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On the Weak Solution to the Oseen–Type Problem Arising from Flow around a Rotating Rigid Body

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Abstract: - Considering time-periodic Oseen flow around a rotating body in $\mathbb{R}^3$ we prove *a priori* estimates in L^q-spaces of weak solution for the whole space problem. After a time-dependent change of coordinates the problem is reduced to a stationary Oseen equation with the additional terms $(\omega \times x) \cdot \nabla u$ and $\omega \times u$ in the momentum equation where ω denotes the angular velocity. Assuming that force f has a form $f = \nabla \cdot F$, we prove L^q-estimates of weak solution using a theory of Littlewood–Paley decomposition and of maximal operators.

Key-Words: - Rigid body, Weak solution, Marcinkiewicz theory, Paley–Littlewood theory, Existence, Uniqueness

1 Introduction

Over the past years, there has been a great impulse in studying the motion of a rigid body. The first systematic study on this subject initiated with the pioneering work of G. Kirchhoff [K], Lord Kelvin [T] regarding the motion of one or more bodies in a frictionless liquid. After that many mathematicians have furnished significant contributions to this fascinating field under different assumptions on the body and on the fluid. We wish to quote the work of H. Brenner [B] concerning the steady motion of one or more bodies in a linear viscous liquid in the Stokes approximation, further H. F. Weinberger [W1]-[W2] and D. Serre [S] regarding the fall of a body in an incompressible Navier-Stokes fluid under the action of gravity. Recently see e.g. G. P. Galdi, A. L. Silvestre [GS], R. Farwig, T. Hishida, D. Müller [FHM], R. Farwig [Fa1,Fa2], T. Hishida [H1]–[H3], M. Hieber, M. Geissert, H. Heck, [HGH], S. Kračmar, Š. Nečasová, P. Penel, [KNPe1,KNPe2], R. Farwig, M. Krbec, Š. Nečasová [FKN1, FKN2], R. Farwig, J. Neustupa [KN].

Before describing the main results, we would like to introduce some basic problems of practical interest. The orientation of long bodies in liquid of different nature is a fundamental issue in many practical interest. A first, fundamental step in modelling and the orientation of long bodies in liquids is to investigate experimantally their free fall behavior (sedimentation), both in Newtonian and viscoelastic liquids see [Le], [PC]. The addition of short fiber-like particles to a polymer matrix is well-known to enhance the mechanical properties of the *composite material*, see [Ad]. Typical sizes of a fiber are hundred micrometers in diameter and a centimeter in length [Ad]. The degree of enhancement depends strongly on the orientation of the fibers and the fiber orientation is in turn caused by the flow occurring in the mold ; see [LYKU]. Very important is *separation of macromolecules by electrophoresis*. Electrophoresis is a dominant analytical separation technique in the biological technique in the biological sciences [GRO]. Modern applications include weight determination of proteins [HR], DNA sequencing [Tr], and diagnosis of genetic disease [Bo]. Electrophoresis involves the motion of charged particles (macromolecules) in solution, under the influence of an electric field. Certain types of macromolecules have a symmetric and rigid straight-rod shape and several hundreds nanometers in lenght [GRO]. The orientation of the molecules plays an important role, since it is responsible for the loss of separability during steady-field gel electrophoresis [GRO], [TMW]. We would like to mention flow-induced microstructures. Particle pair interactions are a fundamental mechanism that enter strongly in all practical applications of particulate flows [Jo],[Ro]. They are due to inertia and normal stresses and are maximally different in Newtonian and viscoelastic liquids [JLPF]. In the most well-studied case of fluidized spheres, the principal interaction between a neighboring pair is described by the mechanism of drafting, kissing and tumbling in Newtonian liquids, and or drafting, kissing and chain-

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ing in viscoelastic liquids [Jo1], [Jo2]. A first, fundamental step in modelling and the orientation of long bodies in liquids is to investigate experimentally their free fall behavior (sedimentation), both in Newtonian and viscoelastic liquids see [Le], [PC], [PGG].

