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Mutual Inductance of Thin Coaxial Circular Coils with Constant Current Density in Air (Filament Method)

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CANADA

Abstract: - This paper deals with an efficient and fast approach for determining the mutual inductance between thin current loops (filamentary coils, thin cylindrical coils, thin disk coils and all possible combinations). This approach is based on the filament method where conductors are approximated by a set of Maxwell's coils. The resulting equations are expressed in term of the complete elliptical integrals of the first and second kind and permit fast calculation of the mutual inductance for systems of interests. These new formulas are accurate and simple for a wide range of applications. Computational cost and accuracy of this technique will be also discussed.

Key-Words: - Computational electromagnetics, mutual inductance, filament circular coils, disk coils, thin wall solenoids.

1 Introduction

Circular coils are widely used in various electromagnetic applications such as magnetically coupled coils at radio frequencies (RF), in modern medicine and telemetric system applied in biomedical engineering, naval and spacecraft magnetics, superconducting magnetic storage (SMES) problems, magnetic resonance applications, coil guns, tubular linear motors, current reactors, transmission lines and VLSI systems. The mutual inductance as a fundamental electrical parameter for coil can be calculated by applying the Biot-Savart law directly or using other alternate methods [1-20]. Before the advent of digital computers, calculations relied almost entirely on various power series [3-5]. Exact methods based on elliptic integral solutions for current loops, thin current cylinders, thin disks, massive coils have existed since at least the time of Maxwell but were laborious without computers. Today FEM and BEM methods are routinely used for electromagnetic problems, but these methods have problems of accuracy in handling sharp surface singularities unless a high density of elements is used, [20]. The purpose of this paper is to present an elliptic integral-based solution for different coaxial current loops for witch the mutual inductance will be calculated. The Maxwell's circular coils are used to replace the current loops that will be treated in this paper. This method is well known as the filament method, which leads to accurate expressions of the mutual inductance. The rapidity of this calculation is very important judging from the computational cost.

2 Problem Formulation

The main idea of this method, called the Filament Method, is the using of Maxwell's coils where coils are divided into coaxial circular coils, [3-5], Fig. 1 for which the mutual inductance is given by the expression,

$$M_{Maxwell} = \frac{2\mu_0\sqrt{R_I R_{II}}}{k}[(1-\frac{k^2}{2})K(k)-E(k)] \qquad (1)$$

$$k^2 = \frac{4R_I R_{II}}{(R_I + R_{II})^2 + c^2}$$

In order to account for the finite dimensions of the coils, massive solenoids are considered to be subdivided into meshes of filamentary coils [7-12] as shown at Fig. 2.

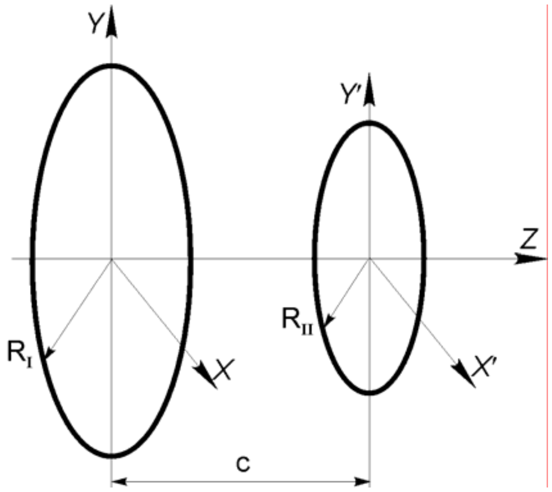

Fig 1. Maxwell's coils

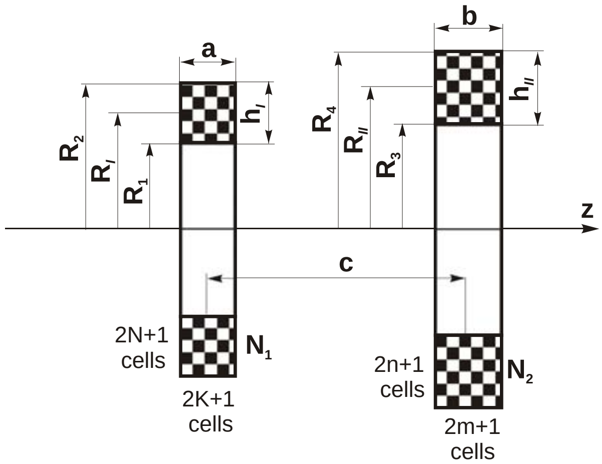

Fig.2. Configuration of mesh coils: Two circular coils of rectangular cross section

