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Numerical Simulation of an Oldroyd-B Fluid with a Preconditioned Domain Decomposition Method

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Abstract: This paper deals with the numerical implementation of a preconditioned domain decomposition method to approximate the solution of a non-Newtonian viscoelastic Oldroyd-B model. The governing equations can be decomposed into a Navier-Stokes system and a transport equation and a modified Schwarz scheme, involving block preconditioners for the Navier-Stokes equations, is used to solve iteratively the decoupled problems. Numerical results are provided for steady flow in the two-dimensional lid driven cavity.

Key–Words: Oldroyd-B model, finite elements, domain decomposition method, preconditioners.

1 Introduction

Domain Decomposition Methods (DDM) applied to the numerical solution of large-scale algebraic systems arising from the approximation of partial differential differential equations have been intensively studied by many authors (see e.g [8], [3], [11] and the references cited therein). They are based on a decomposition of the spatial domain of the problem into several subdomains, which may or may not overlap, and consist on solving reduced size subproblems on these subdomains, while enforcing suitable continuity requirements at the corresponding interfaces. Such reformulations are usually motivated by the need to create new solvers for parallel computers. A multi-domain approach can also account for the solution of heterogeneous models related to physical problems which can be defined in complex geometries. Another important feature is that domain decomposition methods lead to the construction of optimal (mesh size independent) preconditioners.

The aim of this paper is to study the numerical implementation of a preconditioned DDM applied to a non-Newtonian viscoelastic Oldroyd-B model in the steady case. The constitutive equations lead to a highly non-linear system of partial differential equations of mixed type which can be decoupled into a Navier-Stokes system and a transport equation. The preconditioned DDM developed here involves two block preconditioners for the Navier-Stokes equations (see e.g. [1], [7]).

The paper is organized as follows. The governing fluid equations are introduced in section 2. In section 3 we present the domain decomposition method for both the Navier-Stokes system and the transport equation, using a modified Schwarz multiplicative scheme. Appropriate block preconditioners for the Navier-Stokes system are introduced in section 4. Numerical results for an Oldroyd-B flow in a two-dimensional lid driven cavity are presented and discussed in section 5. Finally we summarize the conclusions of this work.

2 Governing Equations

We consider steady isothermal flows of incompressible Oldroyd-B fluids in a bounded domain $\Omega \subset I\!R^2$ with a polygonal boundary $\partial\Omega$

. For these fluids, the extra-stress tensor is related to the kinematic variables through

$$\mathbf{S} + \lambda_1 \overset{\nabla}{\mathbf{S}} = 2\mu\Big(\lambda_2 \overset{\nabla}{D\mathbf{u}} + D\mathbf{u}\Big), \qquad (1)$$

where $\mathbf{u}$ is the velocity field, $D\mathbf{u} = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^t)$ denotes the symmetric part of the velocity gradient, μ is the constant viscosity and $\lambda_1 > 0$, $\lambda_2 > 0$ are respectively the relaxation and retardation times. The symbol ∇ denotes the objective derivative of Oldroyd type defined by

$$\overset{\nabla}{\mathbf{S}} = \mathbf{u} \cdot \nabla\, \mathbf{S} - \mathbf{S}\, \nabla\mathbf{u} - (\nabla\mathbf{u})^t\, \mathbf{S}.$$

The Cauchy stress tensor is given by $\mathbf{T}=-p\mathbf{I} + \mathbf{S}$, where p represents the pressure and $\mathbf{S}$ is the extra stress tensor. The equations of conservation of momentum and mass hold in the domain Ω,

$$\rho\,\mathbf{u}\cdot\nabla\mathbf{u}+\nabla p=\nabla\cdot\mathbf{S},\qquad \nabla\cdot\mathbf{u}=0, \tag{2}$$

where $\rho > 0$ is the (constant) density of the fluid. Decomposing the extra-stress tensor $\mathbf{S}$ into the sum of its Newtonian part $\boldsymbol{\tau}_s = 2\mu\frac{\lambda_2}{\lambda_1}\,D\mathbf{u}$ and its viscoelastic part $\boldsymbol{\tau}$, we rewrite (1)-(2) as

$$\begin{cases} -\frac{\lambda_2}{\lambda_1}\mu\,\Delta\mathbf{u}+\rho\mathbf{u}\cdot\nabla\mathbf{u}+\nabla p=\nabla\cdot\boldsymbol{\tau}, \\ \nabla\cdot\mathbf{u}=0, \\ \boldsymbol{\tau}+\lambda_1\overset{\nabla}{\boldsymbol{\tau}}=2\mu\left(1-\frac{\lambda_2}{\lambda_1}\right)Du. \end{cases}$$