2 Formulation of the problem

In this paper we consider a three-dimensional rigid body rotating with angular velocity $\omega = \tilde{\omega}(0,0,1)^T$, $\tilde{\omega} \neq 0$ and assume that the complement is filled with a viscous incompressible fluid modelled by the Navier-Stokes equations. We consider the viscous flow either past a rotating body $K \subset\subset \mathbb{R}^3$ with axis of rotation ω and with the velocity $u_\infty = ke_3 \neq 0$ at infinity or around a rotating body K which is moving in the direction of its axis of rotation. After changing of the coordinate system, considering that $u_\infty || \omega$ and linearizing in u we get the following system

$$\begin{aligned} -\nu\Delta u + k\,\partial_3 u - (\omega \wedge x)\cdot\nabla u + \omega\wedge u + \nabla p & \\ = f & \quad (1) \\ \operatorname{div} u = 0 & \quad (2) \\ u \rightarrow 0 \text{ as } |x| \rightarrow \infty. & \quad (3) \end{aligned}$$

The linear system (1)–(3) has been analyzed in L^q–spaces, $1 < q < \infty$, in [Fa1,Fa2] proving the *a priori* - estimate

$$\|\nu\nabla^2 u\|_q + \|\nabla p\|_q \leq c\|f\|_q, \quad (4)$$

$$\|k\partial_3 u\|_q + \|(\omega\wedge x)\cdot\nabla u + \omega\wedge u\|_q \leq c\left(1 + \frac{k^4}{\nu^2|\omega|^2}\right)\|f\|_q \quad (5)$$

with the constant $c > 0$ independent of ν, k, ω.

We introduce notation and then we will give a formulation of our problem.

Given a domain $\Omega = \mathbb{R}^3$, the class $C_0^\infty(\Omega)$ consists of C^∞ functions with compact supports contained in Ω. By $L^q(\Omega)$ we denote the usual Lebesgue space with norm $\|\cdot\|_{q,\Omega}$,

$$L_0^q(\Omega) = \{u \in L^q(\Omega) : \int_\Omega u\,dx = 0\},$$

$$\mathcal{D}(\Delta_q) = W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega).$$

We define the homogeneous Sobolev spaces

$$\begin{aligned} \widehat{W}^{1,q}(\mathbb{R}^3) &= \overline{C_0^\infty(\mathbb{R}^3)}^{\|\nabla\cdot\|_{q,\mathbb{R}^3}} \\ &= \{v \in L^q_{\text{loc}}(\mathbb{R}^3);\ \nabla v \in L^q(\mathbb{R}^3)^3\}/\mathbb{R}, \end{aligned} \quad (6)$$

and their dual space

$$\widehat{W}^{-1,q}(\mathbb{R}^3) = (\widehat{W}^{1,q/(q-1)}(\mathbb{R}^3))^*, \quad (7)$$

with norm $\|\cdot\|_{-1,q,\mathbb{R}^3}$.

Remarks: We would like to mention that the dual space of a domain from Assumption I (ii), (iii) have to be define in the following way

$$W^{-1,q}(\Omega) = (W^{1,q'}(\Omega))^*.$$

Let us consider the problem (1)- (3).

Definition 1.
Let $1 < q < \infty$. Given $f \in \widehat{W}^{-1,q}(\mathbb{R}^3)^3$, we call $\{u,p\} \in \widehat{W}_0^{1,q}(\mathbb{R}^3)^3 \times L^q(\mathbb{R}^3)$ weak solution to (1)–(3) if

$$(i) \qquad \nabla\cdot u = 0 \quad \textit{in} \quad L^q(\mathbb{R}^3),$$

$$(ii) \qquad (\omega\wedge x)\cdot\nabla u - \omega\wedge u \in \widehat{W}^{-1,q}(\mathbb{R}^3),$$

