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Globally in time existence theorem for the Navier-Stokes flow in the exterior of a rotating obstacle

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Abstract: Consider the Navier-Stokes fluid filling the whole 3-dimensional space exterior to a rotating obstacle with constant angular velocity ω. By using a coordinate system attached to the obstacle, the problem is reduced to an equivalent one in a fixed exterior domain. It is proved that the reduced problem possesses a unique global solution which goes to a stationary flow as $t \to \infty$ when ω and the initial disturbance are small in a sense.

Key–Words: Navier-Stokes flow, rotating body, exterior domain, global solution, stability, decay

1 Introduction

Let us consider the motion of a viscous fluid filling an infinite space exterior to a rigid body, that moves in a prescribed way such as rotation and translation. In order to understand the rotation effect mathematically, this paper studies the purely rotating case. Thus, suppose that the body is rotating about y_3-axis with constant angular velocity $\omega = (0,0,a)^T, a \in \mathbb{R}$; here and hereafter, all vectors are column ones. Let Ω be an exterior domain in $\mathbb{R}^3$ with smooth boundary $\partial\Omega$. Unless the body is axisymmetric, the domain occupied by the fluid varies with time t, and it is described as

$$\Omega(t) = \{y = \mathcal{O}(at)x; x \in \Omega\},$$

where

$$\mathcal{O}(t) = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We consider the Navier-Stokes equation

$$\partial_t \widetilde{u} + \widetilde{u} \cdot \nabla_y \widetilde{u} = \Delta_y \widetilde{u} - \nabla_y \widetilde{p}, \quad \mathrm{div}_y \widetilde{u} = 0,$$

for $y \in \Omega(t), t > 0$, subject to the boundary and initial conditions

$$\widetilde{u}|_{\partial\Omega(t)} = \omega \times y, \qquad \widetilde{u} \to 0 \text{ as } |y| \to \infty,$$

$$\widetilde{u}(y,0) = u_0(y),$$

where $\widetilde{u}(y,t) = (\tilde{u}^1, \tilde{u}^2, \tilde{u}^3)^T$ and $\widetilde{p}(y,t)$ are respectively unknown velocity and pressure of the fluid; u_0 is the given initial velocity; $\omega \times y = a(-y_2, y_1, 0)^T$ is the velocity of the rotating body so that the boundary condition is the usual nonslip one. A reasonable way from both mathematical and physical points of view is to take the frame $x = \mathcal{O}(at)^T y$ attached to the body ([2], [8], [15]). The following change of functions is thus made:

$$u(x,t) = \mathcal{O}(at)^T \widetilde{u}(y,t), \qquad p(x,t) = \widetilde{p}(y,t).$$

The problem is then reduced to

$$\partial_t u + u \cdot \nabla u = \Delta u - M_a u - \nabla p, \quad \mathrm{div}\, u = 0, \tag{1}$$

in the fixed domain $\Omega \times (0,\infty)$ subject to

$$u|_{\partial\Omega} = \omega \times x, \qquad u \to 0 \text{ as } |x| \to \infty, \tag{2}$$

$$u(x,0) = u_0(x), \tag{3}$$

where

$$M_a = -(\omega \times x) \cdot \nabla + \omega \times, \quad \omega = (0,0,a)^T. \tag{4}$$

We prove that the problem (1) with (2), (3) possesses a unique global solution $u(t)$ which goes to a stationary flow u_s as $t \to \infty$ when ω and $u_0 - u_s$ are small in a sense. Thus the first step is to find a solution u_s of the stationary problem

$$-\Delta u_s + M_a u_s + \nabla p_s + u_s \cdot \nabla u_s = 0, \qquad \text{div}\, u_s = 0,$$

in Ω subject to

$$u_s|_{\partial\Omega} = \omega \times x, \qquad u_s \to 0 \text{ as } |x| \to \infty.$$

Look at the linear part of the first equation of (1). The crucial drift operator $(\omega \times x) \cdot \nabla$ has a variable coefficient growing at infinity and causes the following difficulties, which indicate that the term $(\omega \times x) \cdot \nabla u$ is never subordinate to the viscous term Δu even if $|\omega|$ is small:

– the semigroup generated by the operator $\Delta - M_a$ is not an analytic one in, say, L_2 ([15], [16]);

– the essential spectrum of the operator $\Delta - M_a$ in L_2 consisits of a set of equally spaced half lines parallel to the negative half real line in the complex plane ([7]);

– the pointwise estimate of the fundamental solution of the operator $\Delta - M_a$ is slightly worse than $1/|x - y|$ for large (x, y) ([6], [17]).

