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On the addition of discrete fuzzy numbers

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Abstract: - The main goal of this paper is the representation of a family of solutions for the addition of fuzzy discrete numbers. The inclusion in this family of the method recently stablished by Wang[4] is proved.

Key-Words: - Fuzzy numbers, discrete fuzzy numbers, Fuzzy arithmetic, Fuzzy sets

1 Introduction

The concept of fuzzy number appears in Zadeh[1] and Chang[1] to study the properties of probability functions. Several models of membership function have been used. In fact, the concept of fuzzy number tackles the problem of what the knowledge of a number is. A fuzzy number can be characterized by the verbal expression: "about the number n" and its membership function determines an interval where, although the number n can be natural or integer, any real number can be taken as the value of the knowledge and another interval where each real number satisfies the specifications with a degree less than 1. So, a *convex* membership such as a trapezoidal or a triangular shape is taken often as a suitable definition.

Nevertheless, the knowledge about a real number can be such that it implies that the number is natural or its support is a discrete subset of R. In this case the expression "about the number n" determines a membership function on the discrete support with an interval of "normality".

For these fuzzy discrete numbers, the addition defined trough the extension principle or by operating the elements of the levels or r-cuts on the support sometimes is not a fuzzy discrete number,i.e. the convexity is not satisfied[2,4].

To avoid this drawback, Wang et Al. [3,4]define arithmetic operations on fuzzy discrete numbers by means of a slight transformation of the extension principle. In other direction, Qui([5]) establishes algebraic operations, a norm and a metric from it.

In this paper, we show that the Wang's method can be included in a more general family, where we have used as relevant property the following one: when we consider each fuzzy discrete number as the value over the elements of its support of any fuzzy number such that its support is an enlargement of the discrete one, the addition yields a convex discrete fuzzy set.

2 Preliminars.

2.1 Definition

A fuzzy subset of the real line $\Re$ with membership function $\tilde{u}:\Re\rightarrow[0,1]$ is a *fuzzy number* if its support is an interval $[a,b]$ and there exist real numbers s, t with $a\leq s\leq t\leq b$ fulfilling:

1. $\tilde{u}(x)=1$ for $s\leq x\leq t$
2. $\tilde{u}(x)\leq\tilde{u}(y)$ for $a\leq x\leq y\leq s$
3. $\tilde{u}(x)\geq\tilde{u}(y)$ for $t\leq x\leq y\leq b$
4. $\tilde{u}(x)$ is upper semicontinuous

We will denote the set of fuzzy numbers by FN

2.2 Definition

A fuzzy subset of the real line $\Re$ with membership function $u:\Re\rightarrow[0,1]$ is a *fuzzy discrete number* if its support is finite, i.e. there exist $x_1,...,x_n\in\Re$ with $x_1<x_2<...<x_n$, such that $\sup u=\{x_1,...,x_n\}$ and there exist natural numbers s, t with $1\leq s\leq t\leq n$ fulfilling:

1. u(x_i)=1 for any natural number and with $s \le i \le t$
2. $u(x_i) \le u(x_j)$ for each natural number i, j with $1 \le i \le j \le s$
3. $u(x_i) \ge u(x_j)$ for each natural number i, j with $t \le i \le j \le n$

We will denote the set of fuzzy discrete numbers by FDN

2.3 Remark

- A fuzzy discrete number is not a fuzzy number but it satisfies the conditions 1. 2. 3. of a fuzzy number [2.1] over its support.
- The r-cut or r-level of a fuzzy number is a closed interval of $\Re$, [8]

2.4 Addition of fuzzy numbers

Two common ways of defining addition operation are either making use of the Zadeh's extension principle [8] or employing the r-cut representation of fuzzy numbers [8]. In the first one, given $\tilde{u},\tilde{v} \in FN$, the fuzzy number $\tilde{u} \oplus \tilde{v}$ is defined for all z∈ R by

$$(\tilde{u} \oplus \tilde{v})\ (z) = \sup_{z=x+y} \min(\tilde{u}(x), \tilde{v}(y))$$

In the second case, we define, $\tilde{u} \oplus \tilde{v}$, by defining its r-cut, $[\tilde{u} \oplus \tilde{v}]^r = [\tilde{u}]^r + [\tilde{v}]^r$ for any $r \in (0,1]$. Then [8] $\tilde{u} \oplus \tilde{v}$ can be expressed, as $\sup(\tilde{u} \oplus \tilde{v})$ = $\bigcup_{r\in[0,1]} [\tilde{u} \oplus \tilde{v}]^r$.