$\{u,p\}$ satisfies (1) in the sense of distributions, that is

$$\begin{aligned} \langle\nabla u, \nabla\varphi\rangle - \langle(\omega\wedge x)\cdot\nabla u - \omega\wedge u, \varphi\rangle \quad (8) \\ + k\left\langle\frac{\partial u}{\partial x_3}, \varphi\right\rangle - \langle p, \nabla\cdot\varphi\rangle = \langle f, \varphi\rangle, \\ \varphi \in C_0^\infty(\mathbb{R}^3), \end{aligned}$$

where $< .,. >$ denotes the duality pairings.
$\{u,p\}$ satisfies (8) for all $\varphi \in \widehat{W}^{1,q/(q-1)}(\mathbb{R}^3)$.

Theorem 1. *Let $1 < q < \infty$ and suppose*

$$f \in \widehat{W}^{-1,q}(\mathbb{R}^3)^3, \quad (9)$$

then the problem (1)-(3) possesses a weak solution $\{u,p\} \in \widehat{W}^{1,q}(\mathbb{R}^3) \times L^q(\mathbb{R}^3)$

$$\begin{aligned} \|\nabla u\|_q + \|p\|_q + \|(\omega\wedge x)\cdot\nabla u - \omega\wedge u\|_{-1,q} \\ \leq C\|f\|_{-1,q}, \end{aligned} \quad (10)$$

with some $C > 0$, depends on q. The solution is unique in $\widehat{W}^{1,q}(\mathbb{R}^3)^3$ up to a constant multiple of ω for u.

3 Proof of the main theorem

We give a sketch of the proof of Theorem 1, for more details see [KNPe2].

For a rapidly decreasing function $u \in \mathcal{S}(\mathbb{R}^n)$ let

$$\mathcal{F}u(\xi) = \hat{u}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} u(x)\, dx,$$

with $\xi \in \mathbb{R}^n$, be the Fourier transform of u. Its inverse is denoted by $\mathcal{F}^{-1}$. Moreover, we define the centered Hardy-Littlewood maximal operator

$$\mathcal{M}u(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |u(y)|\, dy, \quad x \in \mathbb{R}^n,$$

for $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ where Q runs through the set of all cubes centered at x.
Due to the geometry of the problem it is reasonable to introduce cylindrical coordinates $(r, x_3, \theta) \in (0,\infty)\times \mathbb{R}\times[0,2\pi)$. Then the term $(\omega \wedge x)\cdot\nabla u = -x_2\partial_1 u + x_1\partial_2 u$ may be rewritten in the form

$$(\omega \wedge x)\cdot\nabla u = \partial_\theta u$$

using the angular derivative ∂_θ applied to $u(r, x_3, \theta)$.

Now we will solve (1) – (3) explicitly using Fourier transforms and multiplier operators. Working first of all formally or in the space $\mathcal{S}'(\mathbb{R}^n)$ of tempered distributions we apply the Fourier transform $\mathcal{F} = \widehat{\ }$ to (1) – (3). With the Fourier variable $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$ and $s = |\xi|$ we get from (1) – (3).

$$\begin{aligned} &(\nu s^2 + ik\xi_3)\widehat{u} - \partial_\varphi \widehat{u} + \tilde{\omega} e_3 \times \widehat{u} + i\xi\widehat{p} = \widehat{f}, \\ &i\xi\cdot\widehat{u} = 0. \end{aligned} \tag{11}$$

Here $(\omega\times\xi)\cdot\nabla_\xi = -\xi_2\partial/\partial\xi_1 + \xi_1\partial/\partial\xi_2 = \partial_\varphi$ is the angular derivative in Fourier space when using cylindrical coordinates for $\xi \in \mathbb{R}^n$. Since $i\xi\cdot\widehat{u} = 0$ implies $i\xi\cdot(\partial_\varphi\widehat{u} - \omega\times\widehat{u}) = 0$, the unknown pressure p is given by $-|\xi|^2\widehat{p} = i\xi\cdot\widehat{f}$, i.e.,

$$\widehat{\nabla p}(\xi) = i\xi\cdot\widehat{p} = \frac{(\xi\cdot\widehat{f})\widehat{f}}{|\xi|^2}.$$

Hence u may be considered as a (solenoidal) solution of the reduced problem

$$-\nu\Delta u + k\partial_3 u - \partial_\theta u + \omega\wedge u = f \quad \text{in} \quad \mathbb{R}^n, \tag{12}$$

where by f we mean $f = f - \nabla p$.

