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Jordan Representation of Perfect Reconstruction Filter Banks using Nilpotent Matrices

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Abstract: This paper contains the factorization of the polyphase matrix of finite impulse response perfect reconstruction filter banks into unimodular factors containing finite Jordan nilpotent structures and the associated transform matrices. An important contribution of the paper is the proposal of a systematic procedure for the construction of the transform matrices. The factorization is based on the M-channel lifting scheme which is essentially a prime factorization. It leads to simple implementation structures maintaining the computational complexity unchanged.

Keywords: paraunitary filter banks; prime factorization; Jordan form; nilpotent matrix; lifting technique.

1. Introduction

The complete and minimal factorization of biorthogonal polyphase matrices is important in designing filter banks and wavelets. This has been studied extensively in the literature [1, 2, 3]. In [4] Schuller and Swelden proposed a new design method for biorthogonal filter banks based on the factorization of the polyphase matrix of a perfect reconstruction filter bank (PRFB) into unimodular factors containing Jordan nilpotent forms with invertible transform matrices in between. This design is complete, meaning that, any perfect reconstruction filter bank can be represented in the nilpotent matrix formalism. Such a design leads to an all-FIR filter bank with unequal lengths for analysis and synthesis filters to adapt to perceptual limits; that is, long analysis filters for good frequency selectivity and short synthesis filters for noise reduction. A systematic procedure for the generation of the transform matrices along with the finite unimodular Jordan matrices has not been proposed so far. In this paper we present a scheme that makes use of the lifting factorization technique [5, 6] to arrive at the finite unimodular Jordan structure and the associated transform matrices. This is essentially a prime factorization.

The paper is organized as follows. Section 2 gives the polyphase representation of the M-channel filter bank and Section 3 carries a brief description of the basic theory involved. In Section 4, factorization is presented along with a brief review of the M-channel lifting factorization. The factorization is illustrated with an example in Section 5.

2. Polyphase Representation

Consider an M-band analysis/synthesis filter bank depicted in Fig.1. The input is an M- dimensional block vector given by,

$$\mathbf{x}(m) = [x(mM+M-1),\ldots,x(mM)]^{\mathbf{T}} \qquad (1)$$

where 'm' is the block index. If $\mathbf{X}(z)$, $\mathbf{Y}(z)$ and $\hat{\mathbf{X}}(z)$ are the Z-transforms of the input vector, subband signal vector and the reconstructed signal vector respectively, and $\mathbf{E}(z)$ and $\mathbf{R}(z)$ are the analysis and synthesis polyphase matrices respectively, then

$$\mathbf{Y}(z) = \mathbf{E}(z)\,\mathbf{X}(z) \tag{2}$$

for the analyzer and,

$$\hat{\mathbf{X}}(z) = \mathbf{R}(z)\,\mathbf{Y}(z) \tag{3}$$

for the synthesizer. The filter bank has the PR property if,

$$\mathbf{R}(z) = z^{-d}\mathbf{S}^{n_a}(z)\mathbf{E}^{-1}(z) \tag{4}$$

where $\mathbf{S}(z)$ is the shift matrix which circularly shifts the elements of a vector or matrix by one sample and d is the amount by which the output is delayed to allow for causal filters ,

$$\mathbf{S}(z) = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & 0.. & 0 \\ .. & .. & .. & .. & .. \\ .. & .. & .. & .. & 1 \\ z & 0 & 0 & .. & 0 \end{bmatrix} \tag{5}$$

n_a denotes the amount of shift. The range of n_a is limited to $0 \le n_a \le M$, for a M-band filter bank.

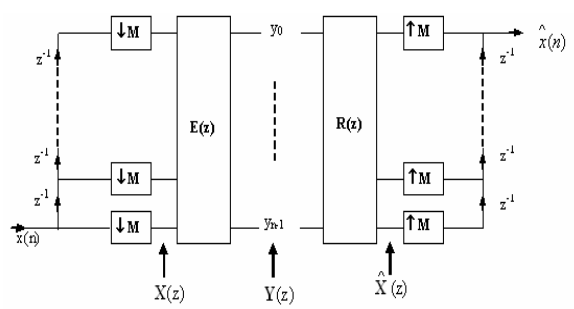

Fig. 1: Block schematic of the polyphase representation of M-Channel filter bank

3. Basic Theory

The basic building block for the design of the filter bank in the nilpotent matrix formalism is a unimodular matrix polynomial $[\mathbf{I} + \mathbf{A}(z)]$ with $\mathbf{A}(z)$ a nilpotent matrix polynomial [7] with order of nilpotency l. The synthesis polyphase matrix is $[\mathbf{I} + \mathbf{A}(z)]^{-1}$ which is also a unimodular polynomial matrix expressed using the Taylor series expansion as

$$[\mathbf{I}+\mathbf{A}(z)]^{-1} = \mathbf{I} + \sum_{i=1}^{l-1} (-\mathbf{A}(z))^{i} \tag{6}$$

Since $\mathbf{A}(z)$ is a nilpotent matrix polynomial with order of nilpotency l the expansion in equation (6) will contain l-1 terms only. Thus, if the analysis filters are causal FIR, then the synthesis filters are also causal FIR. The analysis and synthesis filters result with unequal lengths, for $l > 2$.