2.5 Addition of fuzzy discrete numbers

Let $u,v \in FDN$. As the following example shows, it is possible that:

$$u \oplus v \notin FDN$$

Let $u = \{0.3/1, 1/2, 0.5/3\}$ and $v = \{0.4/4, 1/6, 0.8/8\}$ be fuzzy discrete numbers. Then

$u \oplus v = \{0.3/5, 0.4/6, 1/8, 0,5/9, 0.8/10, 0.5/11\} \notin FDN$,

because the convexity is not true for 0.8/10.

2.6 Wang's method for the addition of fuzzy discrete numbers

Using the r-level sets of discrete fuzzy numbers, Wang[4] obtains a kind of representation of discrete fuzzy numbers. From this representation, for any u,v∈ FDN, he defines a unique fuzzy discrete number, $u \underset{W}{\oplus} v$, such that it has for r-level

$[u \underset{W}{\oplus} v]^r =$

$\{x \in \sup(u) + \sup(v) : min([u]^r + [v]^r) \le x \le max([u]^r + [v]^r)\}$

where $min([u]^r + [v]^r) = min\{x : x \in [u]^r + [v]^r\}$ and $max([u]^r + [v]^r) = max\{x : x \in [u]^r + [v]^r\}$ for any r∈ [0,1] .

Thus, Wang[4] defines a closed operation (addition) in the set of fuzzy discrete numbers, FDN. Furthermore, he proves that for any couple u,v∈ FDN if u⊕v∈ FDN then u⊕v=$u \underset{W}{\oplus} v$.

2.7 Wang's method exemple

Let $u = \{0.4/15, 1/19, 1/25, 0.5/30\}$ and $v = \{0.2/1, 1/2, 1/3, 0.3/4\}$ be fuzzy discrete numbers. Then

$$u \underset{W}{\oplus} v = \{0.2/16, 0.4/17, 0.4/18, 0.4/19, 0.4/20, 1/21, 1/22, 1/23, 1/26, 1/27, 1/28, 0.5/29, 0.5/31, 0.5/32, 0.5/33, 0.3/34\}$$

since,

$[u \underset{W}{\oplus} v]^1 = \{21, 22, 23, 26, 27, 28\}$

$[u \underset{W}{\oplus} v]^{0.5} = \{21, 22, 23, 26, 27, 28, 29, 31, 32, 33\}$

$[u \underset{W}{\oplus} v]^{0.4} = \{17,18,19,20,21,22,23,26,27,28,29,31,32,33\}$

$[u \underset{W}{\oplus} v]^{0.3} = \{17,18,19,20,21,22,23,26,27,28,29,31,32,33,34\}$

$[u \underset{W}{\oplus} v]^{0.2} = \{16,17,18,19,20,21,22,23,26,27,28,29,31,32,33,34\}$

3 Associaton of fuzzy numbers to a fuzzy discrete number

3.1 Addition of fuzzy numbers

For each $u \in FDN$ with support, $sup(u) = \{x_1, ...x_s, ..., x_t, ..., x_n\}$ and $u(x_i) = 1$ for every natural number i such that $s \le i \le t$, we consider the set of fuzzy numbers $FN(u)$ in which if $\tilde{u} \in FN(u)$ then $\tilde{u}$ fulfils the following properties:

1. $sup(\tilde{u}) = [x_1, x_n]$
2. If $x_i \in sup(u)$ then $\tilde{u}(x_i) = u(x_i)$ for every $i = 1, ..., n$
3. For every $x \in [x_i, x_{i+1}]$ with $1 \le i \le i+1 \le s$ the relations $u(x_i) \le \tilde{u}(x) \le u(x_{i+1})$ hold.

4. If $x \in [x_i, x_{i+1}]$ with $t \le i \le i+1 \le n$ the relations $u(x_i) \ge \tilde{u}(x) \ge u(x_{i+1})$ hold.

3.2 Examples

- For each $u \in FDN$ with support $\sup(u) = \{x_1, ..., x_s, ..., x_t, ..., x_n\}$ with $x_1 < ... < x_s < ... < x_t, ... < x_n$, and $u(x_p) = 1$ for every natural number p, $s \le p \le t$. Let $\tilde{u}_l$ be the fuzzy number defined as follows:

$$\tilde{u}_l(x) = \begin{cases} \frac{u(x_{i+1}) - u(x_i)}{x_{i+1} - x_i}(x - x_{i+1}) + u(x_{i+1}) & \text{if} \quad x \in [x_i, x_{i+1}], \text{with } x_{i+1} \le x_s \\ 1 & \text{if} \quad x \in [x_s, x_t] \\ \frac{u(x_i) - u(x_{i+1})}{x_i - x_{i+1}}(x - x_i) + u(x_i) & \text{if} \quad x \in [x_i, x_{i+1}], \text{with } x_i \ge x_t \end{cases}$$

It is straightforward to prove that $\tilde{u}_l \in FD(u)$.

