Proceedings of the 5th WSEAS International Conference on Signal Processing, Istanbul, Turkey, May 27-29, 2006 (pp6-9)

Conditions for the Existence of Convolution Representations

IRWIN W. SANDBERG
Department of Electrical and Computer Engineering
The University of Texas at Austin
1 University Station C0803
Austin, Texas 78712 - 0240
USA

Abstract: - For an important family of linear shift-invariant multidimensional maps that take (measurable) bounded inputs into bounded outputs, necessary and sufficient conditions are given for the existence of convolution representations with integrable impulse responses.

Key-Words: - linear systems, convolutions, multidimensional systems, impulse responses, shift-invariant systems, bounded measurable inputs.

1 Introduction

In the signal-processing literature, $x(\alpha)$ typically denotes a function. In the following we distinguish between a function x and $x(\alpha)$, the latter meaning the value of x at the point (or time) α. Sometimes a function x is denoted by $x(\cdot)$, and also we use Hx to mean $H(x)$. This notation is often useful in studies of systems in which signals are transformed into other signals.

A recent paper [1] considers continuous-time linear time-invariant systems governed by a relation $y = Hx$ in which x is an input, y is the corresponding output, and H is the system map that takes inputs into outputs. It was assumed that inputs and outputs are complex-valued functions defined on the set $I\!R$ of real numbers. As is well known, it is a widely-held belief of long standing that the input-output properties of H are completely described by its impulse response. Using a standard interpretation of what is meant by a system's impulse response, it is shown in [1] that this belief is incorrect in a simple setting in which x is drawn from the linear space of bounded uniformly-continuous complex-valued functions defined on $I\!R$. More specifically, it was shown that there is an H of the kind described above, even a causal H, whose impulse response is the zero function, but which takes certain inputs into nonzero outputs.[1] It is clear that such H's do not possess convolution representations.

Here, in Section 2, we consider the most general family of linear shift-invariant maps that take (measurable) bounded inputs defined on $I\!R^d$ into bounded outputs defined on $I\!R^d$, where d is an arbitrary positive integer. We give, in Theorem 1 of Section 2, necessary and sufficient conditions for the existence of convolution representations with integrable impulse responses. In the Appendix a related result is described concerning cases in which inputs and outputs are defined on the half-line $[0, \infty)$.

[1]Another result along these lines is described in Section 2 (see Theorem 2). See also [2] for a different approach. For related results, concerning discrete-time systems, including a representation theorem for input-output maps, see [3] and the references cited there.

2 The Representation Theorem

2.1 Preliminaries

Throughout this section, d is an arbitrary positive integer and $L_\infty(I\!R^d)$ denotes the normed linear space of complex-valued bounded Lebesgue-measurable functions x defined on $I\!R^d$, with the norm given by

$$\|x\| = \sup_{\alpha \in I\!R^d} |x(\alpha)|. \tag{1}$$

The expression $L_1(I\!R^d)$ stands for the normed linear space of Lebesgue integrable complex-valued functions x defined on the set $I\!R^d$ of real d-vectors, with the usual norm given by

$$\|x\|_1 = \int_{I\!R^d} |x(\alpha)|\, d\alpha. \tag{2}$$

As usual, when $L_1(I\!R^d)$ is regarded as a metric space, the elements of $L_1(I\!R^d)$ are understood to be equivalence classes. By convergence in $L_1(I\!R^d)$, we mean convergence to an element of $L_1(I\!R^d)$ with respect to the norm in $L_1(I\!R^d)$. We use $\mathcal{H}$ to stand for the family of all linear shift-invariant maps H from $L_\infty(I\!R^d)$ into itself, such that the restriction of H to $BL_1(I\!R^d)$ is a continuous map into $BL_1(I\!R^d)$, in which $BL_1(I\!R^d)$ denotes the linear space of bounded $L_1(I\!R^d)$ functions, with the norm defined by (2).