Up to now, particularly in the last decade, a lot of efforts have been made on the problems above or some related ones; see [2], [3], [10], [12], [13], [15] for the nonstationary flow, [4], [5], [6], [8], [9], [11], [17], [26] for the stationary one. Among them, the stationary solutions of [9] and [5] can be taken as the basic flow around which a global solution exists since their solutions enjoy so good asymptotic behavior at infinity that one can expect the stability. In fact, Galdi [9] derived pointwise estimates

$$|u_s(x)| \leqq c/|x|, \qquad |\nabla u_s(x)| + |p_s(x)| \leqq c/|x|^2$$

of a unique stationary solution provided that ω is small enough and that, in case the external force $f = \text{div}\, F$ is present, it has some decay properties and is also small in a sense. Another outlook on the pointwise estimates above in a different framework by use of function spaces has been recently provided by Farwig and Hishida [5] when the external force $f = \text{div}\, F$ is taken from a larger class $F \in L_{3/2,\infty}(\Omega)$, where $L_{q,\infty}(\Omega)$ is the weak-L_q space, one of the Lorentz spaces introduced below. To be more precise, a stationary solution of class

$$u_s \in L_{3,\infty}(\Omega), \qquad (\nabla u_s, p_s) \in L_{3/2,\infty}(\Omega) \tag{5}$$

has been uniquely constructed for small ω and $\|F\|_{L_{3/2,\infty}(\Omega)}$, subject to

$$\|u_s\|_{L_{3,\infty}(\Omega)} + \|(\nabla u_s, p_s)\|_{L_{3/2,\infty}(\Omega)} \leqq C\left(|\omega| + \|F\|_{L_{3/2,\infty}(\Omega)}\right). \tag{6}$$

This result can be ragarded as a generalization of [21] and [25] to the rotating body problem.

The solvability of the initial value problem (1), (2), (3) was studied in [2], [10], [13] and [15]. Borchers [2] constructed weak solutions for u_0 in $L_2(\Omega)$. As usual, we do not know the uniqueness of weak solutions. Later on, in [15] the existence of a unique solution locally in time was proved when, roughly speaking, u_0 possesses the regularity $W_2^{1/2}(\Omega)$. This local existence result has been recently extended to the general L_q-theory by Geissert, Heck and Hieber [13] to replace $W_2^{1/2}(\Omega)$ by $L_3(\Omega)$. Galdi and Silvestre [10] showed the unique existence of local and global strong solutions by the Galerkin method. Their global solution was constructed around a stationary solution u_s of Galdi [9] and the stability of the solution u_s was also proved. To be more precise, if ω is small and if $u_0 - u_s$ is taken from $W_2^2(\Omega)$ with small W_2^1-norm, together with $u_0|_{\partial\Omega} = \omega \times x$ and $(\omega \times x) \cdot \nabla(u_0 - u_s) \in L_2(\Omega)$, then there is a global solution $u(t)$ which satisfies $\|\nabla(u(t) - u_s)\|_{L_2(\Omega)} \to 0$ as $t \to \infty$.

Our goal is to prove the stability of the stationary solution u_s of [5], [9] for small ω and $u_0 - u_s \in L_{3,\infty}(\Omega)$. Let (u_s, p_s) be the stationary solution of class (5) subject to (6) (with $F = 0$ for simplicity). Set

$$v(x,t) = u(x,t) - u_s(x), \qquad \pi(x,t) = p(x,t) - p_s(x),$$

and $v_0(x) = u_0(x) - u_s(x)$. Then our stability problem is reduced to the global existence and decay of solutions to

$$\partial_t v + v \cdot \nabla v + u_s \cdot \nabla v + v \cdot \nabla u_s = \Delta v - M_a v - \nabla \pi, \qquad \text{div}\, v = 0, \tag{7}$$

in $\Omega \times (0, \infty)$ subject to

$$v|_{\partial\Omega} = 0, \qquad v \to 0 \text{ as } |x| \to \infty, \tag{8}$$

$$v(x, 0) = v_0(x). \tag{9}$$

Although the global solution of [10] is more regular than ours, new contribution of our global existence theorem is to deduce the definite decay rates of $v(t)$, see (14) below, which seem to be optimal.

