Proceedings of the 9th WSEAS International Conference on Applied Mathematics, Istanbul, Turkey, May 27-29, 2006 (pp193-198)

The numerical solution for Singular Integro- Differential Equations in Generalized Hölder spaces

Iurie Caraus
Moldova State University
Faculty of Mathematics and Informatics
Mateevici 60 str., Chisinau
Moldova

Nikos E. Mastorakis
WSEAS
A. I. Theologou 17-23
15773, Zographou, Athens,
Greece

Abstract: We have suggested the numerical schemes of collocation methods for approximative solution of singular integro- differential equations with kernels of Cauchy type. The equations are defined on the arbitrary smooth closed contours of complex plane. The researched methods are based on Fejér points. Theoretical background of collocation methods has been obtained in Generalized Hölder spaces.

Singular Integro- differential equations, Fejér points, Generalized Hölder spaces:

1 Introduction

Singular integral equations with Cauchy kernels (SIE) and Singular integro- differential equations with Cauchy kernels (SIDE) model many problems in elasticity theory, aerodynamics, mechanics, thermoelasticity, queueing system analysis, etc.[7]-[11]

The general theory of SIE and SIDE has been widely investigated in last decades [12]-[16].

The exact solution for SIDE can only be found in rare specific cases. That is why the necessity exists to elaborate numerical methods for solving SIDE with the corresponding theoretical background.

The problem of numerical solution for SIDE by collocation methods has been studied in [17]-[18]. The equations have been defined on the unit circle.

The case, however, when the contour of integration can be an arbitrary closed smooth curve (not unit circle), has not been studied enough. Transition to another contour, different from the standard one, implies many difficulties. It should be noted that the conformal mapping from the arbitrary contour to the unit circle using the Riemann function does not solve the problem, but only makes it more difficult.

We note theoretical background of collocation methods for SIDE in classical Hölder spaces has been obtained in [5], [19]. The equations have been defined on arbitrary smooth closed contours.

2 Theorem on approximation of functions by Lagrange polynomials

Let Γ be an arbitrary smooth closed contour bounding a simply- connected region F^+ of complex plane, let $t = 0 \in F^+$, $F^- = C \setminus \{F^+ \cup \Gamma\}$, C is the complex plane.

Let $z = \psi(w)$ be a Riemann function, mapping conformably and unambiguously the outside unit circle $\{|w| = 1\}$ on the surface F^-, so that $\psi(\infty) = \infty$, $\psi^{'}(\infty) = 1$.

By $\omega(\delta)$ $(\delta \in (0, l], l = \max\limits_{t^{'},t^{''}\in\Gamma} |t^{'} - t^{''}|)$we denote the arbitrary module of continuity and by $H(\omega)$ we denote the generalized Hölder space [1],[2] with norm

$$||g||_\omega = ||g||_C + H(g; \omega); \quad (1)$$

$$||g||_c = \max_{t\in\Gamma} |g(t)|, \quad H(g; \omega) = \sup_{s\in(0,l]} \left(\frac{\omega(g; \delta)}{\omega(\delta)}\right);$$

here the $\omega(g; \delta)$is the module of continuity for function $g(t)$on Γ.

We consider only the spaces $H(\omega)$ with the module of continuity satisfying the Bari- Stechkin conditions: [2]

$$\int_0^h \frac{\omega(\xi)}{\xi} d\xi < \infty, \quad (2)$$

or

$$\int_0^\delta \frac{\omega(\xi)}{\xi} d\xi + \delta \int_\delta^h \frac{\omega(\xi)}{\xi^2} d\xi = O(\omega(\delta)), \delta \to 0, \quad (3)$$

By $H^{(r)}(\omega)$, $r \geq 0$ ($H^{(0)}(\omega) = H(\omega)$) we denote the space of r times continuous- differentiable functions. The derivatives of the $r-th$ order for these functions are elements of space $H(\omega)$. The norm on $H^{(r)}(\omega)$ is given by formula

$$||g||_{\omega,r} = \sum_{k=0}^{r} ||g^{(k)}||_c + H(g^{(r)};\omega). \qquad (4)$$

Remind that if $\omega(\delta) = \delta^{\alpha}$, $\alpha \in (0,1]$, then $H(\omega) = H_{\alpha}$ is the Hölder space with exponent α.