4. Factorization

In this section we develop a factorization of the polyphase matrix of a PR filter bank into Jordan factors of the form $[\mathbf{I} + z^{p}\mathbf{J}]$, $p\ \varepsilon\ \{1,-1\}$ and $\mathbf{J}$ the Jordan nilpotent matrix.

The factorization is based on the lifting scheme [5, 6]. In Section 4.1 we give a brief review of the lifting factorization along with an illustration of the method for generating the transform matrices based on this factorization.

4.1. Lifting Scheme

Daubechies and Swelden [5] proved that the polyphase matrix of a PR filter bank can be factored into a unit upper and lower triangular 2 x 2 matrices and a diagonal matrix. These triangular matrices are the lifting steps. Since the lifting factorization has unit diagonal scaling it is easy to invert the matrix. The 2-band lifting scheme has been extended to the M-channel case by Chen and Kevin [6]. As in the 2-band case Euclidean Algorithm is the basis for the M-channel lifting factorization. Repeated size reduction using the Monic Euclidean Algorithm and Gaussian elimination results in the M-channel lifting steps.

4.1.1. M-channel lifting Factorization

An M-channel lifting step from channel j to channel i ($j,i = 1,\ldots,M$) with multipliers $\alpha(z)$, is defined as [6],

$$\Gamma_{i,j}[\alpha(z)] = \mathbf{I} + \alpha(z)\,\mathbf{e}_i\mathbf{e}_j^{T} \tag{7}$$

where $\mathbf{I}$ is an M x M identity matrix and $\mathbf{e}_i$ is a Mx1 vector with 1 at the i^{th} position. Since $\mathbf{E}(z)$ is the polyphase

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matrix of a FIR filter bank, $\alpha(z)$ is FIR. $(\Gamma_{i,j}[\alpha(z)])$ is a unimodular triangular matrix. Its inverse is also a simple lifting step

$$(\mathbf{\Gamma}_{i,j}[\alpha(z)])^{-1} = [\mathbf{I} - \alpha(z)\,\mathbf{e}_i \mathbf{e}_j^{\,T}] = \mathbf{\Gamma}_{i,j}[-\alpha(z)] \qquad (8)$$

Let $\mathbf{E}(z)$ be the polyphase matrix of a PRFB with $det\,[\mathbf{E}(z)] = z^{-k}$; $k \in Z$. Then there exist two sets of FIR M-channel simple lifting steps, $\mathbf{B}(z)$ and $\mathbf{F}(z)$, and a diagonal matrix, $\boldsymbol{\lambda}(z)$, where

$$\boldsymbol{\lambda}(z) = \mathrm{diag}\,(1,1,\ldots,z^{-k}) \qquad (9.a)$$

or $$\boldsymbol{\lambda}(z) = \mathrm{diag}(z^{-k_1}, z^{-k_2}, \ldots, z^{-k_M}) \qquad (9.b)$$

with $k_1+k_2+...k_M = k$,

$$\mathbf{E}(z) = \mathbf{B}(z)\,\boldsymbol{\lambda}(z)\,\mathbf{F}(z) \qquad (9.c)$$

This is a variant of the Smith Normal form [7]. In equation (9.c),

$$\mathbf{B}(z) = \prod_{i=1}^{M-1} \mathbf{B}_i(z) \qquad (10)$$

where $\mathbf{B_i}(z)$ corresponds to the Gaussian elimination

$$\mathbf{F}(z) = \prod_{k=M-1}^{1} \mathbf{F}_k(z) \qquad (11)$$

where $\mathbf{F}_k(z)$ corresponds to the application of the Monic Euclidean Algorithm.