The cross-sectional area of the first coil of the mean radius R_I is divided into (2*K* + 1) by (2*N* + 1) cells, and that of the second of the mean radius R_{II} into (2*m* + 1) by (2*n* + 1) cells. Each cell in the first coil contains one filament, and the current density in the coil cross section is assumed to be uniform, so that the filament currents are equal. The same conclusion is valuable for the second coil. It means that it is possible to take into consideration the pair of filamentary unit turn coils for which the mutual inductance is given by (1) where '*i*' and '*j*' are the corresponding thin coils of the first and second massive solenoids, respectively.

To determine the total mutual inductance between the coaxial circular coils of rectangular cross section, the following reasoning is employed. The magnetic flux linking the second coil, the filament '*j*', due to entire set of the first current filaments of which current density with previous assumption of uniform distribution being

$$I_i = \frac{N_1 I_1}{(2K+1)(2N+1)} \tag{2}$$

is given by the flux $\sum_i \Phi_{ij}$, and the total magnetic flux linking the entire second coil is given by $\sum_j (\sum_i \Phi_{ij})$. Since each secondary filament is only a fraction

$$\frac{N_2}{(2m+1)(2n+1)} \tag{3}$$

of the second coil turns, the flux linkage Φ_l with the second winding is given by

$$\Phi_l = \frac{N_2}{(2m+1)(2n+1)} \sum_j (\sum_i \Phi_{ij}) = \frac{N_2}{(2m+1)(2n+1)} \sum_j (\sum_i I_i M_{ij}) \tag{4}$$

Hence the mutual inductance is given by

$$M = \frac{\Phi_l}{I_1} \tag{5}$$

expressions (2)-(5) give,

$$M = \frac{N_1 N_2 \sum_j (\sum_i M_{ij})}{(2K+1)(2N+1)(2m+1)(2n+1)} \tag{6}$$

Finally, the mutual inductance can be expressed in the following form,

$$M = \frac{N_1 N_2 \sum_{g=-K}^{g=K} \sum_{h=-N}^{h=N} \sum_{p=-m}^{p=m} \sum_{l=-n}^{l=n} M(g,h,p,l)}{(2K+1)(2N+1)(2m+1)(2n+1)} \tag{7}$$

where *M*(*g*,*h*,*p*,*l*) is the mutual inductance between two filamentary coils (Maxwell's coils), [2-5] that form the coil of rectangular cross section,

$$M(g,h,p,l)=\frac{\mu_0\sqrt{R_{11}(h)R_{22}(l)}}{k}[(2-k^2)K(k)-2E(k)]$$

where

$$R_{11}(h)=R_I+\frac{h_I}{(2N+1)}h,\ h=-N,...,0,...,N$$

$$R_{22}(l)=R_{II}+\frac{h_{II}}{(2n+1)}l,\ \ l=-n,...,0,...,n$$

$$R_I=\frac{R_1+R_2}{2},\ R_{II}=\frac{R_3+R_4}{2}$$

$$h_I=R_2-R_1,\ h_{II}=R_4-R_3$$

$$z(g,p)=c-\frac{a}{(2K+1)}g+\frac{b}{(2m+1)}p$$

$$g=-K,...,0,...,K\ ;\ g=-m,...,0,...,m$$

$$k^2(g,h,p,l)=\frac{4R_{11}(h)R_{22}(l)}{(R_{11}(h)+R_{22}(l))^2+z^2(g,p)}$$

$E(k)$ and $K(k)$ are complete elliptic integrals of the first and second kinds, [21-22].
The formula (1) is the basic formula to obtain all possible combinations of coaxial circular coils either finite or infinite thickness.

