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Symbolic Computation of Generalized Transient Visco-Elastic Flow with Variable Viscosity inside a Movable Tube using Computer Algebra

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Abstract: - Using Computer Algebra Software certain work about symbolic computational rheology is realized. The explicit solutions for the transport equation corresponding to the transient flow within a movable tube, for certain visco-elastic fluid with rheological memory and variable viscosity is derived. Four different cases are considered and the corresponding solutions are given in terms of Bessel functions. The first case corresponds to the periodic laminar flow without memory and constant viscosity. The second case is concerned with the periodic laminar flow without memory but with variable viscosity. The third case corresponds to the transient laminar flow with memory and constant viscosity. And the fourth case is concerned with the transient laminar flow with memory but with variable viscosity. The transport equation that is used here, results from the combination of the general form of the equation of continuum mechanics with a rheological model of the Kelvin kind with a generalization using rheological memory. For the problems with periodic flow, the method of solution is the separation of variables and for the problems with transient flow, the method of solution is the Laplace Transform Technique with the application of the Bromwich integral and the residue theorem. The solutions are obtained by means of two algorithm for computer algebra, one for the case of periodic flow and other for the transient flow.. As results we obtain the explicit forms of the velocity fields of the fluid for the four considered cases and also we made a stability analysis of our results both for periodic flow as for transient flow.

Key-Words: - Symbolic Computational Rheology, Computer Algebra, Generalized Maxwell Model, Periodic flow, transient flow, Laplace Transform, Residue Theorem, Bessel Functions, Routh-Hurwitz Theorem.

1. Introduction

Recently we have presented some examples on Symbolic Computational Rheology [1]. In fact the Mathematical Rheology is a source of very interesting computational problems. It is possible at the present time to speak about the Computational Rheology (CR) as a new emergent scientific discipline. Within the domain of CR we can observe two lines of development. The first line is named Numerical Computational Rheology (NCR) and the second line is named Symbolic Computational Rheology (SCR). The NCR is concerned with numerical solutions of non-linear mathematical rheological problems using appropriate software for numerical computation. The SCR is dedicated to obtain analytical solutions for linear mathematical rheological problems using Computer Algebra Sotware (CAS) for symbolic computation [2]. The domain of NCR is the dominant paradigm but the present authors think that the SCR is a land practically unexplored and very interesting. As we said previously, we continue here the work on SCR that was initiated at [1]. In reference [1], certain linear problem for impulse transport for a certain non-newtonian fluid with visco-elastic properties and some kind of rheological memory, was solved using Maple [3]. The mathematical model was a modified Navier-Stokes equation. Now in the present work we consider a more general situation with a more complex mathematical model. Specifically we combine the general movement equations of continuum media with an extended rheological model of the Maxwell kind with memory. We consider both periodic and transient flow in movable tube. Also we consider the cases with homogeneous and inhomogeneous viscosity.

The method that we apply to study the periodic flow is the method of separation of variables but the method that we use to study the transient flow is the Laplace Transform Technique with the Bromwich integral and the residue theorem [4]. The stability analysis of the transient flow is realized using the Routh-Hurwitz theorem. All calculations are implemented using certain Maple algorithm. The computations are very long and heavy as to be made by hand, using only pen and paper. We confirm here the great importance of CAS for SCR.

2 The Mathematical Problem

Here we consider the problem of transient laminar flow for a generalized viscous-elastic fluid inside a movable circular tube with a pressure gradient, with rheological memory and variable viscosity. The two equations that describe the system are [5]

$$\rho\left(\frac{\partial}{\partial t}\mathrm{v}(r,t)\right)+\left(\frac{d}{dz}\mathrm{P}(z)\right)+\frac{\sigma_{r,z}(r,t)+r\left(\frac{\partial}{\partial r}\sigma_{r,z}(r,t)\right)}{r}=0 \qquad (1)$$

$$\sigma_{r,z}(r,t)+\mu(r)\left(\frac{\partial}{\partial r}\mathrm{v}(r,t)\right)+\tau_0\left(\frac{\partial}{\partial t}\sigma_{r,z}(r,t)\right)+\chi\int_0^t \mathrm{e}^{(-\varepsilon\,(t-\tau))}\,\sigma_{r,z}(r,\tau)\,d\tau=0 \qquad (2)$$

where (1) is a general equation of continuum mechanics (Newton Law for continuum media) [5] and (2) is generalized Maxwell model [5] for a incompressible and visco-elastic fluid with rheological memory and with a spatially variable viscosity. In (1), ρ is the density of the fluid, v(r,t) is the velocity of fluid at distance *r* from the axis of the tube of radius *a* at time *t*, P(z) is the pressure of fluid at the plane z and $\sigma_{r,z}$ (r,t) is the stress on fluid at a distance *r* at time *t*. In the equation (2), μ(r) is the viscosity of fluid which is variable on the space, τ_0 is certain characteristic time of the fluid and χ and $\in$ are the parameters that specify the rheological memory assumed exponentially decreasing.