The matrices $\mathbf{B}(z)$ and $\mathbf{F}(z)$ are the cascades of $[\mathbf{I} + \alpha(z)\,\mathbf{e}_i\mathbf{e}_j^T]$. The matrix $\mathbf{e}_i\mathbf{e}_j^T$ is nilpotent with order of nilpotency 2. The Jordan form of $\mathbf{e}_i\mathbf{e}_j^T$ is given by $\mathbf{J}_\varphi$. That is,

$$\mathbf{e}_i\,\mathbf{e}_j^{\,T} = \mathbf{W}_i\,\mathbf{J}_\varphi\,\mathbf{W}_i^{-1} \qquad (12.\ a)$$

and the cascade is,

$$[\mathbf{I} + \alpha(z)\,\mathbf{e}_i\mathbf{e}_j^T] = \mathbf{W}_i\,[\mathbf{I} + \alpha(z)\,\mathbf{J}_\varphi]\,\mathbf{W}_i^{-1} \qquad (12.\ b)$$

The Jordan form of the matrix $\mathbf{e}_i\mathbf{e}_j^{\,T}$ is,

$$\mathbf{J}_\varphi = \begin{bmatrix} 0 & 1 & 0\ldots & 0 \\ 0 & 0 & 0\ldots & \vdots \\ 0 & \vdots & \ddots & 0 \\ 0 & 0 & 0\ldots & 0 \end{bmatrix} \qquad (13)$$

and $\mathbf{W}_i$ and $\mathbf{W}_i^{-1}$ are the transform matrices. The detailed steps for the reduction of the cascades of $[\mathbf{I} + \alpha(z)\,\mathbf{e}_i\,\mathbf{e}_j^T]$ into the finite Jordan nilpotent form along with the associated transform matrices are given in Appendix A. We express $\mathbf{B}(z)$ and $\mathbf{F}(z)$ as cascades of the basic building blocks containing the least order nilpotent matrices along with the transform matrices associated with the finite Jordan structure. The Jordan nilpotent form leads to a simple implementation structure. Even though this factorization is not of minimal delay it is a low delay one. This procedure is not based on degree reduction, but on size reduction. Since this factorization is based on lifting, any higher order matrix can also be decomposed into lower order ones. The transform matrices are essentially non-singular and can be expressed as in (14) [8], where $\mu \neq 0$, $\mathbf{A}'$ is a non singular matrix, and the permutation matrix $\mathbf{P}$ swaps row 1 and row i. The structure has the minimum number of parameters. A lifting factorization for $\mathbf{A}$ can be obtained by recursively applying this structure on $\mathbf{A}'$.

$$\mathbf{A} = \mathbf{HDLP}$$
$$= \underbrace{\begin{bmatrix} 1 & r_1 & \ldots & r_{M-1} \\ 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \end{bmatrix}}_{\mathbf{H}} \underbrace{\begin{bmatrix} \mu & 0 \\ 0 & \mathbf{A}' \end{bmatrix}}_{\mathbf{D}}$$
$$\underbrace{\begin{bmatrix} 1 & 0 & \ldots & 0 \\ l_1 & 1 & & \\ \vdots & & \ddots & \\ l_{M-1} & & & 1 \end{bmatrix}}_{\mathbf{L}} \underbrace{\begin{bmatrix} 0 & \ldots & 1 & 0 \\ 0 & 1 & \vdots & 0 \\ 1 & \cdots & 0 & \\ 0 & \ldots & & 1 \end{bmatrix}}_{\mathbf{P}} \qquad (14)$$

Since the factors have unit diagonal scaling the polyphase matrix corresponding to the FIR synthesizer is obtained by directly inverting each of the building blocks and the associated transform matrices. The proposed factorization is illustrated for an order one matrix in the next section. The same procedure can be extended to higher order polyphase matrices as well.

5. Example

Consider an order-one analysis polyphase matrix [9],

$$\mathbf{E}(z) = 1/9\begin{bmatrix} 8+z^{-1} & -2+2z^{-1} & -2+2z^{-1} \\ -2+2z^{-1} & 5+4z^{-1} & -4+4z^{-1} \\ -2+2z^{-1} & -4+4z^{-1} & 5+4z^{-1} \end{bmatrix}$$

of a 3-channel FIR PR filter bank with

$$det[\mathbf{E}(z)] = -81z^{-1}.$$

$\mathbf{E}(z)$ can be factorized by applying the proposed procedure as,

$$\mathbf{E}(z) = 1/9\begin{bmatrix} 0 & 0 & -1 \\ 0.44 & 0.0556 & 0.5 \\ -0.44 & 0.4444 & 0 \end{bmatrix} \left[\mathbf{I} + z\mathbf{J}_{\varphi}\right]$$

$$\begin{bmatrix} 0 & 0 & -2.25 \\ 0 & 1 & 0 \\ -1 & 1 & 0 \end{bmatrix} \lambda(z) \begin{bmatrix} -0.22 & -0.22 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\left[\mathbf{I} + z^{-1}\mathbf{J}_{\varphi}\right] \begin{bmatrix} 4.5 & 0 & 0 \\ -0.5 & -1 & -1 \\ 0.5 & 1 & 2 \end{bmatrix},$$

where $\lambda(z) = \begin{bmatrix} -9 & 0 & 0 \\ 0 & 9z^{-1} & 0 \\ 0 & 0 & 9 \end{bmatrix}.$

Figures (2) and (3) show the implementation structure of the polyphase matrix of the 3-channel filter bank given in the example.