3 Problem Solution

3.1 Two coaxial thin disk coils

Let us consider the system: two coaxial thin disk coils with N_1 and N_2 being number of turns, respectively. Dimensions of this system are shown in Fig. 3.
This system can by obtained from (7) replacing $a = b = 0$, $R_{11} = R_1$, $R_{22} = R_2$, $R_{33} = R_3$, $R_{44} = R_4$ for which the corresponding mutual inductance is,

$$M=\frac{N_1N_2}{(2N+1)(2n+1)}\sum_{h=-N}^{h=N}\sum_{l=-n}^{l=n}M(h,l) \qquad (8)$$

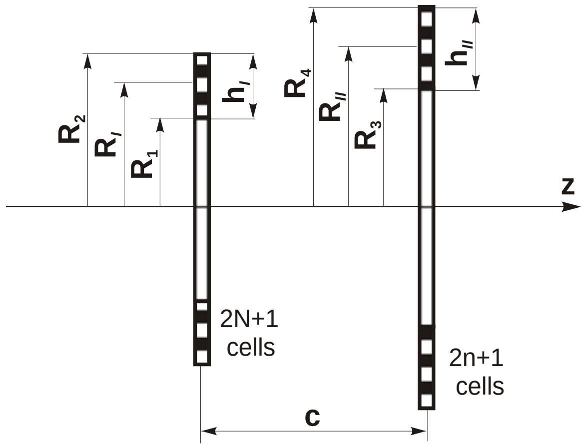

Fig 3. Configuration of mesh matrix: Two thin disk coils

where

$$M(h,l)=\frac{\mu_0\sqrt{R_{11}(h)R_{22}(l)}}{k(h,l)}[(2-k^2)K(k)-2E(k)]$$

$$R_{11}(h)=R_I+\frac{h_I}{2N+1}h,\ h=-N,...,0,...,N$$

$$h_{II}=\frac{R_4-R_3}{2},\ \ R_{II}=\frac{R_4+R_3}{2}$$

$$R_{11}(h)=R_I+\frac{h_I}{2N+1}h,\ h=-N,...,0,...,N$$

$$h_I=\frac{R_2-R_1}{2},\ R_I=\frac{R_2+R_1}{2},\ \ z=c$$

$$k^2(h,l)=\frac{4R_{11}(h)R_{22}(l)}{(R_{11}(h)+R_{22}(l))^2+z^2}$$

3.2 Thin wall solenoid and thin circular disk coil

Let us looking into the system: the thin wall solenoid and the thin disk coil with N_1, N_2 being number of turns, respectively. Dimensions of this system are shown in Fig. 4.
This system can by obtained from (7) replacing $h_I = 0$, $R_{11} = R$, $b = 0$ for which the corresponding mutual inductance is,

$$M=\frac{N_1N_2}{(2K+1)(2n+1)}\sum_{g=-K}^{g=K}\sum_{l=-n}^{l=n}M(g,l) \qquad (9)$$

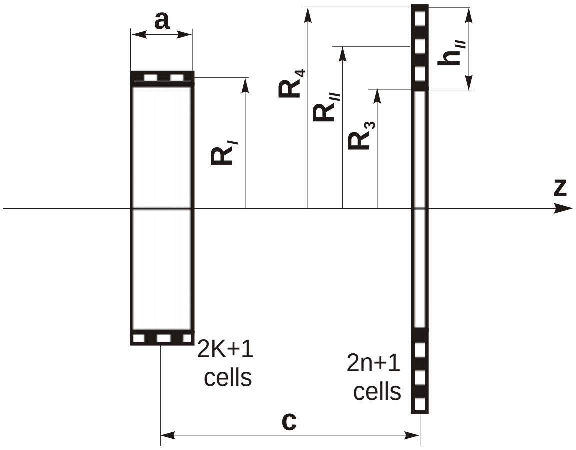

Fig 4. Configuration of mesh matrix: Thin wall solenoid-thin disk.

where

$$M(g,l)=\frac{\mu_0\sqrt{R_{11}R_{22}(l)}}{k(g,l)}[(2-k^2)K(k)-2E(k)]$$

$$R_{22}(l)=R_{II}+\frac{h_{II}}{2n+1}l,\ l=-n,...,0,...,n$$

$$R_{11}=R_I=R,\qquad h_{II}=R_4-R_3$$

$$z(g)=c-\frac{a}{2K+1}g,\quad g=-K,...,0,...,K$$

$$k^2(g,l)=\frac{4R_{11}R_{22}(l)}{(R_{11}+R_{22}(l))^2+z(g)^2}$$

3.3 Filamentary circular coil and circular thin disk coil

Let us examine the system: the filamentary coil and the thin disk coil with N_2 being number of turns. Dimensions of this system are shown in Fig. 5.