The mathematical problem that is proposed here consists in to obtain the analytical solution for the system of equations (1) and (2) with the following initial and boundary conditions

$$\mathrm{v}(r,0)=\mathrm{f}(r) \qquad (3)$$

$$\sigma_{r,z}(r,0)=\mathrm{g}(r) \qquad (4)$$

$$\mathrm{v}(a,t)=\mathrm{h}(t) \qquad (5)$$

The equation (3) says that fluid starts from the rest. The equation (4) gives the initial profile of stresses within the fluid. The equation (5) says that the velocity of fluid at the wall of tube is justly the velocity of tube like a whole. At general f(r) , g(r) and h(t) are arbitrary functions.

3 Problem Solution

Our rheological model (1)-(5) is a linear model but it can not be solved at general by means of the method of separation of variables. It is necessary to apply the Laplace Transform method, making the inverse transform by means of the theorem of residues [4], when the symbolic computation of the transient flow is required.. Since the necessary manipulations to solve our rheological model are too voluminous as for to be made by hand with pencil and paper, it is necessary to apply some type of system of computer algebra that allows symbolic computation.

3.1. Method of Solution

We intend to solve (1)-(5) both for the periodic flow as for the transient flow including the cases with constant a variable viscosity. For the case of periodic flow we use the method of separation of variables. For the case of transient flow we use the method of Laplace transform with Bromwich-residues . The two methods are implemented at fully using two algorithms of computer algebra

A sketch of the algorithm for computation of the solution for periodic flow is as follows.

1. Assume the following forms of periodic flow with frequency *w* and without memory

$$\sigma_{r,z}(r,t)=\sigma_0(r)\,\mathrm{e}^{(w\,t\,I)} \qquad (6)$$

$$\mathrm{v}(r,t)=\mathrm{v}(r)\,\mathrm{e}^{(w\,t\,I)} \qquad (7)$$

$$\frac{\partial}{\partial z}\mathrm{P}(z,t)=-b_0\,\mathrm{e}^{(w\,t\,I)} \qquad (8)$$

$$\chi=0 \qquad (9)$$

2. Substitute (6) ,(7) and (9) on (2) and express the stress on terms of derivatives of velocity.
3. Substitute the obtained stress in the equation (1) and solve this equation with the

boundary condition $v(a) = \nu$ and the finitude condition $v(0)$ = finite.

4. With the obtained velocity profile to compute the total flux in the tube.
5. Determine the analytical form of the pressure gradient that it is necessary to apply to maintain the periodic flow with a determined total flux.

This algorithm is easily turned into a Maple algorithm.

Now, for the case of transient flow, a sketch of the algorithm is as follows:

The inputs of the algorithm are: *Sys*, that represents the equations (1)-(2); *I.C.* that represents the initial conditions (3)-(4); *B.C.* that represents the boundary condition (5); and *F.C.* that represents a certain finitude condition for the solutions.

The output for the algorithm is the explicit solution of (1)-(5) it is to say the analytical forms fro v(r,t) and $\sigma_{r,z}$ (r,t). Our second algorithm operates at the following way: The inputs *Sys*., *I.C.*, and *B.C*, by means of a *Laplace Transformer* are turned into a transformed system denoted *T.Sys*. and a transformed boundary condition denoted *T.B.C.* Then, *T.Sys*, *T.B.C.* and *F.C.* are processed by a certain *Dsolver* that generates the transformed solution denoted *Tsol*.

Next, *Tsol* is processed by means of an *inverser* with residue theorem, and we obtain the explicit form of the solution, denote *sol*. Finally, a stability analysis based on Routh-Hurwitz theorem is realized and the algorithm is finished.

With the analytical profiles of velocities that are obtained, the total fluxes within the tube are computed and other rheological magnitudes are calculated, such as the necessary gradients to maintain the periodic flow or the distribution of stresses within the fluid for the case of transient flow.