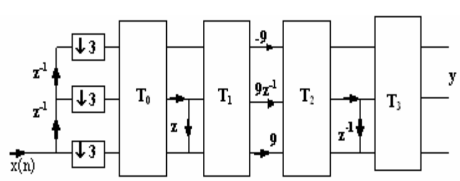

Fig. 2: The block schematic of the analysis filter bank for M = 3 and nilpotency $l = 2$

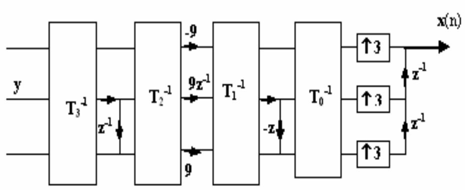

Fig. 3: The block schematic of the synthesis filter bank for M = 3 and nilpotency $l = 2$

The constant lifting matrices and the matrices associated with the Jordan forms are cascaded to form the transform matrices, $\mathbf{T}_i$ (for i = 0,… ,3). These transform matrices are sparse invertible matrices.

Remark: In this example, the transform matrices $\mathbf{T}_0$ and $\mathbf{T}_1$ correspond to $\mathbf{B}(z)$ whereas $\mathbf{T}_2$ and $\mathbf{T}_3$ correspond to $\mathbf{F}(z)$.

6. Conclusion

We have shown that the polyphase matrices of perfect reconstruction filter banks can be represented using unimodular matrices as the basic building blocks. These unimodular matrices contain Jordan nilpotent factors. This factorization of the polyphase matrix yields simple structures for the implementation of the perfect reconstruction filter banks and retains the computational complexity unchanged.

Appendix A

(a) Representation of B(z) in the finite Jordan form

The r^{th} cascade of $\mathbf{B}(z)$ is given by,

$$\mathbf{B}_r(z) = \prod_{l=r+1}^{M} \mathbf{\Gamma}_{l,r}[\alpha_l(z)] \tag{15}$$

where,

$$\mathbf{\Gamma}_{l,r}[\alpha_l(z)] = [\mathbf{I} + \alpha_l(z)\mathbf{e}_l\,\mathbf{e}_r^{T}] \tag{16}$$

$\alpha_l(z)$ are all FIR. Thus,

$$\mathbf{B}(z) = \prod_{i=1}^{M-1}\prod_{l=i+1}^{M} [\mathbf{I} + \alpha_l(z)\mathbf{e}_l\,\mathbf{e}_i^{T}] \tag{17}$$

Expanding equation (17),

$$\mathbf{B}(z) = \prod_{l=2}^{M}\left[\mathbf{I} + \alpha_l(z)\mathbf{e}_l\,\mathbf{e}_1^{T}\right]\prod_{l=3}^{M}\left[\mathbf{I} + \alpha_l(z)\mathbf{e}_l\,\mathbf{e}_2^{T}\right]$$

$$\ldots\left[\mathbf{I} + \alpha_M(z)\,\mathbf{e}_M\,\mathbf{e}_{M-1}^{T}\right] \tag{18}$$

Now replacing $\mathbf{e}_i\,\mathbf{e}_j^{T}$ in (18) by its Jordan form,

$$\prod_{l=2}^{M}\left[\mathbf{I}+\alpha_l(z)\mathbf{e}_l\,\mathbf{e}_1^{T}\right]=\mathbf{W}_1\left[\mathbf{I}+\alpha_2(z)\mathbf{J}_{\varphi}\right]\mathbf{W}_1^{-1}$$

$$\ldots\mathbf{W}_{M\text{-}1}\left[\mathbf{I}+\alpha_M(z)\mathbf{J}_{\varphi}\right]\mathbf{W}_{M\text{-}1}^{-1} \quad (19)$$