In particular, if u is the fuzzy discrete number defined by: $u = \{0.2/1, 1/2, 1/3, 0.3/4\}$

$$\tilde{u}_l(x) = \begin{cases} \frac{8x - 6}{10} & if \quad x \in [1,2] \\ 1 & if \quad x \in [2,3] \\ \frac{31 - 7x}{10} & if \quad x \in [3,4] \end{cases}$$

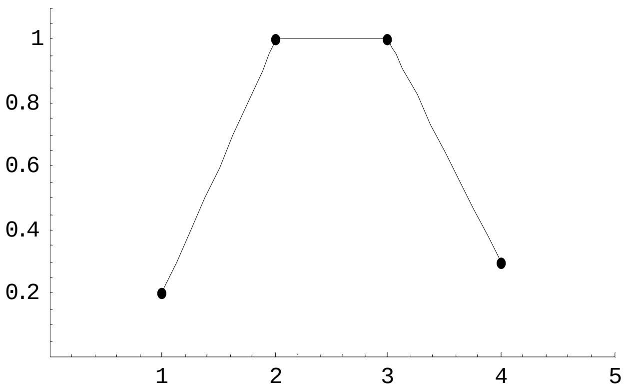

fig.1 fuzzy number $\tilde{u}_l$

- For each $u \in FDN$ with support $\sup(u) = \{x_1, ..., x_s, ..., x_t, ..., x_n\}$ with $x_1 < ... < x_s < ... < x_t, ... < x_n$, and $u(x_p) = 1$ for every natural number p, $s \le p \le t$. Let $\tilde{u}_\alpha$ be the fuzzy number defined for each i=1,...,n-1

$$\tilde{u}_\alpha(x) = \begin{cases} u(x_i) & if \quad x \in [x_i, x_{i+1}), \text{amb } x_{i+1} \le x_s \\ 1 & if \quad x \in [x_s, x_t] \\ u(x_{i+1}) & if \quad x \in (x_i, x_{i+1}], \text{amb } x_i \ge x_t \end{cases}$$

It is easy to prove that $\tilde{u}_\alpha \in FN(u)$.

In particular, if u is the fuzzy discrete number defined by $u = \{0.2/1, 1/2, 1/3, 0.3/4\}$

$$\tilde{u}_\alpha(x) = \begin{cases} 0.2 & if \quad x \in [1,2) \\ 1 & if \quad x \in [2,3] \\ 0.3 & if \quad x \in (3,4] \end{cases}$$

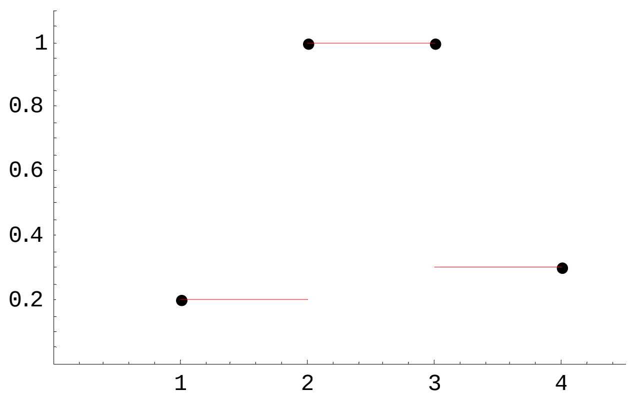

fig.2 Fuzzy number $\tilde{u}_\alpha$

4 Towards a closed addition of fuzzy discrets numbers

4.1 Definition:

Let u be a fuzzy discrete number with support, $\sup(u) = \{x_1, ..., x_s, ..., x_t, ..., x_n\}$ and $u(x_p) = 1$ for every natural number p, $s \le p \le t$. Let v be another fuzzy discrete number with support $\sup(v) = \{y_1, ..., y_m, ..., y_k, ..., y_r\}$ and $v(y_p) = 1$ for every natural number p, $m \le p \le k$.