The space $H(\omega)$ is a Banach nonseparable space. So the approximation of the whole class of functions by norm (1) with the help of finite- dimensional approximation is impossible. But in some subset of $H(\omega)$ the problem can be solved in the affirmative.

Let t_j ($j = \overline{0,2n}$) be a set of distinct points on Γ. By U_n we denote the operator, which maps any function $g(t)$ defined on Γ into its interpolating Lagrange polynomial defined by using the nodes t_j :

$$(U_n g)(t) = \sum_{j=0}^{2n} g(t_j) l_j(t), \quad t \in \Gamma, \qquad (5)$$

$$l_j(t) = \prod_{k=0,k\neq j}^{2n} \frac{t-t_k}{t_j-t_k} \cdot \left(\frac{t_j}{t}\right)^n \equiv \sum_{r=-n}^{n} \Lambda_r^{(j)} t^r, \quad j = \overline{0,2n}. \qquad (6)$$

The following theorem gives the deviation of Lagrange polynomials and function in generalized Hölder spaces [3]:

Theorem 1. *Let $\omega_1(\delta)$ and $\omega_2(\delta)$ ($\delta \in (0,l]$) be modules of continuity satisfying (2) or (3) and the function $\Phi(\delta) = \omega_1(\delta)/\omega_2(\delta)$ is non-decreasing on $(0,l]$. The points make a system of Fejér knots on Γ [4]:*

$$t_j = \psi\left(exp\left(\frac{2\pi i}{2n+1}(j-n)\right)\right), \quad j = \overline{0,2n}. \qquad (7)$$

Here $z = \psi(w)$ is the Riemann function for contour Γ. Then for any function $g(t) \in H(\omega_1)$, the following estimate holds:

$$||g - U_n g||_{\omega_2} \leq (d_1 + d_2 \ln n)\Phi\left(\frac{1}{n}\right) H(g;\omega_1).$$

By d_k, $k = 1, 2, \ldots$ we denote certain constants independent of n.

3 Numerical schemes of the collocation methods

In complex space $H_{\omega}(\Gamma)$ we consider the singular integro- differential equation (SIDE)

$$(Mx \equiv) \sum_{r=0}^{q} [\tilde{A}_r(t) x^{(r)}(t) + \tilde{B}_r(t) \frac{1}{\pi i} \int_{\Gamma} \frac{x^{(r)}(\tau)}{\tau - t} d\tau + $$
$$+ \frac{1}{2\pi i} \int_{\Gamma} K_r(t,\tau) \cdot x^{(r)}(\tau) d\tau] = f(t), \quad t \in \Gamma, \qquad (8)$$

where $\tilde{A}_r(t), \tilde{B}_r(t), f(t)$ and $K_r(t,\tau)$ ($r = \overline{0,q}$) are known functions on Γ; $x^{(0)}(t) = x(t)$ is the unknown function; $x^{(r)}(t) = \dfrac{d^r x(t)}{dt^r}$ ($r = \overline{1,q}$); q is a positive integer.

We seek a solution of equation (8) in the class of functions, satisfying the conditions

$$\frac{1}{2\pi i} \int_{\Gamma} x(\tau) \tau^{-k-1} d\tau = 0, \quad k = \overline{0,q-1}. \qquad (9)$$

Using the Riesz operators $P = \frac{1}{2}(I+S)$, $Q = I - P$, (where I is an identical operator and S is a singular operator (with Cauchy nucleus)) we rewrite the equation (8) in the following form convenient for consideration:

$$(Mx) \equiv \sum_{r=0}^{q} [A_r(t)(Px^{(r)})(t) + B_r(t)(Qx^{(r)})(t) + $$
$$\frac{1}{2\pi i} \int_{\Gamma} K_r(t,\tau) x^{(r)}(\tau) d\tau] = f(t), \quad t \in \Gamma, \qquad (10)$$

where $A_r(t) = \tilde{A}_r(t) + \tilde{B}_r(t)$, $B_r(t) = \tilde{A}_r(t) - \tilde{B}_r(t)$, $\quad r = \overline{0,q}$.

Equation (10) with conditions (9) will be denoted as "problem (10)-(9)".