Theorem 2. *Let* $1 < q < \infty$ *and* $f \in \widehat{W}^{-1,q}(\mathbb{R}^3)$. *Then the equation*

$$\begin{aligned} &Lu \equiv \\ &-\Delta u + \frac{\partial u}{\partial x_3} - (\omega\wedge x)\cdot\nabla u + \omega\wedge u = f \ \text{ in } \mathbb{R}^3 \end{aligned} \tag{13}$$

possesses a weak solution $u \in \widehat{W}^{1,q}(\mathbb{R}^3)$ *subject to the estimate*

$$\begin{aligned} &\|\nabla u\|_{q,\mathbb{R}^3} + \|(\omega\wedge x)\cdot\nabla u - \omega\wedge u\|_{-1,q,\mathbb{R}^3} \\ &\quad \le C\|f\|_{-1,q,\mathbb{R}^3}, \end{aligned} \tag{14}$$

with some $C > 0$, *depends on* q. *The solution is unique in* $\widehat{W}^{1,q}(\mathbb{R}^3)^3$ *up to a constant multiple of* ω *for* u.

In Fourier space – using cylindrical coordinates $(s, \varphi, \xi_3) \in \bar{\mathbb{R}}_+\times[0,2\pi]\times\mathbb{R}$, $s = \sqrt{\xi_1^2+\xi_2^2}$, for $\xi = (\xi_1,\xi_2,\xi_3)^T$ as well and note that $\widehat{\partial_\varphi u} = \partial_\varphi\widehat{u}$, $\partial_\varphi u = (e_3\times x)\cdot\nabla u$ and $\widehat{u}$ satisfies the equation

$$\frac{1}{\tilde{\omega}}(\nu|\xi|^2 + ik\xi_3)\widehat{u} - \partial_\varphi\widehat{u} + e_3\wedge\hat{u} = \frac{1}{\tilde{\omega}}\widehat{f}$$

with respect to φ. Denoting $\widehat{v}(\varphi) = O^T_{e_3}(\varphi)\widehat{u}(s,\varphi,\xi_3)$ then we are looking for the solution of the following problem:

$$\frac{1}{\tilde{\omega}}(\nu|\xi|^2 + ik\xi_3)\widehat{v} - \partial_\varphi\widehat{v} = \frac{1}{\tilde{\omega}}O^T_{e_3}(\varphi)\widehat{f} \tag{15}$$

After some calculation we obtain

$$\begin{aligned} &\widehat{u}(\xi) \equiv \\ &\equiv \int_0^\infty e^{-\nu|\xi|^2 t} O^T_\omega(t)(\mathcal{F}f(O_\omega(t). - kte_3))(\xi)dt. \end{aligned} \tag{16}$$

Finally note that $e^{-\nu|\xi|^2 t}$ is the Fourier transform of the heat kernel

$$E_t(x) = \frac{1}{(4\pi\nu t)^{3/2}} e^{-|x|^2/4\nu t} \tag{17}$$

yielding

$$u(x) = \int_0^\infty E_t * O^T_\omega(t) f(O_\omega(t). - kte_3)(x)dt. \tag{18}$$

Note that $F = f - \nabla p$ is solenoidal so that the identity $i\xi\cdot\widehat{F} = 0$ implies that also u is solenoidal.
An essential step is to show

$$\|\nabla u\|_{q,\mathbb{R}^3} \le C\|G\|_{q,\mathbb{R}^3}$$

for the force of the form $f = \nabla\cdot G$ with $G \in C_0^\infty(\mathbb{R}^3)^9$ on account of the density property see Lemma 1.