Let $A: FDN \to FN$ be a function such that $A(u) \in FN(u)\ \forall u \in FDN$. We consider the fuzzy set, which will be called $u \underset{A}{\oplus} v$, defined as follows :

- $\sup(u \underset{A}{\oplus} v) = \left\{ \begin{matrix} z \in R \text{ such that } z = x + y, \\ x \in \sup(u), y \in \sup(v) \end{matrix} \right\}$
- $(u \underset{A}{\oplus} v)(z) = (A(u) \oplus A(v))(z)\ \forall\ z \in \sup(u \underset{A}{\oplus} v)$

4.2 Remark

- It is straightforward to prove that $u \underset{A}{\oplus} v \in FDN$.
- If $A(u)=\tilde{u}_l$ and $A(v)=\tilde{v}_l$ defined in [3.2] then $u \underset{A}{\oplus} v$ will be called $u \underset{l}{\oplus} v$
- If $A(u)=\tilde{u}_\alpha$ and $A(v)=\tilde{v}_\alpha$ defined in [3.2] then $u \underset{A}{\oplus} v$ will be called $u \underset{\alpha}{\oplus} v$

4.3 Examples:

- Given the fuzzy discrete numbers $u=\{0.4/15, 1/19, 1/25, 0.5/30\}$ and $v=\{0.2/1, 1/2, 1/3, 0.3/4\}$. Let's consider the association of the fuzzy numbers $A(u)=\tilde{u}_l \in FN(u)$, $A(v)=\tilde{v}_l \in FN(v)$ defined in [3.2]. Namely:

$$\tilde{u}_l = \begin{cases} \dfrac{3x-37}{20} & \text{if} \quad x \in [15,19] \\ 1 & \text{if} \quad x \in [19,25] \\ \dfrac{35-x}{10} & \text{if} \quad x \in [25,30] \end{cases}$$

$$\tilde{v}_l = \begin{cases} \dfrac{8x-6}{10} & if \quad x \in [1,2] \\ 1 & if \quad x \in [2,3] \\ \dfrac{31-7x}{10} & if \quad x \in [3,4] \end{cases}$$

The addition described in [2.4] yields the fuzzy number:

$$(\tilde{u}_l \oplus \tilde{v}_l)(x) = \begin{cases} \dfrac{8x-126}{10} & if \quad x \in [16, 16.25] \\ \dfrac{24x-314}{190} & if \quad x \in [16.25, 21] \\ 1 & if \quad x \in [21, 28] \\ \dfrac{276-7x}{80} & if \quad x \in [28, \frac{236}{7}] \\ \dfrac{241-7x}{10} & if \quad x \in [\frac{236}{7}, 34] \end{cases}$$

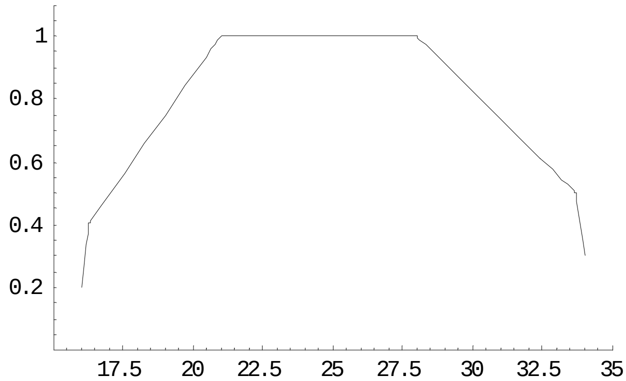

fig.3 Fuzzy number $(\tilde{u}_l \oplus \tilde{v}_l)$

And by means the discretization defined in [4.1]

$u \oplus_l v = \{0.2/16,\ \frac{45}{95}/17,\ \frac{59}{95}/18,\ \frac{71}{95}/19,\ \frac{83}{95}/20,\ 1/21,\ 1/22,\ 1/23,\ 1/26,\ 1/27,\ 1/28, \frac{73}{80}/29,\ \frac{59}{80}/31,\ \frac{13}{20}/32,\ \frac{9}{16}/33,\ 0.3/34\}$