We consider the nondimensional form of this system by introducing the following quantities

$$x=\frac{\tilde{x}}{L},\ \ u=\frac{\tilde{u}}{U},\ \ p=\frac{\tilde{p}L}{\mu U},\ \ \boldsymbol{\tau}=\frac{\tilde{\boldsymbol{\tau}}L}{\mu U},$$

where the symbol $\tilde{}$ is attached to dimensional parameters (L represents a reference length and U a characteristic velocity of the flow). We also set

$$\varepsilon = 1-\frac{\lambda_2}{\lambda_1},$$

and introduce the Reynolds number and the Weissenberg number

$$\mathcal{R}e=\frac{\rho UL}{\mu},\qquad \mathcal{W}e=\frac{\lambda_1 U}{L}.$$

The nondimensional system takes the form

$$\begin{cases} -(1-\varepsilon)\,\Delta\mathbf{u}+\mathcal{R}e\,\mathbf{u}\cdot\nabla\mathbf{u}+\nabla p=\nabla\cdot\boldsymbol{\tau}, \\ \nabla\cdot\mathbf{u}=0, \\ \boldsymbol{\tau}+\mathcal{W}e\overset{\nabla}{\boldsymbol{\tau}}=2\varepsilon\,D\mathbf{u}. \end{cases} \tag{3}$$

and is composed of a Navier-Stokes system for $(\mathbf{u}, p)$ coupled with a transport equation for $\boldsymbol{\tau}$. This system is supplemented with a Dirichlet boundary condition

$$\mathbf{u}=\mathbf{g}\qquad \text{on } \partial\Omega \quad \text{with} \quad \mathbf{g}\cdot\mathbf{n}=0,$$

$\mathbf{n}$ being the unit outward normal to the boundary.

3 Finite Element Approximation

In this section we apply a domain decomposition method for both the Navier-Stokes equations and the transport equation in system (3). Let us consider a nonoverlapping decomposition of Ω into subdomains Ω_1 and Ω_2, satisfying the following conditions

$$\Omega=\Omega_1\cup\Omega_2,\quad \Omega_1\cap\Omega_2=\emptyset \quad \text{and} \quad \overline{\Omega_1}\cap\overline{\Omega_2}=\Gamma,$$

where Γ denotes the interface between Ω_1 and Ω_2.

3.1 Domain decomposition method

For fixed $\boldsymbol{\tau}$, the first two equations in (3) define a Navier-Stokes system in the variables $(\mathbf{u}, p)$. Here we use the Hood-Taylor $P_2 - P_1$ finite element method for the approximation of $(\mathbf{u}, p)$ (see e.g [2]). The corresponding discrete spaces V_h and Q_h satisfy the discrete inf-sup condition. For fixed $(\mathbf{u}, p)$, the third equation in (3) is a transport equation in $\boldsymbol{\tau}$. As in [10]; the tensor $\boldsymbol{\tau}$ is approximated by using test functions in $\boldsymbol{\Sigma}_h = (Q_h)^4$. Let N^v and N^p be respectively the dimensions of V_h and Q_h. The discrete formulation of the Navier-Stokes equations leads to the following system

$$\mathbf{A}\mathbf{U}=\mathbf{F}(\mathbf{T},\mathbf{U}) \tag{4}$$

where the matrix $\mathbf{A}$ is defined by

$$\mathbf{A}=\begin{bmatrix} \mathbf{A}_1 & \mathbf{0} & \mathbf{B}_1 \\ \mathbf{0} & \mathbf{A}_2 & \mathbf{B}_2 \\ \mathbf{B}_1^t & \mathbf{B}_2^t & \mathbf{0} \end{bmatrix} \tag{5}$$