Assumptions I: Let $\Omega \subset \mathbb{R}^N$, $N \ge 2$, be a domain with boundary $\partial\Omega \in C^{1,1}$ and suppose one of the following cases
(i) Ω is bounded
(ii) Ω is an exterior domain, i.e. , a domain having a

compact nonempty complement
(iii) Ω is a perturbed half space, i.e., there exists some open ball B such that $\Omega \setminus B = \mathbb{R}^n_+ \setminus B$.

For a bounded domain Ω with boundary in $C^{0,1}$ Bogovski [B1], [B2] constructed a bounded linear operator $R : L^q_0(\Omega) \to W^{1,q}_0(\Omega)^n$ such that $u = Rg$ is a solution of

$$\begin{aligned} \operatorname{div} u &= g \text{ in } \Omega \\ u = 0 &\text{ on } \partial\Omega. \end{aligned} \tag{19}$$

satisfying $\|Rg\|_{W^{1,q}(\Omega)} \leq c\|g\|_q$. Additionally R maps $W^{1,q}_0(\Omega) \cap L^q_0(\Omega)$ into $W^{2,q}_0$, see [W].

Lemma 1. (Farwig, Sohr)
Let $\Omega = \mathbb{R}^n$ or let $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ be a domain satisfying Assumption I, further let $1 < q < \infty$. Then there exists a linear bounded operator $R : W^{1,q} \cap \widehat{W}^{-1,q}(\Omega) \to \mathcal{D}(\Delta_q)^n$ if Ω is unbounded or $R : W^{1,q}(\Omega) \cap L^q_0(\Omega) \to \mathcal{D}(\Delta_q)^n$ if Ω is bounded such that $u = Rg$ is a solution of (2.10) for all $g \in W^{1,q} \cap \widehat{W}^{-1,q}(\Omega)$ or $g \in W^{1,q}(\Omega) \cap L^q_0(\Omega)$ respectively; $u = Rg$ satisfies the estimates

$$\|u\|_q \leq C\|g\|_{-1,q}$$

and

$$\|u\|_{W^{2,q}} \leq c(\|\nabla g\|_q + \|g\|_{-1,q}),$$

where $c = c(\Omega, q) > 0$ is a constant.

Proof: see [FS].

In our case for all $f \in \widehat{W}^{-1,q}(\Omega)$, there is $G \in L^q(\Omega)$ such that

$$\nabla \cdot G = f \tag{20}$$
$$\|G\|_{q,\Omega} \leq C\|f\|_{-1,q,\Omega} \tag{21}$$

with some $C > 0$.
As a result, the space $\{\nabla \cdot G; G \in C^\infty_0(\Omega)^n\}$ is dense in $\widehat{W}^{-1,q}(\Omega)$.

Let us derive the L^q estimate of the operator T defined by

$$\begin{aligned} &TG(x) = \nabla u(x) = -\textstyle\int_{\mathbb{R}^3} \nabla_x \nabla_y \Gamma(x,y) : G(y)dy, \\ &\Gamma(x,y) = \textstyle\int_0^\infty E_t O^T_w(t)dt, \end{aligned} \tag{22}$$

The following proposition indicates that fundamental solution does not define a classical Calderon-Zygmund integral operator and we need to use Littlewood-Paley theory.

Proposition 1. There is no constant $C > 0$ such that

$$|x - y|\Gamma(x,y)| \leq C, \forall (x,y) \in R^3 \times R^3.$$

Proof : see [H] or [FHM].