The concept of an impulse response plays a role in the interpretation of our main result. As is well known, the d-dimensional extension of the concept of an impulse function as described by P. Dirac, while often useful in engineering and scientific applications, is unsatisfactory from the viewpoint of mathematics. It is unsatisfactory because according to the usual theory of integration,

$$\int_{I\!R^d} q(\alpha)\, d\alpha = 0$$

for any complex-valued function q defined on $I\!R^d$ with $q(\alpha) = 0$ for $\|\alpha\|_d > 0$, even if $q(0) = \infty$ is allowed.[2] It is also well known, at least for $d = 1$, that an alternative approach (see, for example, [4]) involves envisioning a sequence of progressively taller and narrower unit-integral functions centered at $\alpha = 0$.[3] In this spirit, but with no attempt to adhere strictly to the concept of a generalized function, we introduce the following definition.

The symbol $\mathcal{Q}$ denotes the family of $BL_1(I\!R^d)$-valued maps q defined on $(0, 1)$ such that, with $q(\epsilon)$ denoted by q_ϵ,

$$\int_{I\!R^d} q_\epsilon(\alpha)\, d\alpha = 1 \text{ for } \epsilon \in (0,1),$$

$$\sup_\epsilon \int_{I\!R^d} |q_\epsilon(\alpha)|\, d\alpha < \infty,$$

and

$$\lim_{\epsilon \to 0} \int_{\|\alpha\|_{(d)} > \xi} |q_\epsilon(\alpha)|\, d\alpha = 0, \;\; \xi > 0.$$

Note that q given by the familiar expression

$$\begin{aligned} q_\epsilon(\alpha) &= 1/\epsilon, \;\; |\alpha| \le \epsilon/2 \\ &= 0, \;\;\; \text{otherwise} \end{aligned}$$

is an element of $\mathcal{Q}$ for $d = 1$.

It is reasonable to say, roughly speaking, that an element H of $\mathcal{H}$ has an impulse response (or what might more accurately be called a "q-response limit") if for every $q \in \mathcal{Q}$ we have Hq_ϵ well defined for each $\epsilon \in (0, 1)$, with $\lim_{\epsilon \to 0} Hq_\epsilon$ existing in a meaningful sense and not dependent on q. Our main theorem, Theorem 1, is given in the following section. For the type of H addressed in the theorem, and under conditions (i)–(iii) of the theorem, H has an impulse response h in the precise sense that condition (ii) of the theorem holds [the limit in (ii) turns out to be independent of q].

Let η be a number in $[1, \infty)$. For each positive σ, let W_σ be the map from $L_\infty(I\!R^d)$ to $BL_1(I\!R^d)$ defined by $(W_\sigma x)(\alpha) = w_\sigma(\alpha)x(\alpha)$, in which $w_\sigma \in BL_1(I\!R^d)$ with $w_\sigma(\alpha) = 1$ for $\|\alpha\|_{(d)} \le \sigma$ and $|w_\sigma(\alpha)| \le \eta$, $\|\alpha\|_{(d)} > \sigma$. (e.g., w_σ could be taken to be equal to $2 - \sigma^{-1}\|\alpha\|_{(d)}$ for $\sigma < \|\alpha\|_{(d)} < 2\sigma$, and equal to 0 for $\|\alpha\|_{(d)} \ge 2\sigma$). Finally, given $a \in L_\infty(I\!R^d)$, we say that b_σ assumed equal almost everywhere to an element of $L_\infty(I\!R^d)$ for $\sigma \in (0, \infty)$ converges in $M_\infty(I\!R^d)$ to a as $\sigma \to \infty$ if we have

$$\int_A |b_\sigma(\beta) - a(\beta)|\, d\beta \to 0 \;\text{ as }\; \sigma \to \infty$$

for every bounded Lebesgue measurable subset A of $I\!R^d$, in which case we write $a = \lim_{\sigma \to \infty} b_\sigma$ (with the sense of convergence understood; it is easily checked that $M_\infty(I\!R^d)$ limits are essentially unique). We do not distinguish between $M_\infty(I\!R^d)$ limits that agree almost everywhere.