We seek an approximative solutions of problem (10)- (9) in the form of a polynomial

$$x_n(t) = \sum_{k=0}^{n} \xi_k^{(n)} t^{k+q} + \sum_{k=-n}^{-1} \xi_k^{(n)} t^k, \quad t \in \Gamma, \qquad (11)$$

with unknown coefficients $\xi_k^{(n)} = \xi_k$ ($k = \overline{-n,n}$); obviously $x_n(t)$, constructed by formula (11) satisfies condition (9).

According to the collocation method, we determine the unknowns ξ_k ($k = \overline{-n,n}$) from the condition into inversion in zero the expression $R(t) = (Mx_n)(t) - f(t)$ in $2n+1$ different points t_j $j = \overline{0,2n}$, on Γ :

$$R_n(t_j) = 0, \quad j = \overline{0,2n}. \qquad (12)$$

Using the formulae [5]

$$(Px)^{(r)}(t) = (Px^{(r)})(t), \quad (Qx)^{(r)}(t) = (Qx^{(r)})(t), \tag{13}$$

the relations

$$(t^{k+q})^{(r)} = \frac{(k+q)!}{(k+q-r)!} t^{k+q-r}, \quad k = \overline{0,n};$$

$$(t^{-k})^{(r)} = (-1)^r \frac{(k+r-1)!}{(k-1)!} t^{-k-r}, k = \overline{1,n}, \tag{14}$$

from (12) we obtain the system of linear algebraical equations (SLAE):

$$\sum_{r=0}^{q} \{A_r(t_j) \sum_{k=0}^{n} \frac{(k+q)!}{(k+q-r)!} t^{k+q-r} \xi_k +$$
$$+ B_r(t_j) \sum_{k=1}^{n} (-1)^r \frac{(k+r-1)!}{(k-1)!} \times$$
$$\times t_j^{-k-r} \xi_{-k} + \frac{1}{2\pi i} \cdot \sum_{k=0}^{n} \frac{(k+q)!}{(k+q-r)!} \times$$
$$\times \int_\Gamma K_r(t_j, \tau) \tau^{k+q-r} d\tau \cdot \xi_k +$$
$$+ \sum_{k=1}^{n} (-1)^r \frac{(k+r-1)!}{(k-1)!} \cdot \frac{1}{2\pi i} \times$$
$$\times \int_\Gamma K_r(t_j, \tau) \tau^{-k-r} d\tau \cdot \xi_{-k} \} =$$
$$= f(t_j), \; j = \overline{0,2n}, \tag{15}$$

where $t_j (j = \overline{0,2n})$ is the set of distinct points on Γ and $A_r(t) = \tilde{A}_r(t) + \tilde{B}_r(t)$, $B_r(t) = \tilde{A}_r(t) - \tilde{B}_r(t)$, $r = \overline{0,q}$.

Let $\overset{o}{H}{}^{(q)} (\omega_2)$ be subspace of space $H^{(q)}(\omega_2)$, the elements from $\overset{o}{H}{}^{(q)} (\omega_2)$ satisfy conditions (9) and the norm is defined either as in $H^{(q)}(\omega_2)$.

Theoretical background of collocation methods is given in the following theorem:

Theorem 2. *Let the following conditions be satisfied:*

1. *the functions* $A_r(t), B_r(t), f(t)$ *and* $K_r(t,\tau)$ $r = \overline{0,q}$ *belong to the space* $H(\omega_1)$;
2. $A_q(t) \neq 0$, $B_q(t) \neq 0$, $t \in \Gamma$;
3. *the index of function* $t^q B_q^{-1}(t) A_q(t)$ *is equal to zero;*
4. *the operator* $M : \overset{o}{H}{}^{(q)} (\omega_2) \to H(\omega_2)$ *is a invertible and linear;*
5. *the points* t_j $j = \overline{0,2n}$ *form a system of Fejér knots (7) on* Γ.
6. *the function* $\Phi(\sigma) = \dfrac{\omega_1(\delta)}{\omega_2(\delta)}$ *is nondecreasing on* $(0, l]$.

Then, beginning with $n \geq n_1$ *the SLAE (15) of collocation method has the unique solution* ξ_k $(k = \overline{-n,n})$. *The approximate solutions* $x_n(t)$ *constructed by formula (11) converge in the norm of space* $H^{(q)}(\omega_2)$ *as* $n \to \infty$ *to the exact solution* $x(t)$ *of problem (10)-(9). Furthermore, the following error estimate is true:*

$$||x - x_n||_{\omega_2,q} = O\left(\Phi\left(\frac{1}{n}\right) \ln n\right) \tag{16}$$

4 Auxiliary results

We will formulate one result from [6], establishing the equivalence (in sense of solvability) of SIDE (9) and the singular integral equation (SIE). This result we will use for proving the theorems of convergence.