with

$$(\mathbf{A}_\ell)_{ij}=(1-\varepsilon)\big(\nabla\varphi_i^\ell,\nabla\varphi_j^\ell\big)_\Omega,\quad 1\le i,j\le N^v$$

$$(\mathbf{B}_\ell)_{ij}=\big(\tfrac{\partial\varphi_i^\ell}{\partial x_\ell},\psi_j\big)_\Omega,\quad 1\le i\le N_v,\ 1\le j\le N^p$$

and

$$(\mathbf{F}_\ell(\mathbf{T},\mathbf{U}))_i=\big(\nabla\cdot\mathbf{T}_{\ell,.}-\mathcal{R}e\,\mathbf{U}\cdot\nabla\mathbf{U}_\ell,\varphi_i^\ell\big)_\Omega,\qquad 1\le i\le N^v$$

Similarly the approximate transport equation can be rewritten as

$$\mathbf{S}(\mathbf{U})\mathbf{T}=\mathbf{G}(\mathbf{U},\mathbf{T})$$

where

$$\big(\mathbf{S}(\mathbf{U})\big)_{ij}=\big(\boldsymbol{\phi}_j+\mathcal{W}e\,\mathbf{U}\cdot\nabla\boldsymbol{\phi}_j,\boldsymbol{\phi}_i\big)_\Omega,$$

$$\big(\mathbf{G}(\mathbf{U},\mathbf{T})\big)_i=\mathcal{W}e\,\big((\nabla\mathbf{U})^T\mathbf{T}+\mathbf{T}(\nabla\mathbf{U}),\boldsymbol{\phi}_i\big)_\Omega -2\varepsilon\,\big(D(\mathbf{U}),\boldsymbol{\phi}_i\big)_\Omega.$$

The (global) fixed point iteration scheme can be written as follows:

- For a given initial condition $(\mathbf{U}^0,\mathbf{T}^0)$, find $\mathbf{U}^{m+1}$ solution of

$$\mathbf{A}\mathbf{U}^{m+1}=\mathbf{F}(\mathbf{T}^m,\mathbf{U}^m). \tag{6}$$

- Calculate the new iterate $\mathbf{T}^{m+1}$ solution of

$$\mathbf{S}(\mathbf{U}^{m+1})\mathbf{T}^{m+1}=\mathbf{G}(\mathbf{U}^{m+1},\mathbf{T}^m) \tag{7}$$

- Calculate $\mathbf{U}^{m+2}$.

Reordering the numeration of the nodes, we rewrite the system (6) as follows

$$\begin{bmatrix} \mathbf{A}_{11} & 0 & \mathbf{A}_{1\Gamma} \\ 0 & \mathbf{A}_{22} & \mathbf{A}_{2\Gamma} \\ \mathbf{A}_{\Gamma 1} & \mathbf{A}_{\Gamma 2} & \mathbf{A}^1_{\Gamma\Gamma} + \mathbf{A}^2_{\Gamma\Gamma} \end{bmatrix} \begin{bmatrix} \mathbf{U}_1^{m+1} \\ \mathbf{U}_2^{m+1} \\ \mathbf{U}_\Gamma^{m+1} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_1 \\ \mathbf{F}_2 \\ \mathbf{F}_\Gamma \end{bmatrix} \quad (8)$$

where each one of the involved submatrices has the same structure as $\mathbf{A}$, and where $\mathbf{U}_\ell^m$ is defined on $\Omega_\ell \setminus \Gamma$ ($\ell = 1, 2$), and $\mathbf{U}_\Gamma^m$ is defined on the interface Γ (to simplify the presentation, the index m is dropped in the terms appearing on the right-hand side). System (7) can also be rewritten in an equivalent form