Remark: We would like to mention very important property that the terms $\omega \wedge x\nabla u, \omega \wedge u$ cannot be estimated separately in general case but only in case that we require special type of compatibility condition on the f

$$\frac{1}{2\pi}\int_0^{2\pi} O(\theta)^T f(r, x_3, \theta)d\theta = 0 \text{ for a.a. } r > 0, x_3 \in \mathbb{R}.$$

For more details see [FHM].
Now, we derive the L^q estimate of the operator T.

$$\begin{aligned} &\widehat{\nabla u(\xi)} = \\ &= \textstyle\int_0^\infty \widehat{\psi}_{\nu t}(\xi) O^T_\omega(t) \mathcal{F} G(O_\omega(t). - kte_3)(\xi)\frac{dt}{t} \\ &= \textstyle\int_0^\infty \widehat{\psi}_t(\xi) O^T_{\omega/\nu}(t) \mathcal{F} G(O_{\omega/\nu}(t). - \frac{k}{\nu}te_3)(\xi), \end{aligned} \tag{23}$$

where

$$\widehat{\psi}(\xi) = \frac{1}{(2\pi)^{3/2}}|\xi|^2 e^{-|\xi|^2}$$

and

$$\widehat{\psi}_t(\xi) = \widehat{\psi}(\sqrt{t}\xi) \ \text{ for } t > 0$$

are the Fourier transforms of a function $\psi \in \mathcal{S}(R^3)$ and of $\psi_t(x) = t^{-3/2}\psi(x/\sqrt{t}), t > 0$, resp.

We define $\psi \in \mathcal{S}(\mathbb{R}^n)$ by its Fourier transform

$$\widehat{\psi}(\xi) = (2\pi)^{-n/2}|\xi|^2 e^{-\nu|\xi|^2} = (\widehat{-\Delta) E}_1(\xi) \tag{24}$$

and for all $t > 0$

$$\begin{aligned} &\psi_t(x) = t^{-n/2}\psi(\tfrac{x}{\sqrt{t}}), \\ &\widehat{\psi}_t(\xi) = \widehat{\psi}(\sqrt{t}\xi) = (2\pi)^{-n/2} t|\xi|^2 e^{\nu t|\xi|^2}. \end{aligned} \tag{25}$$

We define the operator

$$TG(x) \equiv \int_0^\infty \psi_t(\xi) O^T_{\omega/\nu}(t) G(O_{\omega/\nu}(t). - \frac{k}{\nu}te_3)(\xi)\frac{dt}{t}. \tag{26}$$

To decompose $\widehat{\psi}_t$ choose $\widetilde{\chi} \in C^\infty_0(\frac{1}{2}, 2)$ satisfying $0 \leq \widetilde{\chi} \leq 1$ and

$$\sum_{j=-\infty}^{\infty} \widetilde{\chi}(2^{-j}s) = 1 \quad \text{ for all } \quad s > 0.$$

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Then define χ_j for $\xi \in \mathbb{R}^n$ and $j \in \mathbb{Z}$ by its Fourier transform

$$\widehat{\chi}_j(\xi) = \widetilde{\chi}(2^{-j}|\xi|), \quad \xi \in \mathbb{R}^n,$$

yielding $\sum_{j=-\infty}^{\infty} \widehat{\chi}_j = 1$ on $\mathbb{R}^n \setminus \{0\}$ and

$$\begin{aligned} &\mathrm{supp}\widehat{\chi}_j \subset A(2^{j-1}, 2^{j+1}) := \\ &\{\xi \in \mathbb{R}^n : 2^{j-1} \le |\xi| \le 2^{j+1}\}. \end{aligned} \tag{27}$$

Using χ_j we define for $j \in \mathbb{Z}$

$$\psi^j = \frac{1}{(2\pi)^{n/2}} \chi_j * \psi \quad (\widehat{\psi}^j = \widehat{\chi}_j \cdot \widehat{\psi}). \tag{28}$$

Obviously, $\sum_{j=-\infty}^{\infty} \psi^j = \psi$ on $\mathbb{R}^n$. We define the linear operator

$$\begin{aligned} &T_j G(x) = \\ &= \int_0^\infty \int_0^\infty \widehat{\psi}_{\nu t}(\xi) O_\omega^T(t) \mathcal{F}G(O_{\omega/\nu}(t). - kte_3)(\xi) \tfrac{dt}{t} \\ &= \int_0^\infty \widehat{\psi}_t(\xi) O_{\omega/\nu}^T(t) \mathcal{F}G(O_{\omega/\nu}(t). - \tfrac{k}{\nu} te_3)(\xi) \tfrac{dt}{t}. \end{aligned} \tag{29}$$

Since formally $T = \sum_{j=-\infty}^{\infty} T_j$, we have to prove that this infinite series converges even in the operator norm on L^q.