This system can by obtained from (7) replacing $N_1 = 1$, $N_2 = N_2$, $h_I = 0$, $R_1 = R_2 = R$, $a = b = 0$ for which the corresponding mutual inductance is,

$$M=\frac{N_2}{(2n+1)}\sum_{l=-n}^{l=n}M(l)\qquad(10)$$

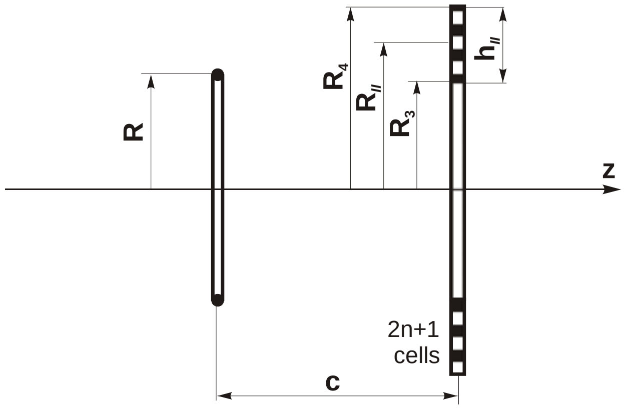

Fig.5. Configuration of mesh matrix: Filamentary circular coil-thin disk coil.

where

$$M(l)=\frac{\mu_0\sqrt{R_{11}R_{22}(l)}}{k(l)}[(2-k^2)K(k)-2E(k)]$$

$$R_{11}=R,\quad R_{II}=\frac{R_4+R_3}{2},\ h_{II}=R_4-R_3$$

$$R_{22}(l)=R_{II}+\frac{h_{II}}{(2n+1)}l,\ l=-n,...,0,...,n$$

$$z=c$$

$$k^2(h)=\frac{4R_{11}R_{22}(l)}{(R_{11}+R_{22}(l))^2+z^2}$$

3.4 Two thin coaxial wall solenoids

Let us consider the system: two thin wall solenoids with N_1, N_2 being number of turns, respectively. Dimensions of this system are shown in Fig. 6.

This system can by obtained from (7) replacing $h_I = h_{II} = 0$, $R_{11} = R_I = R_1$, $R_{22} = R_{II} = R_2$ for which the corresponding mutual inductance is,

$$M=\frac{N_1N_2}{(2K+1)(2m+1)}\sum_{g=-K}^{g=K}\sum_{p=-m}^{p=m}M(g,p)\qquad(11)$$

where

$$M(g,p)=\frac{\mu_0\sqrt{R_{11}R_{22}}}{k(g,p)}[(2-k^2)K(k)-2E(k)]$$

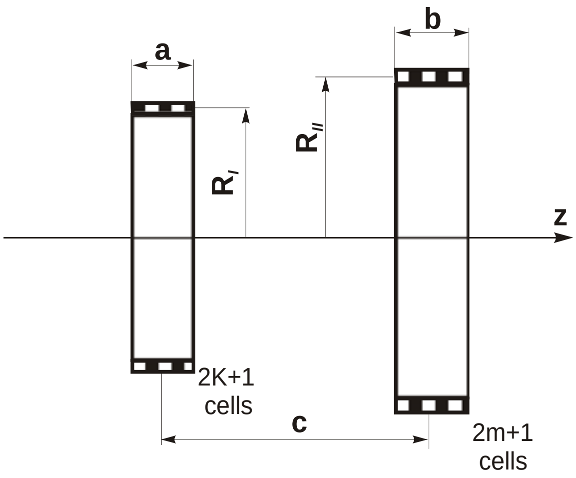

Fig 6. Configuration of mesh matrix: Two coaxial thin solenoids

$$z(g,p) = c - \frac{a}{(2K+1)} g + \frac{b}{(2m+1)} p$$

$$g = -K, ..., 0, ..., K \;,\; p = -m, ..., 0, ..., m$$

$$R_{11} = R_I = R_1, \quad R_{22} = R_{II} = R_2$$

$$k^2(g,p) = \frac{4R_{11}R_{22}}{(R_{11}+R_{22})^2 + z(g,p)^2}$$

3.5 Filamentary circular coil and thin wall solenoid

Finally, let us take into consideration the system: the filamentary coil and the thin wall solenoid with N_2 being number of turns. Dimensions of this system are shown in Fig. 7.

