integration and are to be computed by using the initial conditions. By ADM [4], u is decomposed to

$$u = \sum_{n=0}^{\infty} u_n\,, \tag{5}$$

Using Adomian polynomials, the nonlinear term, Nu, decomposes to

$$Nu = \sum_{n=0}^{\infty} A_n(u_0, u_1, \ldots, u_n). \tag{6}$$

The Adomian polynomials, A_n, can be calculated [6] in the form,

$$A_n = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n} N(v(\lambda))\right]_{\lambda=0}\,, n = 0,1,2,\ldots \tag{7}$$

Now, by substituting (5) and (6) in (4), the following is obtained,

$$u = \sum_{n=0}^{\infty} u_n = u(0) + tu'(0) + L^{-1}g - L^{-1}R\sum_{n=0}^{\infty} u_n - L^{-1}\sum_{n=0}^{\infty} A_n\,, \tag{8}$$

where u_0 is identified as $u(0) + tu'(0) + L^{-1}g$. The Eq. (4) is transformed to a set of recursive relations given by,

$$\begin{aligned} u_0 &= u(0) + tu'(0) + L^{-1}g\,, \\ u_1 &= -L^{-1}Ru_0 - L^{-1}A_0\,, \\ u_2 &= -L^{-1}Ru_1 - L^{-1}A_1\,, \\ &\vdots \\ u_{n+1} &= -L^{-1}Ru_n - L^{-1}A_n\,. \qquad n \ge 1 \end{aligned} \tag{9}$$

After identifying u_0 as $u(0) + tu'(0) + L^{-1}g$, and evaluating it, all the u_n's will become calculable by using I.C.'s. Since the series converges very rapidly, the k-term approximation could be use as a practical solution,

$$\varphi_k = \sum_{i=0}^{k-1} u_i\,. \tag{10}$$

Of course, the exact solution could be obtained by utilizing,

$$\lim_{k\to\infty} \varphi_k = u\,.$$

3 Some Examples

Now, we apply the above mentioned technique to solve some nonlinear problems.

3.1 Duffing Equation

The following problem is motivated by an analogous problem in ordinary differential equations and widely used in many perturbation techniques,

$$\ddot{u} + u + \varepsilon u^3 = 0\,, \tag{11}$$

with initial conditions,

$$u(0) = 1\,, \quad \dot{u}(0) = 5\,. \tag{12}$$

Rewriting the equation in the operator form,

$$L_t(u) + u + \varepsilon N(u) = 0\,. \tag{13}$$

The highest order linear derivative operator is L_t, where L_t is a second order differential operator with respect to t. The nonlinear operator is represented by N. The integral operator, L_t^{-1}, is a twofold integration with respect to t from 0 to t. Applying the L_t^{-1} to equation (13) and using initial condition (12). It yields,

$$u(t) = 1 + 5t - L_t^{-1}(u) - \varepsilon L_t^{-1}N(u)\,. \tag{14}$$

The solution, $u(t)$, can be decomposed as follows,

$$u(t) = \sum_{n=0}^{\infty} u_n(t)\,. \tag{15}$$

Substituting (16) to (15) gives,

$$\sum_{n=0}^{\infty} u_n(t) = 1 + 5t - L_t^{-1}\Big(\sum_{n=0}^{\infty} u_n(t)\Big) - \varepsilon L_t^{-1}N\Big(\sum_{n=0}^{\infty} u_n(t)\Big). \tag{16}$$

The Eq. (14) is transformed to a set of recursive relations given by,

$$\begin{cases} u_0(t) = 1+5t, \\ u_{n+1}(t) = -L_t^{-1}(u_n) - \varepsilon L_t^{-1} N(u_n), \quad n \geq 0 \end{cases} \tag{17}$$

where u_0 is obtained using initial conditions and the u_{n+1} terms are calculated using preceding relations, (17). Hence, the summation of u_n terms is the desired solution which converges rapidly. In practice, we can calculate the first k terms of the summation in place of the whole, so the approximate solution is,

$$\varphi_k = \sum_{n=0}^{k-1} u_n(t). \tag{18}$$

The procedure is clear to follow and to calculate. These calculations can be programmed using symbolic packages such as Maple V or Mathematica.