2 Main Theorems

To state our main theorems, we introduce some function spaces. We adopt the same symbols for vector and scalar function spaces. Let $C_0^\infty(\Omega)$ consist of all C^∞-functions with compact supports in Ω. For $1 \leqq q \leqq \infty$ and $0 \leqq k \in \mathbb{Z}$, we denote by $W_q^k(\Omega)$, with $W_q^0(\Omega) = L_q(\Omega)$, the usual L_q-Sobolev space of order k. Let $1 < q < \infty$ and $1 \leqq r \leqq \infty$. Then the Lorentz spaces are defined by

$$L_{q,r}(\Omega) = \Big(L_1(\Omega), L_\infty(\Omega)\Big)_{1-1/q,r},$$

where $(\cdot,\cdot)$ is the real interpolation functor, see [1]. It is well known that f is in $L_{q,\infty}(\Omega)$ if and only if

$$\sup_{\sigma>0} \sigma \left|\{x \in \Omega; |f(x)| > \sigma\}\right|^{1/q} < \infty$$

and that $L_{q,\infty}(\Omega)$ is the dual space of $L_{q/(q-1),1}(\Omega)$. Note that $C_0^\infty(\Omega)$ is not dense in $L_{q,\infty}(\Omega)$. We next introduce some solenoidal function spaces. Let $C_{0,\sigma}^\infty(\Omega)$ be the class of all C_0^∞-vector fields f which satisfy div $f = 0$ in Ω. For $1 < q < \infty$ we denote by $J_q(\Omega)$ the completion of $C_{0,\sigma}^\infty(\Omega)$ in $L_q(\Omega)$. Then the Helmholtz decomposition of L_q-vector fields holds, see Miyakawa [23]:

$$L_q(\Omega) = J_q(\Omega) \oplus \{\nabla\pi \in L_q(\Omega); \pi \in L_{q,loc}(\overline{\Omega})\}.$$

Let P denote the projection operator from $L_q(\Omega)$ onto $J_q(\Omega)$ associated with the decomposition. Then the operator $\mathcal{L}_a$ is defined by

$$\begin{cases} D(\mathcal{L}_a) = \{u \in J_q(\Omega) \cap W_q^2(\Omega); u|_{\partial\Omega} = 0, \\ \qquad\qquad (\omega \times x)\cdot\nabla u \in L_q(\Omega)\}, \\ \mathcal{L}_a u = -P[\Delta u - M_a u], \end{cases}$$

see (4). It is proved in [13] that the operator $-\mathcal{L}_a$ generates a C_0-semigroup $\{T_a(t)\}_{t\geq 0}$ on the space $J_q(\Omega), 1 < q < \infty$ (see also [14] for the case $q = 2$). We need also the solenoidal Lorentz spaces, which are defined by

$$J_{q,r}(\Omega) = \Big(J_{q_0}(\Omega), J_{q_1}(\Omega)\Big)_{\theta,r}$$

where $1 < q_0 < q < q_1 < \infty, 1 \leqq r \leqq \infty$ and $1/q = (1-\theta)/q_0 + \theta/q_1$. Then $\{T_a(t)\}_{t\geq 0}$ is extended to the semigroup on the space $J_{q,r}(\Omega)$.

In the construction of a global solution to (7), (8), (9), the essential step is to establish the following L_p-L_q estimates of the semigroup $T_a(t)$.

Theorem 1 *Suppose that*

$$\begin{cases} 1 < p \leqq q < \infty & \textit{for } j = 0, \\ 1 < p \leqq q \leqq 3 & \textit{for } j = 1, \end{cases} \tag{10}$$

and let $a_0 > 0$ be arbitrary. Set

$$\kappa = \frac{3}{2}\left(\frac{1}{p} - \frac{1}{q}\right).$$

1. *There is a constant $C = C(p, q, a_0) > 0$ such that*

   $$\|\nabla^j T_a(t) f\|_{L_q(\Omega)} \leqq C t^{-j/2-\kappa} \|f\|_{L_p(\Omega)} \tag{11}$$

   for all $t > 0$, $f \in J_p(\Omega)$ and ω with $|\omega| = |a| \leqq a_0$. For $q = \infty$ and $j = 0$ as well, estimate (11) holds.

2. *Let $1 \leqq r \leqq \infty$. Then there is a constant $C = C(p, q, r, a_0) > 0$ such that*

   $$\|\nabla^j T_a(t) f\|_{L_{q,r}(\Omega)} \leqq C t^{-j/2-\kappa} \|f\|_{L_{p,r}(\Omega)} \tag{12}$$

   for all $t > 0$, $f \in J_{p,r}(\Omega)$ and ω with $|\omega| = |a| \leqq a_0$.

The restriction $q \leqq 3$ for the gradient estimate, which was first proved by Iwashita [19] for the case of the usual Stokes semigroup ($\omega = 0$), is closely related to the decay structure of stationary solutions. Since the decay of our fundamental solution is slightly worse than that of the usual Stokes one as was mentioned, it is hopeless to improve the restriction $q \leqq 3$ for $j = 1$ in Theorem 1. In fact, Maremonti and Solonnikov [22] pointed out that one cannot avoid that restriction for the gradient estimate even when $\omega = 0$.