For later use we cite the following lemma, see [FHM].

Lemma 2. *The functions* ψ^j, ψ_t^j, $j \in \mathbb{Z}$, $t > 0$, *have the following properties:*

(i) $\mathrm{supp}\widehat{\psi}_t^j \subset A\left(\frac{2^{j-1}}{\sqrt{t}}, \frac{2^{j+1}}{\sqrt{t}}\right)$.

(ii) For $m > \frac{n}{2}$ *let* $h(x) = (1 + |x|^2)^{-m}$ *and* $h_t(x) = t^{-n/2} h(\frac{x}{\sqrt{t}})$, $t > 0$. *Then there exists a constant* $c > 0$ *independent of* $j \in \mathbb{Z}$ *such that*

$$\begin{aligned} |\psi^j(x)| &\le c 2^{-2|j|} h_{2^{-2j}}(x), \quad x \in \mathbb{R}^n, \\ \|\psi^j\|_1 &\le c 2^{-2|j|}. \end{aligned}$$

Proof: see [FHM].

To introduce a Littlewood-Paley decomposition of L^q choose $\widetilde{\varphi} \in C_0^\infty(\frac{1}{2}, 2)$ such that $0 \le \widetilde{\varphi} \le 1$ and

$$\int_0^\infty \widetilde{\varphi}(s)^2 \frac{ds}{s} = \frac{1}{2}.$$

Then define $\varphi \in \mathcal{S}(\mathbb{R}^n)$ by its Fourier transform $\widehat{\varphi}(\xi) = \widetilde{\varphi}(|\xi|)$ yielding for every $s > 0$

$$\widehat{\varphi}_s(\xi) = \widetilde{\varphi}(\sqrt{s}|\xi|), \quad \mathrm{supp}\widehat{\varphi}_s \subset A(\frac{1}{2\sqrt{s}}, \frac{2}{\sqrt{s}}) \tag{30}$$

and the normalization $\int_0^\infty \widehat{\varphi}_s(\xi)^2 \frac{ds}{s} = 1$ for all $\xi \in \mathbb{R}^n \setminus \{0\}$.

Theorem 3. *Let* $1 < q < \infty$. *Then there are constants* $c_1, c_2 > 0$ *depending on* q *and* φ *such that for all* $f \in L^q$

$$c_1 \|f\|_q \le \left\| \left(\int_0^\infty |\varphi_s * f(\cdot)|^2 \frac{ds}{s} \right)^{1/2} \right\|_q \le c_2 \|f\|_q$$

where $\varphi_s \in \mathcal{S}(\mathbb{R}^n)$ *is defined by (30).*

Proof: See [St1],[St2].

As a preliminary version of Theorem 2 we prove the following proposition.

Proposition 1. *Let* $j \in \mathbb{Z}$. *The linear operator* T *defined by 26 satisfies the estimate*

$$\|T_j G\|_q \le c \|G\|_q \quad \text{for all} \quad G \in L^q,\ q \in (2, \infty) \tag{31}$$

with a constant $c = c(q) > 0$ *independent of* f.

Proof: see [KNPe2].

Then $T = \sum_{j=-\infty}^{\infty} T_j$ converges in the operator norm on L^q and $\|TG\|_q \le c\|G\|_q$, for every $G \in S(\mathbb{R}^3)^3$ and $q > 2$.