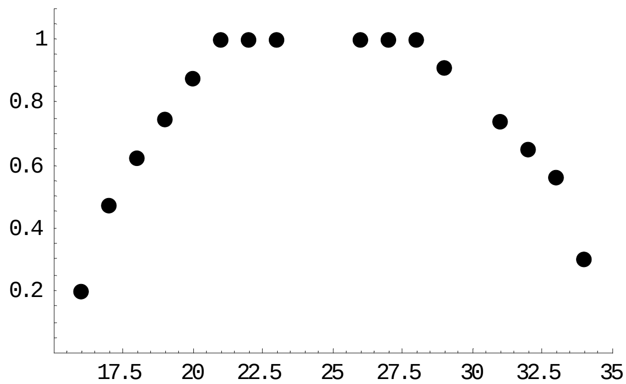

fig.4 Fuzzy discrete number $u \underset{l}{\oplus} v$

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4.4 Theorem:

Let $u, v \in FDN$ be two fuzzy discrete numbers with support, $\sup(u) = \{x_1, ..., x_s, ..., x_t, ..., x_n\}$ and $\sup(v) = \{y_1, ..., y_m, ..., y_k, ..., y_r\}$ .Then $u \underset{W}{\oplus} v$ defined in [2.6]and $u \underset{\alpha}{\oplus} v$ defined in [4.2] are the same fuzzy discrete number.

Proof:

We will prove that, for all $r \in R$

$$\left[u \underset{W}{\oplus} v\right]^r = \left[u \underset{\alpha}{\oplus} v\right]^r$$

We know , for all $r \in R$

a) $[\tilde{u}_\alpha \oplus \tilde{v}_\alpha]^r = [\tilde{u}_\alpha]^r + [\tilde{v}_\alpha]^r$ (see [2.4])

b) $\min(\max)[\tilde{u}_\alpha]^r = \min(\max)[u]^r$

Because, for each $r \in R$ we have

i. $x_s \in [\tilde{u}_\alpha]^r \cap [x_1, x_s] \neq \varnothing$

ii. $x_t \in [\tilde{u}_\alpha]^r \cap [x_t, x_n] \neq \varnothing$

Therefore:

$$\min[\tilde{u}_\alpha]^r = \min\left([\tilde{u}_\alpha]^r \cap [x_1, x_s]\right) = \min\left([u]^r \cap [x_1, x_s]\right) = \min[u]^r$$

$$\max[\tilde{u}_\alpha]^r = \max\left([\tilde{u}_\alpha]^r \cap [x_t, x_n]\right) = \max\left([u]^r \cap [x_t, x_n]\right) = \max[u]^r$$

c) $\left[u \underset{\alpha}{\oplus} v\right]^r \subset [\tilde{u}_\alpha \oplus \tilde{v}_\alpha]^r$

Because, $[\tilde{u}_\alpha \oplus \tilde{v}_\alpha](x) = \left(u \underset{\alpha}{\oplus} v\right)(x)$ if $x \in \sup(u \underset{\alpha}{\oplus} v) \subset \sup\left(\tilde{u}_\alpha \oplus \tilde{v}_\alpha\right)$

d) $$x \in \left[u \underset{W}{\oplus} v\right]^r \Leftrightarrow \begin{cases} x \in \sup\left(u \underset{W}{\oplus} v\right) = \sup(u) + \sup(v) \\ \qquad \text{See } [2.6] \\ \min([u]^r + [v]^r) \leq x \leq \max([u]^r + [v]^r) \end{cases}$$

but, for all $r \in R$:

$$min([u]^r + [v]^r) \leq x \leq max([u]^r + [v]^r) \Leftrightarrow$$

$$\Leftrightarrow min[u]^r + \min[v]^r \leq x \leq max[u]^r + max[v]^r \Leftrightarrow$$

(since b))

$$\overset{\text{since b)}}{\Leftrightarrow} min[\tilde{u}_\alpha]^r + \min[\tilde{v}_\alpha]^r \leq x \leq max[\tilde{u}_\alpha]^r + max[\tilde{v}_\alpha]^r \Leftrightarrow$$

$$\Leftrightarrow min([\tilde{u}_\alpha]^r + [\tilde{v}_\alpha]^r) \leq x \leq max([\tilde{u}_\alpha]^r + [\tilde{v}_\alpha]^r) \Leftrightarrow$$

$$\overset{\text{since c)}}{\Leftrightarrow} min([\tilde{u}_\alpha \oplus \tilde{v}_\alpha]^r) \leq x \leq max([\tilde{u}_\alpha \oplus \tilde{v}_\alpha]^r) \Leftrightarrow$$

$\Leftrightarrow x \in [\tilde{u}_\alpha \oplus \tilde{v}_\alpha]^r$,because $[\tilde{u}_\alpha \oplus \tilde{v}_\alpha]^r$ is an interval.