Using the integral representations [6]

$$\frac{d^q(Px)(t)}{dt^q} = \frac{1}{2\pi i} \int_\Gamma \frac{v(t)}{\tau - t} d\tau, \quad t \in F^+ \tag{17}$$

$$\frac{d^q(Qx)(t)}{dt^q} = \frac{1}{2\pi i} \int_\Gamma \frac{v(t)}{\tau - t} d\tau, \quad t \in F^-$$

we reduce the problem (10)-(9) to the equivalent (in terms of solvability) singular integral equation (SIE)

$$(\Theta v \equiv) C(t) v(t) + \frac{D(t)}{\pi i} \int_\Gamma \frac{v(\tau)}{\tau - t} d\tau +$$
$$+ \frac{1}{2\pi i} \int_\Gamma h(t,\tau) v(\tau) d\tau = f(t), t \in \Gamma \tag{18}$$

for the unknown function $v(t)$ where

$$C(t) = \frac{1}{2} \left[A_q(t) + t^{-q} B_q(t)\right],$$

$$D(t) = \frac{1}{2} \left[A_q(t) - t^{-q} B_q(t)\right], \tag{19}$$

$h(t,\tau)$, is the Hölder function by both variables. The obvious formula for determining $h(t,\tau)$ can be found in [6].

Note that the right-hand sides in (18) and (10) coincide by virtue condition (9).

The equivalence of the existence of the solutions between the SIE (18) and the problem (10)-(9) is the result of the following lemma from [6].

Lemma *The SIE (18) and problem (10)- (9) are equivalent in terms of solvability. That is, for each solution $v(t)$ of SIE (18) there is a solution of problem (10)- (9), determined by formulae*

$$(Px)(t) = \frac{(-1)^q}{2\pi i(q-1)!} \int_\Gamma v(\tau)[(\tau - t)^{q-1} \times$$

$$\times \ln\left(1 - \frac{t}{\tau}\right) + \sum_{k=1}^{q-1} \tilde{\alpha}_k \tau^{q-k-1} t^k] d\tau, \tag{20}$$

$$(Qx)(t) = \frac{(-1)^q}{2\pi i(q-1)!} \int_\Gamma v(\tau)\tau^{-q}[(\tau - t)^{q-1} \times$$

$$\times \ln\left(1 - \frac{\tau}{t}\right) + \sum_{k=0}^{q-2} \tilde{\beta}_k \tau^{q-k-1} t^k] d\tau,$$

($\tilde{\alpha}_k, k = \overline{1, q-1}$ and $\tilde{\beta}_k, k = \overline{0, q-2}$ are real numbers) and vice versa for each solution $x(t)$ of problem (10)- (9) there is a solution $v(t)$

$$v(t) = \frac{d^q(Px)(t)}{dt^q} + t^q \frac{d^q(Qx)(t)}{dt^q},$$

to the SIE (18).

Furthermore for given set of linear- independent solutions of (18), there are corresponding set of linear- independent solutions of the problem (10)- (9) and vise versa. In formulae (20) $\ln(1 - t/\tau)$ and $\ln(1 - \tau/t)$(for given τ) there are the branches that vanish at the points $t = 0$ and $t = \infty$, respectively.

5 Proof of theorem 2

Using the conditions (12)

$$R_n(t_j) = 0, \quad j = \overline{0, 2n} \tag{21}$$

we obtain that the (15) is equivalent to the operator equation

$$U_n M U_n x_n = U_n f, \tag{22}$$

where M is an operator defined in (10). We will show that if $n \geq n_1$ is large enough, then the operator $U_n M U_n$ is reversible. The operator acts from the subspace

$$\overset{o}{X}_n = \left\{ t^q \sum_{k=0}^{n} \xi_k t^k + \sum_{k=-n}^{-1} \xi_k t^k \right\}$$

(the norm is defined as in $H^{(q)}(\omega_2)$) to the space $X_n = \sum_{k=-n}^{n} r_k t^k$, (the norm is defined as in $H(\omega_2)$).