2.2 Criteria for the Existence of Convolution Representations

Our main result is the following.

Theorem 1 : Let H be a linear shift-invariant map of $L_\infty(I\!R^d)$ into itself. Then there is an $h \in L_1(I\!R^d)$ such that

$$(Hx)(\gamma) = \int_{I\!R^d} h(\gamma - \beta)x(\beta)\, d\beta \tag{3}$$

for almost all $\gamma \in I\!R^d$ and every $x \in L_\infty(I\!R^d)$ if and only if

(i) the restriction of H to $BL_1(I\!R^d)$ is a continuous map into $BL_1(I\!R^d)$.

(ii) For each $q \in \mathcal{Q}$, Hq_ϵ converges in $L_1(I\!R^d)$ as $\epsilon \to 0$.

(iii) $Hx = \lim_{\sigma \to \infty}(HW_\sigma x),\; x \in L_\infty(I\!R^d)$ in the sense of convergence in $M_\infty(I\!R^d)$.

Proof:

We use the following result given in [6] (see Theorem 1 there).

Lemma 1: Let H be an element of $\mathcal{H}$ with the property that

$$Hx = \lim_{\sigma \to \infty}(HW_\sigma x), \;\; x \in L_\infty(I\!R^d) \tag{4}$$

in the sense of convergence in $M_\infty(I\!R^d)$, and let q be an element of $\mathcal{Q}$. Then the following two statements are equivalent.

(a) Hq_ϵ converges in $L_1(I\!R^d)$.

(b) There is an element h of $L_1(I\!R^d)$ such that

$$(Hx)(\gamma) = \int_{I\!R^d} h(\gamma - \beta)x(\beta)\, d\beta \tag{5}$$

for almost all $\gamma \in I\!R^d$ and every $x \in L_\infty(I\!R^d)$.

Continuing with the proof of the theorem, suppose that (i)–(iii) are met. Then by (i) and (iii), H belongs to $\mathcal{H}$ with (4) satisfied. Thus, by Lemma 1 and (ii), we have (3). Conversely, suppose that (3) holds with $h \in L_1(I\!R^d)$. Condition (i) is satisfied because $h \in L_1(I\!R^d)$. We see also that (ii) is met, by the following lemma (which is a special case of a corresponding result in [7]).

Lemma 2: Let q and h belong to $\mathcal{Q}$ and $L_1(I\!R^d)$, respectively. Then

$$\int_{I\!R^d} q_\epsilon(\cdot - \beta)h(\beta)\, d\beta \tag{6}$$

is an element of $L_1(I\!R^d)$ for each ϵ, and it converges in $L_1(I\!R^d)$ to h as $\epsilon \to 0$.[4]

[2] More specifically, with $q(0) = \infty$ allowed, the integral is zero as a Lebesgue integral or as an improper Riemann integral. In the remainder of the paper, all integrals are meant to be interpreted as Lebesgue integrals.

[3] There is also a related theory of *distributions* [5] developed by L. Schwartz and S. Sobelov around 1948, in which the delta function is viewed as a linear functional on a certain type of space of infinitely differentiable functions of compact support. Distribution theory frees differential calculus from certain difficulties that arise because of the existence of nondifferentiable functions. No use is made of these ideas in this paper.

[4] Lemma 2 is related to results in [8, p. 149] and [9, p. 72], and [8] gives references concerning other related results.