And

$$\left.\begin{array}{l} x \in \sup\left(u \underset{W}{\oplus} v\right) = \sup(u) + \sup(v) \\ x \in \left[\tilde{u}_\alpha \oplus \tilde{v}_\alpha\right]^r \end{array}\right\} \Leftrightarrow x \in \left[u \underset{\alpha}{\oplus} v\right]^r$$

4.5 Examples

- If $u = \{0.4/15, 1/19, 1/25, 0.5/30\}$ and $v = \{0.2/1, 1/2, 1/3, 0.3/4\}$. Then, the fuzzy numbers $\tilde{u}_\alpha(x)$ and $\tilde{v}_\alpha(x)$ are defined as follow

$$\tilde{u}_\alpha(x) = \begin{cases} 0.4 & if \quad x \in [15, 19) \\ 1 & if \quad x \in [19, 25] \\ 0.5 & if \quad x \in (25, 30] \end{cases}$$

$$\tilde{v}_\alpha(x) = \begin{cases} 0.2 & if \quad x \in [1, 2) \\ 1 & if \quad x \in [2, 3] \\ 0.3 & if \quad x \in (3, 4] \end{cases}$$

Thus we obtain the addition:

$$\tilde{u}_\alpha(x) \oplus \tilde{v}_\alpha(x) = \begin{cases} 0.2 & \text{if} \quad x \in [16,17) \\ 0.4 & \text{if} \quad x \in [17,21) \\ 1 & \text{if} \quad x \in [21,28] \\ 0.5 & \text{if} \quad x \in (28,33] \\ 0.3 & \text{if} \quad x \in (33,34] \end{cases}$$

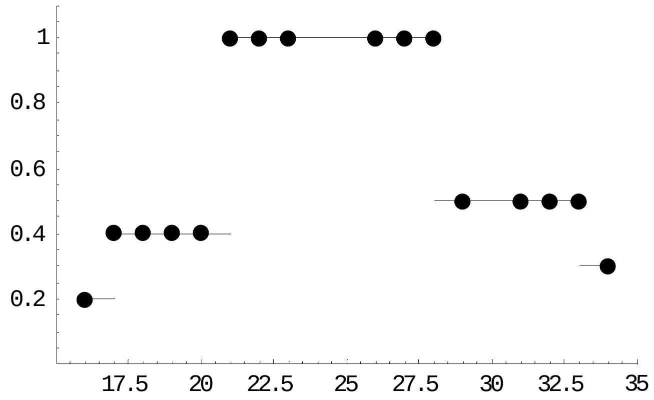

fig.5 Fuzzy number $\tilde{u}_\alpha \oplus \tilde{v}_\alpha$

And the discretization defined in [4.1] produces the fuzzy discrete number:

$$u \oplus_\alpha v = \{0.2/16, 0.4/17, 0.4/18, 0.4/19, 0.4/20, 1/21, 1/22, 1/23, 1/26, 1/27, 1/28, 0.5/29, 0.5/31, 0.5/32, 0.5/33, 0.3/34\}$$

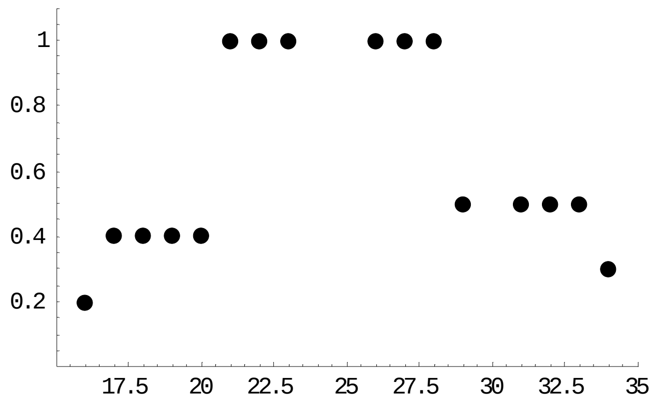

fig.6 Fuzzy discrete number $u \underset{\alpha}{\oplus} v$

4.5 Remark

We can see that the fuzzy discrete number obtained in [2.7] by Wang's method is the same number that we obtain with our method.

5 Conclusion

The consideration of a fuzzy discrete number as a "discretization" of a fuzzy number allows us to use this fuzzy number as a carrier for the addition.

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