In similar way, by using the formulae (17) we represent the functions $\frac{d^q(Px_n)(t)}{dt^q}$ and $\frac{d^q(Qx_n)(t)}{dt^q}$ by Cauchy type integrals with the same density $v_n(t)$:

$$\frac{d^q(Px_n)(t)}{dt^q} = \frac{1}{2\pi i} \int_\Gamma \frac{v_n(t)}{\tau - t} d\tau, \quad t \in F^+ \tag{23}$$

$$\frac{d^q(Qx_n)(t)}{dt^q} = \frac{t^{-q}}{2\pi i} \int_\Gamma \frac{v_n(t)}{\tau - t} d\tau, \quad t \in F^-$$

By Υ_n we denote the polynomial class of the form

$$\sum_{k=-n}^{n} \gamma_k t^k, \quad t \in \Gamma$$

where γ_k are arbitrary complex numbers.

Using formulae (13) and relations (14) we obtain from (23)

$$v_n(t) = \sum_{k=0}^{n} \frac{(k+q)!}{k!} t^k \xi_k +$$

$$+(-1)^q \sum_{k=1}^{n} \frac{(k+q-1)!}{(k-1)!} t^{-k} \xi_{-k}$$

and so $v_n(t) \in \Upsilon_n$.

Using (23), Eq. (22) as well as problem (10)-(9) can be reduced to an equivalent equation (in same sense of solvability)

$$U_n \Theta U_n v_n = U_n f, \tag{24}$$

treated as an equation in the subspace X_n. Obviously, the eq. (24) is the equation of collocation methods for SIE (18). For SIE the collocation method was considered in [2] where sufficient conditions for the solvability and convergence of this method were obtained.

From (23) and $v_n(t) \in \Upsilon_n$ we conclude that if $v_n(t)$ is the solution of equation (24), then $y_n(t)$ is the discrete solution of the system $U_n M U_n x_n = U_n f$ and vise versa. We can determined $y_n(t)$ from relations (20):

$$(Py_n)(t) = \frac{(-1)^q}{2\pi i(q-1)!} \int_\Gamma v_n(\tau)[(\tau - t)^{q-1} \ln(1 - \frac{t}{\tau}) +$$

$$+ \sum_{k=1}^{q-1} \tilde{\alpha}_k \tau^{q-k-1} t^k] d\tau, \tag{25}$$

$$(Qy_n)(t) = \frac{(-1)^q}{2\pi i(q-1)!}\int_\Gamma v_n(\tau)[(\tau-t)^{q-1}\ln(1-\frac{\tau}{t})+$$

$$+\sum_{k=0}^{q-2}\tilde{\beta}_k\tau^{q-k-1}t^k]d\tau,$$

As was mentioned above, the function $y_n(t)$ is determined through $v_n(t)$ from (25) uniquely.

It follows if the equation (24) has a unique solution $v_n(t)$ in subspace X_n, then the following relation

$$y_n(t) = x_n(t) \tag{26}$$

is true.

We will show that for Eq. (24) all conditions of theorem 3[2] are satisfied.

From condition 1 of theorem 2 and from (19) we obtain the condition 1 of theorem 3[2]. From the equality

$$[C(t) - D(t)]^{-1}[C(t) + D(t)] = t^q B_q^{-1}(t)A_q(t),$$

we conclude that indexes of functions of $[C(t) - D(t)]^{-1}[C(t) + D(t)]$ has to be zero which coincides with condition 2 of theorem 3[2]. Note other conditions of theorem 2 coincide with conditions of theorem 3[2].

Because $h(t,\tau) \in H(\omega_1)$ then the exact solution $v(t) \in H(\omega_1)$.

Assumptions 1)-6) in theorem 2 provide the validity of all assumptions of theorem 3 in [2]; therefore the Eq.(24) with $n \geq n_1$ is uniquely solvable. The approximate solutions $v_n(t)$ of (24) converge to the exact solution $v(t)$ of SIE (18) in the norm of the space $H(\omega_2)$ as $n \to \infty$. Hence the operator equation (22) and the SLAE (15) has unique solution for $n \geq n_1$. From theorem 3[2] the following relation holds:

$$||v - v_n||_{\omega_2} = O\left(\Phi\left(\frac{1}{n}\right)\ln n\right) \tag{27}$$

From (20), (25) and (26) we obtain

$$||x - x_n||_{\omega_2,q} \leq c||v - v_n||_{\omega_2}$$

From last relation and from (27) we have (16).