Setting $\mathbf{W}_1=\mathbf{R}_1$, $\mathbf{W}_{M\text{-}1}=\mathbf{R}_{1M-1}$, $\mathbf{W}_1^{-1}\mathbf{W}_2=\mathbf{R}_{11}$ or in general, $\mathbf{W}_{k\text{-}1}^{-1}\mathbf{W}_i=\mathbf{R}_{1k\text{-}1}$ in equation (19),

$$\prod_{l=2}^{M}\left[\mathbf{I}+\alpha_l(z)\mathbf{e}_l\,\mathbf{e}_1^{T}\right]=\mathbf{R}_1\prod_{k=2}^{M}\left[\mathbf{I}+\alpha_k(z)\mathbf{J}_{\varphi}\right]\mathbf{R}_{1k\text{-}1} \quad (20)$$

Thus we obtain $[\mathbf{I}+\alpha_k(z)\mathbf{J}_{\varphi}]$ as the basic building block along with the associated transform matrices $\mathbf{R}_{1k}$ for $k=1, 2,\ldots M\text{-}1$. Expressing each of the blocks in equation (18) as obtained in equation (20),

$$\mathbf{B}(z)=\mathbf{R}_1\prod_{k_1=2}^{M}\left[\mathbf{I}+\alpha_{k_1}(z)\mathbf{J}_{\varphi}\right]\mathbf{R}_{1k_{1\text{-}1}}$$

$$\mathbf{R}_2\prod_{k_2=2}^{M}\left[\mathbf{I}+\alpha_{k_2}(z)\mathbf{J}_{\varphi}\right]\mathbf{R}_{2k_{2\text{-}1}}\ldots$$

$$\mathbf{R}_{MM\text{-}1}\left[\mathbf{I}+\alpha_M(z)\mathbf{J}_{\varphi}\right]\mathbf{R}_{MM\text{-}1}^{-1} \quad (21)$$

with transform matrices in between the basic building blocks, at the beginning and at the end.

(b) Representation of F(z) in the fixed Jordan form

The r^{th} cascade of $\mathbf{F}(z)$ is,

$$\mathbf{F}_{\mathbf{r}}(z)=\left[\prod_{n=1}^{N}\mathbf{Q}_{\mathbf{n}}(z)\right]^{-1} \quad (22)$$

where N is the number of lifting steps.

The matrix $[\mathbf{Q}_n(z)]^{-1}$ is,

$$[\mathbf{Q}_n(z)]^{-1}=\prod_{j=1\ j\neq n}^{M}\mathbf{\Gamma}_{n,j}[q_j(z)] \quad (23)$$

$q_j(z)$ is the element of j^{th} row and n^{th} column of the matrix $\mathbf{Q}_n(z)$ [6].Thus $\mathbf{F}(z)$ can be written as

$$\mathbf{F}(z)=\prod_{k=M-1}^{1}\prod_{n=1}^{N}\prod_{j=1\ j\neq n}^{M}\mathbf{\Gamma}_{n,j}\left[q_j(z)\right] \quad (24)$$

where,

$$\mathbf{\Gamma}_{n,j}\left[q_j(z)\right]=\mathbf{I}+q_j(z)\mathbf{e}_n\,\mathbf{e}_j^{T} \quad (25)$$

Substituting (25) in equation (24) and reducing the matrix $\mathbf{e}_n\,\mathbf{e}_j^{T}$ into the fixed Jordan form as,

$$\mathbf{e}_n\,\mathbf{e}_j^{T}=\mathbf{T}_{n_j}\,\mathbf{J}_{\varphi}\,\mathbf{T}_{n_j}^{-1} \quad (26)$$

with $\mathbf{T}_{n_j}$ and $\mathbf{T}_{n_j}^{-1}$ as the transform matrices, we can express $\mathbf{F}(z)$ as

$$\mathbf{F}(z)=\prod_{k=M-1}^{1}\prod_{n=1\ n\neq j}^{N}\left[\mathbf{I}+q_1(z)\mathbf{e}_n\,\mathbf{e}_1^{T}\right]\ldots$$

$$\ldots\left[\mathbf{I}+q_M(z)\mathbf{e}_n\,\mathbf{e}_M^{T}\right] \quad (27)$$

$$=\prod_{k=M-1}^{1}\prod_{n=1\ n\neq j}^{N}\mathbf{T}_{n1}\left[\mathbf{I}+q_1(z)\,\mathbf{J}_{\varphi}\right]\mathbf{T}_{n1}^{-1}$$

$$\ldots\mathbf{T}_{nM}\left[\mathbf{I}+q_M(z)\,\mathbf{J}_{\varphi}\right]\mathbf{T}_{nM}^{-1} \quad (28)$$

Thus there are N x (M-1) numbers of cascades repeated $(M-1)$ times.

Equations (21) and (28) express the polyphase matrix $\mathbf{E}(z)$ in terms of factors containing a fixed Jordan matrix and the associated transform matrices.

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