By use of the semigroup $T_a(t)$, the problem (7) is converted into

$$v(t) = T_a(t)v_0 - \int_0^t T_a(t-\tau) P \mathrm{div} G(u_s, v(\tau))\, d\tau,$$

where

$$G(u_s, v(t)) = v(t) \otimes v(t) + u_s \otimes v(t) + v(t) \otimes u_s.$$

In view of the class (5) of the stationary solution u_s, however, it is difficult to treat the additional linear terms div$(u_s \otimes v + v \otimes u_s)$. We thus consider the weak formulation

$$\begin{aligned}\langle v(t), \varphi\rangle =&\langle v_0, T_a(t)^*\varphi\rangle \\ &+ \int_0^t \langle G(u_s, v(\tau)), \nabla T_a(t-\tau)^*\varphi\rangle d\tau, \\ &\qquad \forall \varphi \in C_{0,\sigma}^\infty(\Omega),\end{aligned} \tag{13}$$

in terms of the adjoint semigroup $T_a(t)^*$, that is essentially the same as $T_a(t)$ itself since $T_a(t)^* = T_{-a}(t)$ for $a \in \mathbb{R}$; so, Theorem 1 can be applied. We use the $L_{p,r}$-$L_{q,r}$ estimates (12) rather than (11). Let $1 < p < q \leqq 3$ and $1/q = 1/p - 1/3$. Then the interpolation technique developed by Yamazaki [27] combined with (12) for $r = 1$ implies

$$\int_0^\infty \|\nabla T_a(t)f\|_{L_{q,1}(\Omega)} dt \leqq C\|f\|_{L_{p,1}(\Omega)}$$

for $f \in J_{p,1}(\Omega)$. This enables us to deal with the additional linear terms in $G(u_s, v)$ as a perturbation from the semigroup $T_a(t)$. As a result, we obtain the following global existence theorem.

Theorem 2 *Let* $v_0 \in J_{3,\infty}(\Omega)$.

1. *There is a constant* $\delta > 0$ *such that if*

$$|\omega| + \|v_0\|_{L_{3,\infty}(\Omega)} \leqq \delta,$$

then the problem (13) possesses a unique global solution

$$v \in BC\Big((0,\infty); J_{3,\infty}(\Omega)\Big)$$

with

$$w^*\text{-}\lim_{t\to 0} v(t) = v_0 \quad \text{in } J_{3,\infty}(\Omega).$$

2. *Let* $3 < q < \infty$. *Then there is a constant* $\widetilde{\delta}(q) \in (0, \delta]$ *such that if*

$$|\omega| + \|v_0\|_{L_{3,\infty}(\Omega)} \leqq \widetilde{\delta}(q),$$

then the solution $v(t)$ *obtained above enjoys*

$$\|v(t)\|_{L_r(\Omega)} = O\left(t^{-1/2+3/2r}\right) \tag{14}$$

as $t \to \infty$ *for every* $r \in (3, q)$.

3 Conclusions

We have done rigorous mathematical analysis of the Navier-Stokes flow in the exterior of a rotating obstacle with constant angular velocity $\omega = (0, 0, a)^T$. The reduced equation in a reference frame attached to the obstacle involves the drift operator $(\omega \times x) \cdot \nabla$ that is never subordinate to the usual Stokes operator. A unique solution exists globally in time around a stationary flow when ω and the initial disturbance are sufficiently small. Especially, our analysis makes it possible to deduce some optimal asymptotic rates (14) as $t \to \infty$. The most difficult step is to show decay estimates of the semigroup $T_a(t)$ and the proof of them is own interesting, see [18]. The strategy based on some cut-off techniques together with spectral analysis is traced back to Shibata [24] and is similar to that of Iwashita [19] and also Kobayashi and Shibata [20], however, we need several new ideas because the semigroup $T_a(t)$ is never analytic unlike [19], [20]. In particular, it is important to derive the behavior of the resolvent $(\lambda + \mathcal{L}_a)^{-1}$ for large λ along the imaginary axis in the complex plane as well as its regularity for small λ.

Finally, we would like to mention two physical examples which have relation to our theory. One is the particle sedimentation in a viscous liquid and the other is the motion of a viscous fluid around a planet that rotates on its own axis. The former is particularly of practical interest and the problem is to find a falling motion of a rigid body under its own weight in an infinite fluid, see Weinberger [28] and Galdi [8] for details. The body undergoes a rotation and a translation which are to be determined from equilibrium conditions on the boundary; that is, a fluid-body interaction system has to be solved. Indeed the present article is devoted to the fluid motion around a body which moves in a prescribed way, but our study is certainly a step toward an analysis of the problem above.

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