To prove (32) for $q \in (1, 2)$ we use a standard duality argument. The adjoint operator T^* is given by

$$\begin{aligned} &T^* G(x) = \\ &= \int_0^\infty (\psi_t * O_{\omega/\nu}(t) G)(O_{\omega/\nu}^T(t) x + \frac{k}{\nu} te_3) \frac{dt}{t}, \end{aligned} \tag{32}$$

with $g \in \mathcal{S}(\mathbb{R}^3)^3$.

The same argument as above implies that T^* is also a bounded operator on $L^{q/(q-1)}(\mathbb{R}^3)^9$, it implies that T is bounded for $1 < q < 2$. For $q = 2$ we apply Plancherel theorem.

This imply that we proved the following estimate

$$\|\nabla u\|_q \le \|G\|_q. \tag{33}$$

Now, using Farwig- Sohr lemma (Lemma 1) we know that there is $G \in L^q(R^3)^9$ such that

$$\nabla \cdot G = f,\ \|G\|_{q,\mathbb{R}^3} \le C \|f\|_{-1,q,\mathbb{R}^3}.$$

This implies the statement of Theorem 2.

It remains to prove the uniqueness. We use the duality method. We consider the adjoint equation

$$\begin{aligned} &L^* v \equiv \\ &-\Delta v - \frac{\partial u}{\partial x_3} + (\omega \wedge x) \cdot \nabla v - \omega \wedge v + \frac{\partial u}{\partial x_3} = \nabla \cdot F \end{aligned} \tag{34}$$

with $F \in C_0^\infty(\mathbb{R}^3)^9$. This admits the solution

$$\widehat{v}(\xi) = \int_0^\infty e^{-\nu|\xi|^2 t} O_\omega(t)(\mathcal{F}f(O_\omega^T(t).-kte_3))(\xi)dt. \tag{35}$$

Applying the same argument we get

$$\|\nabla v\|_{r,\mathbb{R}^3} \leq C\|F\|_{r,\mathbb{R}^3}, \text{ for all } v \in \widehat{W}^{1,r}(\mathbb{R}^3), \quad r \in (1,\infty). \tag{36}$$

Let $u \in \widehat{W}^{1,q}(\mathbb{R}^3)^3$ be a weak solution of $Lu = 0$ in $\widehat{W}^{1,q}(\mathbb{R}^3)^3$. One can take as a test function to get

$$< Lu, v >= 0.$$

Similarly, one takes u as a test function for (35) in $\widehat{W}^{-1,q/(q-1)}(\mathbb{R}^3)^3$ to obtain

$$< u, L^*v > = < u, \nabla \cdot F > .$$

Therefore,

$$< u, \nabla \cdot F >= 0.$$

Since $F \in (C_0^\infty)^9$ is arbitrary, we obtain $u = 0$ in $\widehat{W}^{1,q}(\mathbb{R}^3)^3$ by Theorem 2. u is a constant vector, but it is a constant multiple of ω because $\omega \wedge u = 0$.

To complete the proof of Theorem 1, we have to show the following lemma

Lemma 3. *Let* $v \in \mathcal{S}(\mathbb{R}^3)$ *be the solution of*

$$-\Delta v + \frac{\partial v}{\partial x_3} - (\omega \wedge x) \cdot \nabla v = 0 \textit{ in } R^3.$$

Then $\operatorname{supp} \widehat{w} \subset \{0\}$.

Proof: This was proved in [Fa2].

Proof of Theorem 1: As we explain before the terms $\omega \wedge x) \times \nabla u - \omega \wedge u$ are divergence free. The pressure is formally obtained from the problem

$$p = -\nabla \cdot (-\Delta)^{-1} f.$$

Since $(-\Delta)^{-1}$ can be justified as a bounded operator from $\widehat{W}^{-1,q}(\mathbb{R}^3)$ to $\widehat{W}^{1,q}(\mathbb{R}^3)$ we get

$$\|\nabla p\|_q \leq c\|f\|_{-1,q},$$

which implies that

$$\|f - \nabla p\|_{-1,q} \leq c\|f\|_{-1,q}.$$

This completes the proof of Theorem 1.

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