$$\begin{bmatrix} \mathbf{S}_{11} & 0 & \mathbf{S}_{1\Gamma} \\ 0 & \mathbf{S}_{22} & \mathbf{S}_{2\Gamma} \\ \mathbf{S}_{\Gamma 1} & \mathbf{S}_{\Gamma 2} & \mathbf{S}^1_{\Gamma\Gamma} + \mathbf{S}^2_{\Gamma\Gamma} \end{bmatrix} \begin{bmatrix} \mathbf{T}_1^{m+1} \\ \mathbf{T}_2^{m+1} \\ \mathbf{T}_\Gamma^{m+1} \end{bmatrix} = \begin{bmatrix} \mathbf{G}_1 \\ \mathbf{G}_2 \\ \mathbf{G}_\Gamma \end{bmatrix} \quad (9)$$

where

$$\mathbf{S}_{ii} \equiv \mathbf{S}_{ii}(\mathbf{U}_i^{m+1}), \qquad \mathbf{S}_{i\Gamma} \equiv \mathbf{S}(\mathbf{U}_\Gamma^{m+1}),$$

$$\mathbf{S}_{\Gamma i} \equiv \mathbf{S}(\mathbf{U}_i^{m+1}), \qquad \mathbf{S}_{\Gamma\Gamma} \equiv \mathbf{S}_{\Gamma\Gamma}(\mathbf{U}_\Gamma^{m+1}),$$

$$\mathbf{G}_i^m \equiv \mathbf{G}_i(\mathbf{U}_i^{m+1}, \mathbf{T}_i^m), \qquad \mathbf{G}_\Gamma^m \equiv \mathbf{G}_\Gamma(\mathbf{U}_\Gamma^{m+1}, \mathbf{T}_\Gamma^m).$$

3.2 The Schwarz method

At each step of the fixed-point algorithm stated in the previous section, we apply the modified Schwarz multiplicative scheme to solve systems (8) and (9). Basically, the idea consists in introducing two matrices that optimize the interface conditions [6].
Let n_Γ^v and n_Γ^p be the number of nodes in Γ for the velocity and pressure, respectively and set $N_\Gamma = 2n_\Gamma^v + n_\Gamma^p$. Let $\{\mathbf{e}_k\}_{k=1,\dots,N_\Gamma}$ be the corresponding canonical basis, and for $i = 1, 2$, let $(\mathbf{W}_{i,in}, \mathbf{W}_{i,\Gamma})$ be the solution of the following problem:

$$\begin{cases} \mathbf{A}_{ii}\,\mathbf{W}_{i,\text{in}} + \mathbf{A}_{i\Gamma}\,\mathbf{W}_{i,\Gamma} = 0 & \text{in } \Omega_i \\ \mathbf{W}_{i,\Gamma} = \mathbf{e}_k & \text{on } \Gamma. \end{cases} \quad (10)$$

The interface matrices $\boldsymbol{\mathcal{M}}_1$ and $\boldsymbol{\mathcal{M}}_2$ (corresponding to the Navier-Stokes system) are then obtained through the following identities:

$$\boldsymbol{\mathcal{M}}_1\,\mathbf{e}_k = -\mathbf{A}_{\Gamma 2}\,\mathbf{W}_{2,\text{in}} - \mathbf{A}^2_{\Gamma\Gamma}\,\mathbf{W}_{2,\Gamma} \quad (11)$$

$$\boldsymbol{\mathcal{M}}_2\,\mathbf{e}_k = -\mathbf{A}_{\Gamma 1}\,\mathbf{W}_{1,\text{in}} - \mathbf{A}^1_{\Gamma\Gamma}\,\mathbf{W}_{1,\Gamma}. \quad (12)$$

Similarly, we construct two interface matrices $\boldsymbol{\mathcal{N}}_1$ and $\boldsymbol{\mathcal{N}}_2$, corresponding to the transport equation.