$$\begin{aligned} u_1(t) &= -\frac{1}{2}t^2 - \frac{5}{6}t^3 - \frac{1}{500}\varepsilon(1+5t)^5, \\ u_2(t) &= (\frac{3}{1000}t^2 + \frac{7}{200}t^3 + \frac{21}{80}t^4 \\ &+ \frac{21}{16}t^5 + \frac{35}{8}t^6 + \frac{75}{8}t^7 + \frac{375}{32}t^8 \\ &+ \frac{625}{96}t^9)\varepsilon^2 + (\frac{1}{1000}t^2 + \frac{1}{120}t^3 \\ &+ \frac{1}{6}t^4 + t^5 + \frac{55}{24}t^6 + \frac{275}{168}t^7)\varepsilon \\ &+ \frac{1}{24}t^4 + \frac{1}{24}t^5, \\ &\vdots \end{aligned} \tag{19}$$

So, the result can be presented as a series solution in the form of,

$$\begin{aligned} \varphi &= \sum_{n=0}^{3} u_n(t) = u_0(t) + u_1(t) + u_2(t) \\ &+ u_3(t) + \ldots \end{aligned} \tag{20}$$

In practice, due to the rapid convergence of the solution, the first few terms will provide the required accuracy.

3.2 Van der Pol Equation

It is an equation describing self-sustaining oscillations in which energy is fed into small oscillations and removed from large oscillations. This equation arises in the study of circuits containing vacuum tubes and widely used in many perturbation techniques,

$$\ddot{u} + u - \varepsilon(1-u^2)\dot{u} = 0, \tag{21}$$

with initial conditions,

$$u(0) = 2, \quad \dot{u}(0) = 0. \tag{22}$$

Rewriting the equation in the operator form,

$$L_t(u) + u + \varepsilon N(u) = 0. \tag{23}$$

Again, the highest order linear derivative operator is L_t. Where L_t is the second order differential operator and N is the nonlinear one. The inverse operator, L_t^{-1} is a twofold integration with respect to t from 0 to t. Applying the operator, L_t^{-1}, to Eq. (23) and employing the initial condition (22) yields,

$$u(t) = 2 - L_t^{-1}(u) - \varepsilon L_t^{-1} N(u). \tag{24}$$

According to ADM [1], $u(t)$ can be decomposed to,

$$u(t) = \sum_{n=0}^{\infty} u_n(t), \tag{25}$$

introducing (25) into (24) gives,

$$\begin{aligned} &\sum_{n=0}^{\infty} u_n(t) = 2 - L_t^{-1}(\sum_{n=0}^{\infty} u_n(t)) \\ &- \varepsilon L_t^{-1} N(\sum_{n=0}^{\infty} u_n(t)). \end{aligned} \tag{26}$$

The following set of relations (27) are recursive and acquired by applying ADM to Eq. (23),

$$\begin{cases} u_0(t) = 2, \\ u_{n+1}(t) = -L_t^{-1}(u_n) - \varepsilon L_t^{-1} N(u_n). \quad n \geq 0 \end{cases} \tag{27}$$

By employing the initial conditions, u_0 is calculated and the other terms, u_{n+1}, are obtained consecutively using the preceded results. The sought solution to the problem is the summation of u_n terms which converges fast. Because of its behavior, fast convergence, it will suffice to calculate only the first k terms as the solution,

$$\varphi_k = \sum_{n=0}^{k-1} u_n(t). \tag{28}$$

As previously mentioned, the packages such as Maple V or Mathematica, are utilized to calculate u_n's in symbolic fashion,

$$
\begin{aligned}
&u_1(t) = -t^2, \\
&u_2(t) = \frac{1}{12}t^4 + \varepsilon t^3, \\
&u_3(t) = -\frac{1}{360}t^6 - \frac{1}{2}\varepsilon t^5 - \frac{3}{4}\varepsilon^2 t^4, \\
&u_4(t) = \frac{1}{20160}t^8 + \frac{13}{120}\varepsilon t^7 \\
&+\frac{113}{120}\varepsilon^2 t^6 + \frac{9}{20}\varepsilon^3 t^5, \\
&\vdots
\end{aligned} \tag{29}
$$