This system can by obtained from (7) replacing $N_1 = 1$, $N_2 = N_2$, $h_I = h_{II} = 0$, $R_{11} = R_I = R_1$, $R_{22} = R_{II} = R_2$, $a = 0$ for which the corresponding mutual inductance is,

$$M = \frac{N_2}{(2m+1)} \sum_{p=-m}^{p=m} M(p) \tag{12}$$

where

$$M(p) = \frac{\mu_0 \sqrt{R_{11}R_{22}}}{k(p)} [(2-k^2)K(k) - 2E(k)]$$

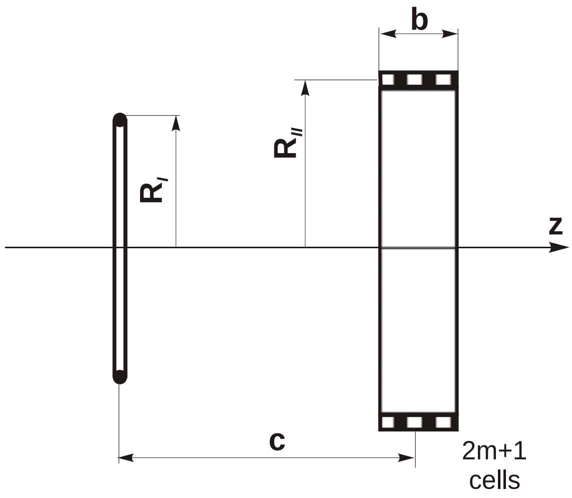

Fig 7. Configuration of mesh matrix: Filamentary circular coil-thin wall solenoid.

$$R_I = R_1 \,,\; R_{II} = R_2$$

$$z(p) = c + \frac{b}{(2m+1)} p, \; p = -m, ..., 0, ..., m$$

$$k^2(p) = \frac{4R_{11}R_{22}}{(R_{11}+R_{22})^2 + z^2(p)}$$

4 Examples

4.1 Example 1

Let us consider two thin disk coils of radii $R_1 = R_3 = 7.63$ *cm*, $R_2 = R_4 = 15.94$ *cm*, $z_Q = 4.68$ *cm* with $N_1 = N_2 = 516$ turns, [3], [11]. Calculate their mutual inductance.
In [3] the mutual inductance is

$$M = 0.0374 \; (H)$$

Applying the exact formula [11] we obtained the mutual inductance,

$$M = 0.03655608 \; (H)$$

Execution time was 0.05999 s.

Table 1 gives values of the mutual inductance using the filament method (8). Also the corresponding computational time and the absolute error of calculation with reference to the exact value [15] are given. The calculation was made in MATLAB programming.

Table1. Comparison of computational efficiency

N / n subdivisions	$M_{Filament}$ *(H)*	*Computational Time (Seconds)*	*Error (%)*
10/10	0.03660395	0.0299	0.13095
30/30	0.03659656	0.0799	0.10876
80/80	0.03659571	0.4100	0.10841
100/100	0.03659566	0.7510	0.10585
500/500	0.03659557	75.2880	0.10803

4.2 Example 2

Let consider the thin wall solenoid of radii R = 10 *cm* and the thin disk of radius $R_1 = 20cm$, $R_2 = 60\ cm$. The length of the solenoid is $(z_2 - z_1)$ where $z_1 = -10\ cm$, $z_2 = 10\ cm$ and the disk is at the plan $z_Q = 60\ cm$ with $N_1 = 100$ and $N_2 = 200$ turns Calculate their mutual inductance.

Applying the exact formula [23] we obtained the mutual inductance,

$$M = 161.76392896\ (\mu H)$$

Execution time was 0.05999 s.

Applying the filament method (9) we obtain the mutual inductance, Table 2 as the corresponding computational time and the absolute error of calculation regarding the exact value [23].