3.2. Results of Computations

Here we present the results of computations for the four considered cases:

1. Periodic flow without memory ($\chi = 0$) with constant viscosity ($\mu(r) = \mu$) and $h(t)= \nu\, e^{i w t}$
2. Periodic flow without memory ($\chi = 0$) with variable viscosity ($\mu(r) = \mu r$) and $h(t)= \nu\, e^{i w t}$
3. Transient flow with memory, constant viscosity ($\mu(r) = \mu$) and $h(t)=\nu\, e^{-\delta t}$
4. Transient flow with memory , variable viscosity (($\mu(r) = \mu r$) and $h(t)=\nu\, e^{-\delta t}$

For this four cases the results that were obtained using our Maple algorithms are as follows.

3.2.1. first case

For the first case the solution is

$$v(r,t) = \left(\frac{(\nu \rho w + b_0 I)\, J_0\!\left(\sqrt{\frac{\rho w(-I+\tau_0 w)}{\mu}}\, r \right)}{J_0\!\left(\sqrt{\frac{\rho w(-I+\tau_0 w)}{\mu}}\, a \right) \rho w} - \frac{b_0 I}{\rho w} \right) e^{(w t I)} \tag{10}$$

where $J_0(x)$ is the Bessel function of the first kind and zero order [6]. Strictly speaking the truly velocity profile is the real part of (10).

3.2.2. second case

For the second case the velocity profile is of the form

$$v(r,t) = v(r)\, e^{(w t I)} \tag{11}$$

where

$$v(r) = \frac{(\sqrt{a}\, b_0 I + \nu \sqrt{a}\, \rho w)\, J_1\!\left(2 \sqrt{\frac{\rho w(-I+\tau_0 w)}{\mu}}\, \sqrt{r} \right)}{\sqrt{r}\, J_1\!\left(2 \sqrt{\frac{\rho w(-I+\tau_0 w)}{\mu}}\, \sqrt{a} \right) \rho w} - \frac{b_0 I}{\rho w} \tag{12}$$

being $J_1(x)$ the Bessel function of the firs kind and order one [6].

3.2.3 third case

For the third case the velocity profile is showed at the first equation of the Figure 1. In such equation, α_n are the zeroes of $J_0(x)$, it is to say $J_0(\alpha_n) = 0$, with $n \geq 1$. The function $\lambda(s)$ is defined as

$$\sqrt{-\frac{\rho\, s\, (s + \varepsilon + \tau_0\, s^2 + \tau_0\, s\, \varepsilon + \chi)}{\mu\, (s + \varepsilon)}} = \lambda(s) \tag{13}$$

and the parameters denoted $S_{i,n}$ are the roots of the cubic equation on s:

$$\sqrt{-\frac{\rho\, s\, (s + \varepsilon + \tau_0\, s^2 + \tau_0\, s\, \varepsilon + \chi)}{\mu\, (s + \varepsilon)}} = \frac{\alpha_n}{a} \tag{14}$$

We use the convention

$$\frac{d}{dS_{i,n}} \lambda(S_{i,n}) = \lim_{s \to S_{i,n}} \frac{d}{ds} \lambda(s) \tag{15}$$

3.2.3 fourth case

For the fourth case the velocity profile is displayed at the first equation of the Figure 1. In such equation, now, α_n are the non-vanishing zeroes of $J_1(x)$, it is to say $J_1(\alpha_n) = 0$, with $n \geq 1$. Again the function $\lambda(s)$ is given by (14) and the parameters $S_{i,n}$ are now the solutions of the following cubic equation

$$\sqrt{-\frac{\rho\, s\,(s+\varepsilon+\tau_0\, s^2+\tau_0\, s\,\varepsilon+\chi)}{\mu\,(s+\varepsilon)}}=\frac{1}{2}\,\frac{\alpha_n}{\sqrt{a}} \tag{16}$$

3.3 Analysis of Results

3.3.1 first case

From the equation (10) the total flux inside the tube is given by the first equation at Figure 3. In this equation BesselJ(0,x) = J_0 (x) and BesselJ(1,x) = J_1(x). Using the computed total flux we can to determine the pressure gradient that is necessary to apply with the aim to maintain the periodic flow with a discharge Q. The result is showed in the first row of the Table 1.

3.3.2 second case

Starting from (12) the total flux inside the tube is calculated and the result is showed by the second equation at Figure 3. For this case the computation of the necessary pressure gradient to maintain the periodic flow is more difficult and such calculation will not be presented here.

3.3.3 third case

The total flux inside the tube for this case is displayed in the second equation of Figure 1. We note that the first equation which gives the velocity profile has three parts: the part 1 is the usual stationary flow in a tube with renormalized viscosity, the part 2 has the same temporal dependence that the boundary condition (5) and the part 3 is the genuine transient flow because the temporal dependence of this part is determined by the rheological properties of the fluid according with (14) and by the Routh-Hurwitz theorem (RHT). The equation (14) can be rewritten as is showed at the second row of Table 1. For hence we have that the first Hurwitz inequality according to the RHT is given by:

$$0<\rho\, a^2\, \tau_0\, \varepsilon+\rho\, a^2$$

The other two Hurwitz inequalities are showed at the second row of Table 1. As the reader can note, given the physical meanings of the parameters, all Hurwitz inequalities are satisfied automatically and then the transient flow is stable, it is to say all roots $S_{i,n}$ have positive real parts.