References:

[1] Gusejnov A.I., Muchtarov Ch.S. Introduction to the theory of nonlinear singular integral equations. Moskva: Izdatel'stvo "Nauka", Glavnaya Redaktsiya Fiziko-Matematicheskoj Literatury. 415 p. R. 3.60 (1980). MSC 200(in Russian)

[2] V. Zolotarevski, Gh. Andriesh. Approximation of functions in generalized Hlder spaces and approximate solution of singular integral equations. Differ. Equations 32, No.9, 1223-1227 (1996); translation from Differ. Uravn. 32, No.9, 1222-1226 (1996).

[3] V. Zolotarevski, Gh. Andriesh. The approximation of functions in generalized Hlder spaces. Computer Science Journal of Moldova, Chisinau, Moldova, V 3, no 3(9), 1995, pp. 300-306

[4] Smirnov V.I.- Lebedev N. A. Functions of a Complex Variable- Constructive Theory. Cambridge, MA: MIT Press 1968.

[5] Zolotarevskii V. A.Finite- dimensional methods for solving singular integral equations on closed integration contours "Stiintsa", Kishinev, 1991. 136 pp. ISBN 5-376-01000-7. (In Russian)

[6] Krikunov Iu.,V. The general boundary Riemann problem and linear singular integro- differential equation, The scientific notes of the Kazani university, 1956, 116(4):pp.3-29, (in Russian)

[7] Cohen J.W. , Boxma O.J. Boundary Value Problems in Queueing System Analysis (North-Holland, Amsterdam, 1983 405 p., ISBN (USA) 0444865675; Russian translation Mir Publishers, Moscow, 1987,

[8] Kalandia A.I. Mathematical methods of two- dimensional elasticity Mir Publishers: 1975, 351 p.

[9] Linkov A.M. Boundary Integral Equations in Elasticity. Theory Kluwer Academic: Dordrecht; Boston, 2002: 268p.

[10] Muskhelishvili N. I. Some basic problems of the mathematical theory ofelasticity: fundamental equations, plane theory of elasticity, torsion, and bending. Groningen, P. Noordhoff; 1953; 704 p.

[11] Ladopoulos E. G. Singular integral equations: linear and non-linear theory and its applications in science and engineering, Springer: Berlin; New York, 2000; 551p.

[12] Ivanov V. V. The theory of approximate methods and their application to the numerical solution of singular integral equations Noordhoff International Publishing, 1976; 330 p. 3.

[13] Gakhov F. D. Boundary values problems. Edited by I. N. Sneddon. Published: Oxford, New York,

Pergamon Press; Reading, Mass., Addison-Wesley Pub. Co. 1966, 561 p.

[14] Muskhelishvili N. I. Singular integral equations: boundary problems of function theory and their application to mathematical physics /. Published: Leyden: Noordhoff International, c1977. Edition: Rev. translation from the Russian / edited by J. R. M. Radok. 447 p.: "Reprint of the 1958 edition."ISBN:9001607004 19.

[15] Vekua N. P. Systems of singular integral equations. Translated from the Russian by A. G. Gibbs and G. M. Simmons. Published: Groningen, P. Noordhoff [1967] Material: 216 p. 23 cm.

[16] Gohberg I. , Krupnik N. Introduction to the theory of one-dimensional singular integral operators. Stiintsa, Kishinev, 1973 (in Russian). German translation: Birkhauser Verlag, Basel, 1979.

[17] Prössdorf Siegfried, Some classes of singular equations Amsterdam; New York: North-Holland Pub. Co; New York: sole distributors for the USA and Canada, Elsevier North-Holland, 1978; 417 p.

[18] Prössdorf S., Michlin S. Singular integral operators. Published: Berlin ; New York : Springer-Verlag,528p.:ISBN:0387159673

[19] Caraus Iu. The numerical solution of singular integro- differential equations in Holder spaces. Conference on Scientific Computation., Geneva, 2002, p. 26- 27.

Acknowledgment: *This research of first author was supported by Central European University ("CEU") Special and Extension Programs. The opinions expressed herein are the author's own and do not necessarily express the views of CEU.*