Finally, let x be an element of $L_\infty(I\!R^d)$, and let A be a bounded Lebesgue measurable subset of $I\!R^d$. Consider

$$\int_A \Big| \int_{I\!R^d} h(\gamma-\beta) w_\sigma(\beta) x(\beta)\, d\beta - \int_{I\!R^d} h(\gamma-\beta)x(\beta)\, d\beta \Big|\, d\gamma,$$

which, aside from a factor of $\sup_\beta |x(\beta)|$, is bounded from above by

$$\int_A \int_{I\!R^d} |h(\gamma-\beta)| \cdot |w_\sigma(\beta) - 1|\, d\beta\, d\gamma$$

$$\leq (1+\eta) \int_A \int_{\|\beta\|_{(d)} > \sigma} |h(\gamma-\beta)|\, d\beta\, d\gamma. \qquad (7)$$

Using the assumptions that A is bounded and that $h \in L_1(I\!R^d)$, the latter implying that

$$\int_{\|\beta\|_{(d)} > \lambda} |h(\beta)|\, d\beta \to 0$$

as $\lambda \to \infty$, we see that the right side of (7) approaches zero as $\sigma \to \infty$. This shows that (iii) is met, and it completes the proof.

2.3 Comments

Condition (iii) is essential (i.e., is not redundant) – even though (ii) has the interpretation that for any $q \in \mathcal{Q}$, H has an impulse response in a very reasonable sense. This is a consequence of the following result in [6], which is along the lines of the theorem in [10].

Theorem 2: There is an $H \in \mathcal{H}$ such that

(a) Hq_ϵ is the zero function for all $\epsilon \in (0,1)$ and every $q \in \mathcal{Q}$.

(b) There are elements x of $L_\infty(I\!R^d)$ such that Hx is not the zero function.[5]

The proof of Theorem 1 in [6] makes clear that when (i)–(iii) hold, h in (3) is unique in $L_1(I\!R^d)$, and that the limit in (ii) is independent of q and equal to h. This, together with Lemma 2, leads immediately to the following variation of Theorem 1, in which attention is focused on h's that are well behaved in the sense that they are bounded.

Theorem $1'$**:** Let H be a linear shift-invariant map of $L_\infty(I\!R^d)$ into itself. Then there is an $h \in BL_1(I\!R^d)$ such that

$$(Hx)(\gamma) = \int_{I\!R^d} h(\gamma-\beta)x(\beta)\, d\beta \qquad (8)$$

for almost all $\gamma \in I\!R^d$ and every $x \in L_\infty(I\!R^d)$ if and only if

(i) the restriction of H to $BL_1(I\!R^d)$ is a continuous map into $BL_1(I\!R^d)$.

(ii) For each $q \in \mathcal{Q}$, Hq_ϵ converges in $BL_1(I\!R^d)$ as $\epsilon \to 0$.

(iii) $Hx = \lim_{\sigma \to \infty}(HW_\sigma x),\ x \in L_\infty(I\!R^d)$ in the sense of convergence in $M_\infty(I\!R^d)$.

Here (ii) can be replaced with: For some $q \in \mathcal{Q}$, Hq_ϵ converges in $BL_1(I\!R^d)$ as $\epsilon \to 0$. And "each" in (ii) of Theorem 1 can also be replaced with "some."

Similarly, there is a continuous h in $BL_1(I\!R^d)$ for which (8) holds for almost all $\gamma \in I\!R^d$ and every $x \in L_\infty(I\!R^d)$ if and only if (i) and (iii) of Theorem $1'$ are satisfied and there is a continuous ℓ in $BL_1(I\!R^d)$ for which Hq_ϵ converges in $BL_1(I\!R^d)$ to ℓ as $\epsilon \to 0$ for some $q \in \mathcal{Q}$. In the Appendix we describe a result related to Theorem 1 concerning cases in which inputs belong to the normed linear space $L_\infty(I\!R_+)$ of complex-valued bounded Lebesgue-measurable functions defined on the nonnegative numbers $I\!R_+$. For those cases, no weighting-operator condition corresponding to (iii) arises.