3.3.4 fourth case

For this case, the total flux is given by the second equation at Figure 2. In this case we have again that the velocity profile is the sum of three parts which are similar to the three parts of the third case. The equation (16) can be rewritten as is displayed at the third row of the Table 1. The first Hurwitz inequality for this case is

$$0<4\,\rho\, a\, \tau_0\, \varepsilon+4\,\rho\, a$$

and the others two inequalities are given at the third row of Table 2. We observe again that the three Hurwitz inequalities are automatically satisfied and for hence the transient flow is stable, it is to say tends to decay when $t \longrightarrow \infty$.

4 Conclusion

This work was a second exploration of the almost virgin land of Symbolic Computational Rheology. The problem that was considered here, was a linear problem, whose solution can be obtained symbolically using certain algorithms of computer algebra. Our principal contributions are the equations (10) and (12) jointly with the figures 1,2, 3 and the Table 1.

The algorithms that were used can be applied to others more complex linear problems with more general boundary conditions. It is evident from this work that computer algebra is very useful to study those problems on mathematical rheology that demands the calculation of the analytical solutions of the modified or generalized Navier-Stokes equations.

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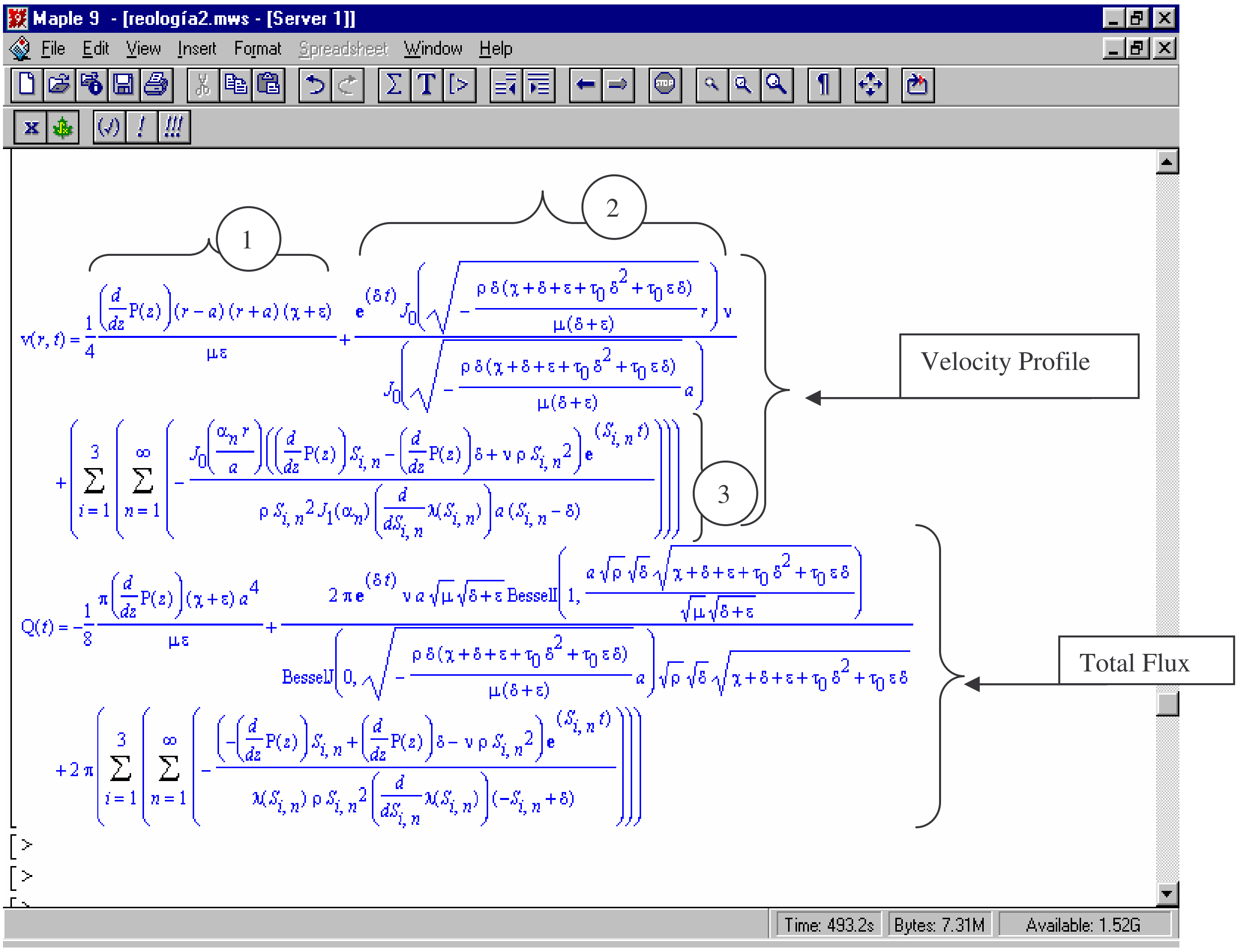