More precisely:

- For $m \geq 0$, solve

1. **The Navier-Stokes system**

   For $k = 0, K$;

   - Given $\mathbf{U}_2^k = (\mathbf{U}_{2,\text{in}}^k, \mathbf{U}_{2,\Gamma}^k)$ and $\mathbf{T}_1^m$

   $$\begin{cases} \text{Find } \mathbf{U}_1^{k+1} = (\mathbf{U}_{1,\text{in}}^{k+1}, \mathbf{U}_{1,\Gamma}^{k+1}) \text{ such that} \\ \mathbf{A}_{11}\,\mathbf{U}_{1,\text{in}}^{k+1} + \mathbf{A}_{1\Gamma}\,\mathbf{U}_{1,\Gamma}^{k+1} = \mathbf{F}_{1,\text{in}} \\ \mathbf{A}_{\Gamma 1}\,\mathbf{U}_{1,\text{in}}^{k+1} + \left(\mathbf{A}^1_{\Gamma\Gamma} - \boldsymbol{\mathcal{M}}_1\right)\,\mathbf{U}_{1,\Gamma}^{k+1} \\ = \mathbf{F}_\Gamma - \mathbf{A}_{\Gamma 2}\mathbf{U}_{2,\text{in}}^k - \left(\mathbf{A}^2_{\Gamma\Gamma} + \boldsymbol{\mathcal{M}}_1\right)\mathbf{U}_{2,\Gamma}^k \end{cases} \quad (13)$$

   - Given $\mathbf{U}_1^{k+1} = (\mathbf{U}_{1,\text{in}}^{k+1}, \mathbf{U}_{1,\Gamma}^{k+1}, \mathbf{T}_2^m)$,

   $$\begin{cases} \text{Find } \mathbf{U}_2^{k+1} = (\mathbf{U}_{2,\text{in}}^{k+1}, \mathbf{U}_{2,\Gamma}^{k+1}) \text{ such that} \\ \mathbf{A}_{22}\,\mathbf{U}_{2,\text{in}}^{k+1} + \mathbf{A}_{2\Gamma}\,\mathbf{U}_{2,\Gamma}^{k+1} = \mathbf{F}_{2,\text{in}} \\ \mathbf{A}_{\Gamma 2}\,\mathbf{U}_{2,\text{in}}^{k+1} + \left(\mathbf{A}^2_{\Gamma\Gamma} - \boldsymbol{\mathcal{M}}_2\right)\,\mathbf{U}_{2,\Gamma}^{k+1} \\ = \mathbf{F}_\Gamma - \mathbf{A}_{\Gamma 1}\mathbf{U}_{1,\text{in}}^{k+1} - \left(\mathbf{A}^1_{\Gamma\Gamma} + \boldsymbol{\mathcal{M}}_2\right)\mathbf{U}_{1,\Gamma}^{k+1} \end{cases} \quad (14)$$

   - Set $\mathbf{U}_1^m = \mathbf{U}_1^{k+1}$ and $\mathbf{U}_2^m = \mathbf{U}_2^{k+1}$

2. **The transport equation**

   For $k = 0, K$;

   - Given $\mathbf{U}_1^m$ and $(\mathbf{T}_{2,\text{in}}^k, \mathbf{T}_{2,\Gamma}^k)$

   $$\begin{cases} \text{Find } \mathbf{T}_1^{k+1} \text{ such that} \\ \mathbf{S}_{11}\,\mathbf{T}_{1,\text{in}}^{k+1} + \mathbf{S}_{1\Gamma}\,\mathbf{T}_{1\Gamma}^{k+1} = \mathbf{G}_{1,\text{in}} \\ \mathbf{S}_{\Gamma 1}\,\mathbf{T}_{1,\text{in}}^{k+1} + \left(\mathbf{S}^1_{\Gamma\Gamma} - \boldsymbol{\mathcal{N}}_1\right)\mathbf{T}_{1\Gamma}^{k+1} \\ = \mathbf{G}_\Gamma - \mathbf{S}_{\Gamma 2}\mathbf{T}_{2,\text{in}}^k - \left(\mathbf{A}^2_{\Gamma\Gamma} + \boldsymbol{\mathcal{N}}_1\right)\mathbf{T}_{2,\Gamma}^k \end{cases}$$