Summation of the first four-term of $u_n(t)$, as in (28), gives a partial sum solution in a series form,

$$
\begin{aligned}
&\varphi_4 = \sum_{n=0}^{3} u_n(t) = 2 + \frac{1}{20160}t^8 - t^2 \\
&-\frac{1}{360}t^6 + \frac{1}{120}t^4 + \varepsilon(-\frac{1}{2}t^5 \\
&+\frac{13}{120}t^7 + t^3) + \varepsilon^2(\frac{113}{120}t^6 - \frac{3}{4}t^4) \\
&+\frac{9}{20}\varepsilon^3 t^5.
\end{aligned} \tag{30}
$$

4 Numerical Results

The obtained ADM results for Duffing and Van der Pol equations are presented in tables 1-3. Tables 1 and 2 contain the ADM results for various points in time while $\varepsilon = 0.1$. As seen in tables 1 and 2, the 3-term and 4-term ADM solutions for both of the aforementioned equations are in good agreement with the corresponding numerical results.

Moreover, when t is increased, it was observed that the error is increased too. It is evident that computation of more terms would results in better approximation. In table 3, the results of Van der Pol equation using ADM with $\varepsilon = 10$ are compared with those of the two-term perturbation solution. It is shown that there is no need to assume that ε is a small parameter as it is required by perturbation theory.

5 Conclusion

In this paper, the standard Adomian decomposition method is used to solve some nonlinear oscillation equations. In comparison with perturbation or linearization methods, this method gives analytical solution in series form which converges rapidly. The reliability and the reduction in the size of computational work is certainly a sign of a wider applicability of the method.

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Table 1 -Comparison of the numerical results with ADM solution for Duffing equation at $\varepsilon = 0.1$ for different values of t

t	3 terms of ADM	4 terms of ADM	Numerical (Runge-kutta)	Numerical solution - 4 terms ADM	Absolute Error (%)
0	0.99980019	0.99980019	1.00000000	0.00374053	0.37405285
0.001	1.00479465	1.00479465	1.00499941	0.00376856	0.37498171
0.01	1.04969418	1.04969418	1.04994019	0.00399432	0.38043308
0.1	1.49265918	1.49265918	1.49314227	0.00234472	0.15703271
0.5	3.18063997	3.18068685	3.17654735	0.00413950	0.13031443
1	3.71608038	3.83005787	3.79760816	0.03244971	0.85447748

Table 2 -Comparison of the numerical results with ADM solution for Van der pol equation at $\varepsilon = 0.1$ for different values of t

t	3 terms of ADM	4 terms of ADM	Numerical (Runge-kutta)	Numerical solution - 4 terms of ADM	Absolute Error (%)
0.001	1.999999	1.999999	1.999998936	6.36E-08	3.1782E-06
0.01	1.999900101	1.999900101	1.999897836	2.2649E-06	0.00011325
0.1	1.99010708	1.990107095	1.99009935	7.7445E-06	0.00038915
0.5	1.76563368	1.765879707	1.765849085	3.06222E-05	0.00173413
1	1.123055555	1.143805158	1.138456445	0.005348713	0.46982149

Table 3 -Comparison of the numerical results with ADM solution for Van der pol equation at $\varepsilon = 10$ for different values of t

t	3 terms of ADM	4 terms of ADM	Numerical (Runge-kutta)	Numerical solution - 4 terms ADM	Perturbation Solution
0	1	2	2	0	29.083333
0.001	1.99999901	1.99999901	1.999998988	2.20E-08	29.0825864959
0.01	1.99990925	1.999909296	1.999909283	1.33E-08	29.0068119317
0.1	1.99245833	1.997052605	1.99544093	0.001611675	16.6393212321