2.4 Conclusion

For the most general family of linear shift-invariant maps that take $L_\infty(I\!R^d)$ into itself, we have given, in Theorem 1 of Section 2, necessary and sufficient conditions for the existence of convolution representations with integrable impulse responses. In the Appendix a related result is described concerning cases in which inputs and outputs are defined on the half-line $[0,\infty)$.

3 Appendix

Here we describe a result related to Theorem 1 concerning cases in which inputs belong to the normed linear space $L_\infty(I\!R_+)$ of complex-valued bounded Lebesgue-measurable functions defined on the nonnegative numbers $I\!R_+$, with the norm given by (1) with $I\!R^d$ replaced with $I\!R_+$. We use $BL_1(I\!R_+)$ to denote the normed linear space of bounded elements of $L_1(I\!R_+)$, with the usual $L_1(I\!R_+)$ norm. The expression $\mathcal{Q}_+$ denotes the family of all $BL_1(I\!R_+)$-valued maps q defined on $(0,1)$ such that, with $q(\epsilon)$ denoted by q_ϵ,

$$\int_{I\!R_+} q_\epsilon(\tau)\, d\tau = 1 \ \text{ for } \ \epsilon \in (0,1),$$

$$\sup_\epsilon \int_{I\!R_+} |q_\epsilon(\tau)|\, d\tau < \infty,$$

and

$$\lim_{\epsilon \to 0} \int_{\tau > \xi} |q_\epsilon(\tau)|\, d\tau = 0,\ \ \xi > 0.$$

Note that q given by the familiar expression

$$\begin{aligned} q_\epsilon(\tau) &= 1/\epsilon, \ \ \tau \leq \epsilon \\ &= 0, \quad \text{otherwise} \end{aligned}$$

[5] In this theorem, "the zero function" means the essentially zero function. There are several variations of the theorem. For example, it is noted in [11] that, for $d = 1$, H can be taken to be causal.

is an element of $\mathcal{Q}_+$. Our result is the following.

Theorem 3 : Let H be a linear shift-invariant causal [6] map of $L_\infty(\mathbb{R}_+)$ into itself. Then there is an $h \in L_1(\mathbb{R}_+)$ such that

$$(Hx)(t) = \int_0^t h(t-\tau)x(\tau)\,d\tau$$

for almost all $t \in \mathbb{R}_+$ and every $x \in L_\infty(\mathbb{R}_+)$ if and only if

(i) the restriction of H to $BL_1(\mathbb{R}_+)$ is a continuous map into $BL_1(\mathbb{R}_+)$.

(ii) For some $q \in \mathcal{Q}_+$, Hq_ϵ converges in $L_1(\mathbb{R}_+)$ as $\epsilon \to 0$.

We omit our proof because it is a simple modification of the proof of Theorem 1, using the following lemma instead of Lemma 1.

Lemma 3 : Let H be a linear shift-invariant causal map from $L_\infty(\mathbb{R}_+)$ into itself, such that the restriction of H to $BL_1(\mathbb{R}_+)$ is a continuous map into $BL_1(\mathbb{R}_+)$, and let $q \in \mathcal{Q}_+$. Then the following two statements are equivalent.

(a) Hq_ϵ converges in $L_1(\mathbb{R}_+)$.

(b) There is an element h of $L_1(\mathbb{R}_+)$ such that

$$(Hx)(t) = \int_0^t h(t-\tau)x(\tau)\,d\tau, \quad t \in \mathbb{R}_+ \ \text{(a.e.)}$$

for every $x \in L_\infty(\mathbb{R}_+)$.

Lemma 3 is a part of Theorem 2 of [12].[7]

3.1 Related Results

There is a related body of results for the less complex cases of linear shift-invariant continuous systems for which outputs are bounded and inputs are drawn from $L_p(\mathbb{R}_+)$ or $L_p(\mathbb{R}^d)$ where $1 \le p < \infty$. In such cases, impulse responses always exist as certain limits, these impulse responses belong to the space $L_m(\mathbb{R}_+)$ or $L_m(\mathbb{R}^d)$, respectively, in which m is the conjugate index of p, and convolution input-output representations always hold. See [7] for the details.