   - Given $\mathbf{U}_2^m$ and $(\mathbf{T}_{1,\text{in}}^k, \mathbf{T}_{1,\Gamma}^k)$,

   $$\begin{cases} \text{Find } \mathbf{T}_2^{k+1} \text{ such that} \\ \mathbf{S}_{22}\,\mathbf{T}_{2,\text{in}}^{k+1} + \mathbf{S}_{2\Gamma}\,\mathbf{T}_{2\Gamma}^{k+1} = \mathbf{G}_{2,\text{in}} \\ \mathbf{S}_{\Gamma 2}\,\mathbf{T}_{2,\text{in}}^{k+1} + \left(\mathbf{S}^2_{\Gamma\Gamma} - \boldsymbol{\mathcal{N}}_2\right)\mathbf{T}_{2\Gamma}^{k+1} \\ = \mathbf{G}_\Gamma - \mathbf{S}_{\Gamma 1}\mathbf{T}_{1,\text{in}}^{k+1} - \left(\mathbf{S}^1_{\Gamma\Gamma} + \boldsymbol{\mathcal{N}}_2\right)\mathbf{T}_{1,\Gamma}^{k+1} \end{cases}$$

   - Set $\mathbf{T}_1^m = \mathbf{T}_1^{k+1}$ and $\mathbf{T}_2^m = \mathbf{T}_2^{k+1}$

3. Set $\mathbf{U}_i^0 = \mathbf{U}_i^m$, $\mathbf{T}_i^0 = \mathbf{T}_i^m$ $(i = 1, 2)$.

4. $m = m + 1$.

To enhance the speed of convergence we next introduce in each subdomain a global block preconditioner.

4 Block Preconditioners

Let us first consider a general framework for the block preconditioning of the Navier-Stokes system (see e.g. [4], [5], [1])

$$\begin{bmatrix} \mathbf{F} & \mathbf{B} \\ \mathbf{B}^T & -\mathbf{C} \end{bmatrix} \mathbf{X} = \mathbf{H}. \quad (15)$$

Let $\mathbf{P}_R$ be a right global preconditioner of the form

$$\mathbf{P}_R = \begin{bmatrix} \mathbf{P}_\mathbf{F} & \mathbf{B} \\ 0 & -\mathbf{P}_\mathbf{S} \end{bmatrix},$$

where $\mathbf{P}_\mathbf{F}$ and $\mathbf{P}_\mathbf{S}$ are respectively preconditioners for $\mathbf{F}$ and $\mathbf{S} = \mathbf{B}^T\mathbf{F}^{-1}\mathbf{B} + \mathbf{C}$ (the corresponding Schur complement). Since

$$\mathbf{P}_R^{-1} = \begin{bmatrix} \mathbf{P}_\mathbf{F}^{-1} & \mathbf{P}_\mathbf{F}^{-1}\mathbf{B}\mathbf{P}_\mathbf{S}^{-1} \\ 0 & -\mathbf{P}_\mathbf{S}^{-1} \end{bmatrix},$$

then the linear system (15) is equivalent to

$$\left\{ \begin{array}{l} \begin{bmatrix} \mathbf{F}\,\mathbf{P}_\mathbf{F}^{-1} & \left[\mathbf{F}\mathbf{P}_\mathbf{F}^{-1} - \mathbf{I}_d\right]\mathbf{B}\mathbf{P}_\mathbf{S}^{-1} \\ \mathbf{B}^t\,\mathbf{P}_\mathbf{F}^{-1} & \mathbf{S}\mathbf{P}_\mathbf{S}^{-1} \end{bmatrix} \mathbf{Y} = \mathbf{H}, \\ \mathbf{P}_R\,\mathbf{X} = \mathbf{Y}. \end{array} \right.$$

In a first step, this approach is used to solve system (10). Taking into account the fact that $\mathbf{A}_{ii}$ $(i = 1, 2)$ has the same structure as $\mathbf{A}$, i.e.