Table2. Comparison of computational efficiency

K	*n*	$M_{Filament}$ *(μH)*	*Computational Time (Seconds)*	*Error (%)*
4	4	161.79107368	0.020	0.016780
50	50	161.76414291	0.150	0.000132
100	100	161.76398298	0.721	0.000008
200	200	161.76394254	5.508	0.000006
300	300	161.76393500	17.024	0.000004
400	400	161.76393236	39.186	0.000002

4.3 Example 3

Let consider the filamentary coil of radius R = 2 *cm* and the thin disk coil of radius $R_1 = 4\ cm$, $R_2 = 6\ cm$ with $N_2 = 100$ turns. Coils are apart $z_Q = 5\ cm$. Calculate their mutual inductance, [12].

Applying the exact formula [12] we obtained the mutual inductance,

$$M = 526.05918238\ nH$$

Execution time was 0.0199 s.

Applying the filament method (10) we obtain the mutual inductance, Table 3 as the corresponding computational time and the absolute error of calculation regarding to the exact value [12].

Table3. Comparison of computational efficiency

N Subdiv.	***M*** *(nH)*	*Computational Time (Seconds)*	*Error (%)*
10	526.07373333	0.02	0.0027660
400	526.05919238	0.03	1.90093e-6
1000	526.05918241	0.04	5.70278e-9
10000	526.05918240	0.20	3.8019e-10
30000	526.05918238	0.62	0.00

4.4 Example 4

Let consider two coaxial thin solenoids of radii $R_1 = 20\ cm$ and $R_2 = 25\ cm$ whose lengths are $(z_2 - z_1)$ and $(z_4 - z_3)$, $z_1 = -5\ cm$, $z_2 = 5\ cm$, $z_3 = 2\ cm$, $z_4 = 18\ cm$, with $N_1 = 100$ and $N_2 = 320$ turns, [3]. Calculate their mutual inductance.

In [2] the mutual inductance is

$$M = 8.458\ mH$$

Applying the exact formula [13] we obtained the mutual inductance,

$$M = 8.47868125\ mH$$

Execution time was 0.0299 s.

Applying the filament method (11) we obtain the mutual inductance, Table 4 as the corresponding computational time and the absolute error of calculation regarding the exact value [13].

Table 4. Comparison of computational efficiency

K	*m*	$M_{Filament}$ *(μH)*	*Computational Time (Seconds)*	*Error (%)*
10	10	8.47866696	0.0199	0.000169
50	50	8.47868067	0.1799	0.000007
100	100	8.47868111	0.6910	0.000002
300	300	8.47868124	18.2170	0.0000001
500	500	8.47868125	71.7640	0.00

4.5 Example 5

Let consider the filamentary coil of radii $R_1 = 7.62\ cm$ that is at plane $z_Q = -2.54\ cm$. The thin solenoid of radii $R_2 = 10.16\ cm$ and the length $z_2 - z_2$, $z_1 = -2.54\ cm$, $z_2 = 2.54\ cm$ with N_2 =

3200 turns, [3]. Calculate their mutual inductance.
In [2] the mutual inductance is

$$M = 372.7\ (\mu H)$$

Applying the exact formula [13] we obtained the mutual inductance,

$$M = 372.81042791\ (\mu H)$$

and the computational time was 0.01999 s.
Applying the filament method (11) we obtain the mutual inductance, Table 5 as the corresponding computational time and the absolute error of calculation regarding the exact value [13].
All preceding results are in very good agreement with already published data.

Table 5. Comparison of computational efficiency

m Subdiv.	***M*** (*µH*)	*Computational Time (Seconds)*	*Error (%)*
10	372.83044365	0.020	0.005369
400	372.81044167	0.030	0.0000037
1000	372.81043011	0.040	0.0000006
10000	372.81042793	0.200	0.0
20000	372.81042791	0.430	0.00

4 Conclusion

Efficient and fast procedures for the calculation of the mutual inductance of thin coaxial current loops are presented. These procedures are based on an elliptic integral-based approach so that one can use them to calculate the mutual inductance of all possible thin coaxial conductors. This method known as the filament method permits to calculate accurately and quickly the mutual inductance of previously mentioned conductors. From obtained results one can conclude that in many cases the mutual inductance has been calculated accurately with few subdivisions of conductors that have directly influence on the computational cost. In some cases when it is necessary to make two summations the calculation of the mutual inductance can be time consuming for achieving a satisfactory accuracy. All results obtained by the proposed approach have been compared with well-known methods. In our examples, a good agreement has been shown with published results. For practical engineering applications this approach can be considered as easy, lucid and rapid. All software is made in MATLAB programming with a personal computer.

Acknowledgment

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