For related material concerning the case in which outputs are bounded and inputs belong to the space $C_0(\mathbb{R}^d)$ of continuous complex-valued functions defined on $\mathbb{R}^d$ that vanish at infinity, see [13]. In [13] one is led to a general system representation that is a uniform limit of a convolution, and a necessary and sufficient condition is given under which the limit reduces to a convolution.

[6]Shift-invariance and causality are defined in the usual way.[12]

[7]We take this opportunity to correct two minor oversights: In Section 2.1 of [12], the phrase "with the norm given by (1)" should be replaced with "with the norm given by (1) with $\mathbb{R}^d$ replaced with $\mathbb{R}_+$." And "η" just above (14) in [6] should be replaced with "2η".

As is well known, there is interest also in the frequency-domain representation of linear systems. In [14] it is shown that not all continuous linear shift-invariant systems are characterized by their frequency responses (even when they exist), but that the members of a certain important large family of linear systems are completely characterized by their suitably-defined frequency responses.

References:

[1] I. W. Sandberg, "Causality and the Impulse Response Scandal," *IEEE Transactions on Circuits and Systems I*, Vol. 50, No. 6, pp. 810–811, 2003.

[2] M. Ciampa, M. Franciosa, and M. Poletti, "A Note on Impulse Response for Continuous, Linear, Time-Invariant, Continuous-Time Systems," *IEEE Transactions on Circuits and Systems I*, Vol. 53, No. 1, pp. 106–113, 2006.

[3] I. W. Sandberg, "Notes on Representation Theorems for Linear Discrete-Space Systems," *Proceedings of the International Symposium on Circuits and Systems*, Orlando, Florida, May 30–June 2, 1999 (four pages on CD).

[4] M. J. Lighthill, *Introduction to Fourier Analysis and Generalized Functions*, Cambridge: Cambridge University Press, 1960.

[5] A. H. Zemanian, *Distribution Theory and Transform Analysis*, Mineola, NY: Dover, 1987. Originally published: New York, McGraw-Hill, 1965 (International Series in Pure and Applied Mathematics).

[6] I. W. Sandberg, "Bounded Inputs and the Representation of Linear System Maps," *Circuits, Systems, and Signal Processing*, Vol. 24, No. 1, pp. 103–115, 2005.

[7] I. W. Sandberg, "Notes on Linear Systems and Impulse Responses," *Circuits, Systems and Signal Processing*, Vol. 23, No. 5, pp. 339–350, 2004.

[8] W. Cheney and W. Light, *A Course in approximation Theory*, Pacific Grove: Brooks/Cole, 2000.

[9] B. E. Petersen, *Introduction to the Fourier Transform and Pseudo-Differential Operators*, Pitman Publishing Inc., Marshfield, Massachusetts: 1983.

[10] I. W. Sandberg, "Continuous Multidimensional Systems and the Impulse Response Scandal," *Multidimensional Systems and Signal Processing*, Vol. 15, No. 3, pp. 295–299, 2004.

[11] ________, "On Causality and the Impulse Response Scandal," *WSEAS Transactions on Circuits and Systems*, Vol. 3, pp. 1741–1744, November 2004.

[12] ________, "On the Representation of Linear System Half-Line Maps," *International Journal of Circuit Theory and Applications*, Vol. 33, No 4, pp. 253–262, 2005.

[13] ________, "Notes on Multidimensional Linear System Representations," *38th Annual Conference on Information Sciences and Systems*, Department of Electrical Engineering, Princeton University, Princeton, NJ, pp. 801–806, March 2004.

[14] ________, "Notes on Linear Systems and Frequency Responses," *International Journal of Circuit Theory and Applications*, Vol. 33, No. 3, pp. 175–181, 2005.