$$\mathbf{A}_{ii} = \begin{bmatrix} \mathbf{A}_{ii}^* & \mathbf{B}_{ii}^* \\ (\mathbf{B}_{ii}^*)^T & \mathbf{0} \end{bmatrix},$$

and choosing $\mathbf{F} = \mathbf{P}_\mathbf{F} = \mathbf{A}_{ii}^*$, $\mathbf{S} = \mathbf{P}_\mathbf{S} = (\mathbf{B}_{ii}^*)^t(\mathbf{A}_{ii}^*)^{-1}\mathbf{B}_{ii}^*$, we can easily see that system (10) is equivalent to

$$\left\{ \begin{array}{l} \begin{bmatrix} \mathbf{I} & 0 \\ (\mathbf{B}_{ii}^*)^t(\mathbf{A}_{ii}^*)^{-1} & \mathbf{I} \end{bmatrix} \mathbf{Y}_{in} = -\mathbf{A}_{i\Gamma}\,\mathbf{e}_k, \\ \begin{bmatrix} \mathbf{A}_\mathbf{ii}^* & \mathbf{B}_\mathbf{ii}^* \\ \mathbf{0} & -\mathbf{S} \end{bmatrix} \mathbf{W}_{i,in} = \mathbf{Y}_{in}. \end{array} \right.$$

The new system is solved using a GMRES method [9]. In this case it is known that the first system converges in two or three iterations and that the convergence is independent of the mesh [7].

In the same spirit, this preconditioning approach is applied to the matrices

$$\mathbf{A}_i = \begin{bmatrix} \mathbf{A}_{ii} & \mathbf{A}_{i\Gamma} \\ \mathbf{A}_{\Gamma i} & \mathbf{A}_{\Gamma\Gamma}^i - \boldsymbol{\mathcal{M}}_i \end{bmatrix}, \quad (16)$$

associated to the Navier-Stokes systems in the subdomains Ω_i $(i = 1, 2)$. The corresponding systems can be formally written as

$$\mathbf{A}_i\,\mathbf{U}_i = \mathbf{F}_i \qquad (i = 1, 2). \quad (17)$$

Let $\mathbf{P}_{A_i}$ be the right preconditioner for $\mathbf{A}_i$ defined by

$$\mathbf{P}_{A_i} = \begin{bmatrix} \mathbf{P}_{A_{ii}} & \mathbf{A}_{i\Gamma} \\ \mathbf{0} & -\mathbf{P}_{S_{i,\Gamma}} \end{bmatrix},$$

where $\mathbf{P}_{A_{ii}}$ and $\mathbf{P}_{S_{i,\Gamma}}$ are preconditioners for $\mathbf{A}_{ii}$ and $\mathbf{S}_{i,\Gamma} = \mathbf{A}_{\Gamma i}(\mathbf{A}_{ii})^{-1}\mathbf{A}_{i\Gamma} - \mathbf{A}_{\Gamma\Gamma}^i + \boldsymbol{\mathcal{M}}_i$. Problem (17) is equivalent to

$$\left\{ \begin{array}{l} \mathbf{A}_i\,\mathbf{P}_{A_i}^{-1}\,\mathbf{Y}_i = \mathbf{F}_i \\ \mathbf{P}_{A_i}\,\mathbf{U}_i = \mathbf{Y}_i \end{array} \right.$$

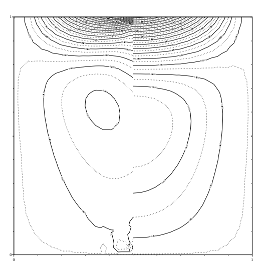
where

$$\mathbf{A}_i\boldsymbol{P}_{A_i}^{-1} = \begin{bmatrix} \mathbf{A}_{ii}\,\mathbf{P}_{A_{ii}}^{-1} & \left(\mathbf{A}_{ii}\,\mathbf{P}_{A_{ii}}^{-1} - \mathbf{I}_d\right)\mathbf{A}_{i\Gamma}\,\mathbf{P}_{S_{i,\Gamma}}^{-1} \\ \mathbf{A}_{\Gamma i}\,\mathbf{P}_{A_{ii}}^{-1} & S_{i,\Gamma}\mathbf{P}_{S_{i,\Gamma}}^{-1} \end{bmatrix}.$$

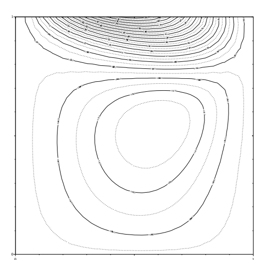
5 Numerical Results

To validate our numerical method we consider the widely used benchmark test of the lid driven cavity flow. The fluid is contained in a squared cavity $\Omega = (0, 1)^2$ where the upper wall moves with a constant velocity $\mathbf{u}_* = (1, 0)$ causing a flow rotation. To handle the corner singularities, we implement a regularized method replacing the tangential velocity on the lid by a parabolic distribution that vanish (together with its first derivative) on the edges or corners where the lid and stationary walls meet.

We consider a vertical decomposition of the cavity into two subdomains $\Omega_1 = (0, 0.5) \times (0, 1)$ and $\Omega_2 = (0.5, 1) \times (0, 1)$.



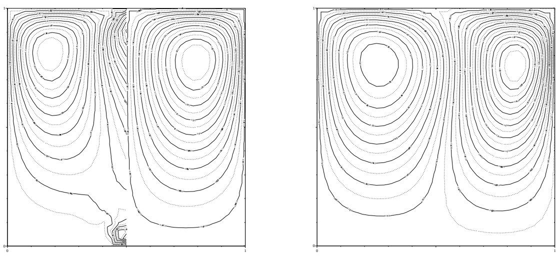
(a) iter=1



(b) iter=31

Figure 1: *Contours of the first component velocity at the initial and final iterations of the PDD Method.*

Figures 1 and 2 show the contours of the velocity field at the initial and final iterations, obtained by using the Preconditioned Domain Decomposition Method (PDDM) presented in the previous sections, and correspond to the case $\mathcal{R}e = 50$ and $\mathcal{W}e = 0.15$. We observe that the first iteration shows a discontinuity on the velocity field at the interface, vanishing when the continuity of the normal stress and velocity field across the interface are achieved (iter=31).



(a) iter=1 (b) iter=31

Figure 2: *Contours of the second velocity component at the first and last iteration of the PDDM.*

Several tests were performed in order to study the influence of the mesh size, as well as the influence of the Weissenberg number on the convergence of our method. Setting the Reynolds number to 50, and varying the viscoleastic parameter, we implement the PDDM for two different meshes and compare the results with those obtained by using two other approaches: the global method with and without preconditioning (GM and PGM). In particular, and as can be seen in Table 1, the CPU time is significantly reduced especially for the refined mesh and for high values of the Weissenberg number. Table 2 shows the corresponding number of iterations.

	GM	PGM	PDDM
Mesh 20×20			
$\mathcal{W}e = 0.05$	346	112	62
0.10	426	112	66
0.15	675	139	84
Mesh 28×28			
$\mathcal{W}e = 0.05$	2793	749	312
0.10	3373	751	343
0.15	5859	997	417

Table 1: Comparison of the CPU time for two different meshes obtained with the methods GM, PGM and PDDM.

	GM	PGM	PDDM
Mesh 20×20			
$\mathcal{W}e = 0.05$	13	15	16
0.10	16	15	18
0.15	25	26	29
Mesh 28×28			
$\mathcal{W}e = 0.05$	14	15	17
0.10	17	15	20
0.15	29	28	31

Table 2: Number of iteration steps corresponding to the global algorithms associated to the methods GM, PGM and PDDM.

6 Conclusion

The aim of this work is to apply a DDM to solve the equations of motion of an Oldroyd-B model using a block preconditioner associated to the Navier-Stokes equations. Numerical results have been obtained on the lid driven cavity benchmark and we conclude that DDM with preconditioners are well adapted to the numerical simulation of this viscoelastic fluid model.
Disregarding the fact that the preconditioners were constructed in an exact form, numerical results indicate a significantly reduction in CPU time when compared to those obtained using a global approximation of the problem. However, if a semi-implicit approximation of the convective term is used, different strategies to construct preconditioners need to be employed, e.g. those based on incomplete factorizations and multigrid methods. It is known that the lid driven cavity is a particular benchmark, where discontinuous conditions on the boundary nodes only allow for convergence with very low values of the Weissenberg number. Imposing the no-slip condition on the two upper corner nodes or using regularization methods we expect convergence for higher Weissenberg numbers. The last approach is just the one corresponding to the preliminary numerical results presented in this work. Further results and more detailed discussion will be object of a forthcoming paper.

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