Figure 1. Results of computations for the third case (transient flow with memory and constant viscosity)

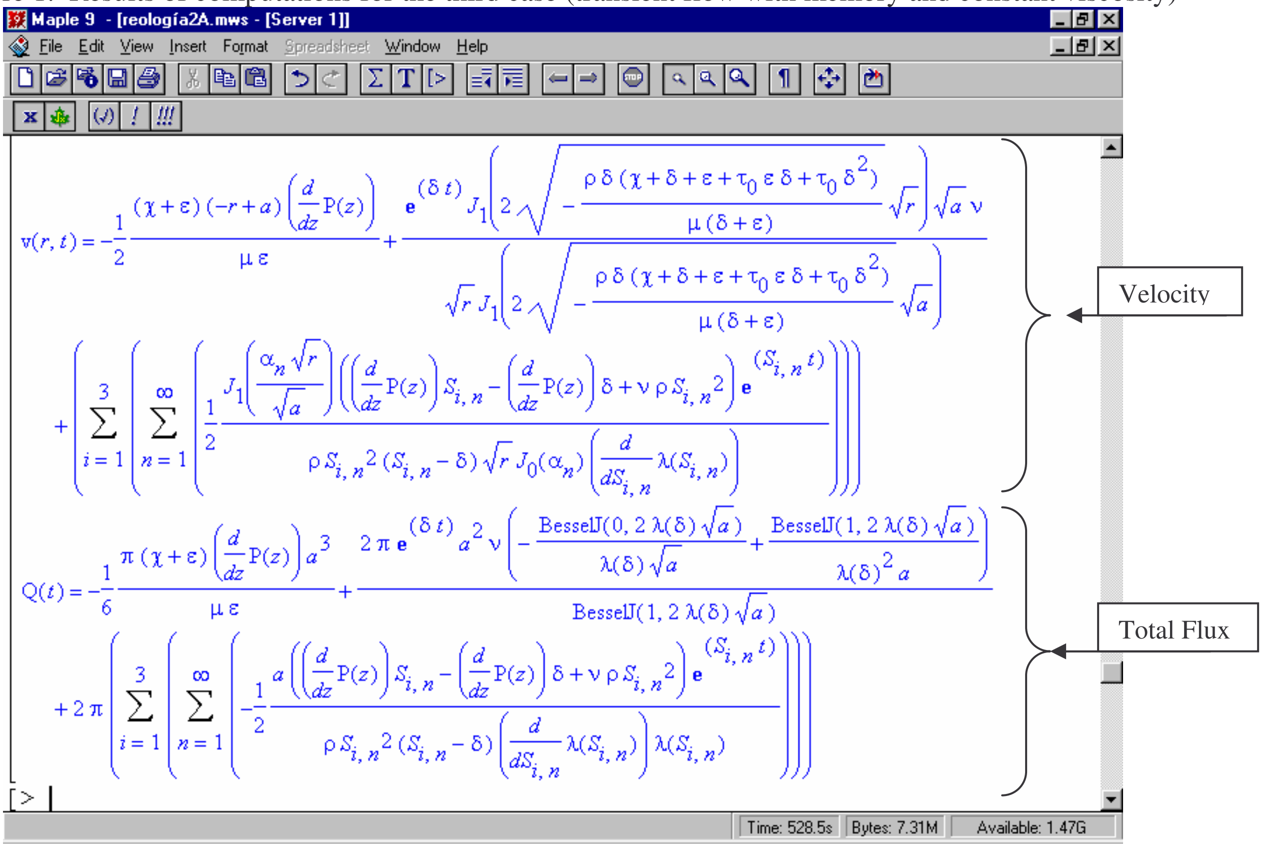

Figure 2. Results of Computation for the fourth case (transient flow with memory and variable viscosity)

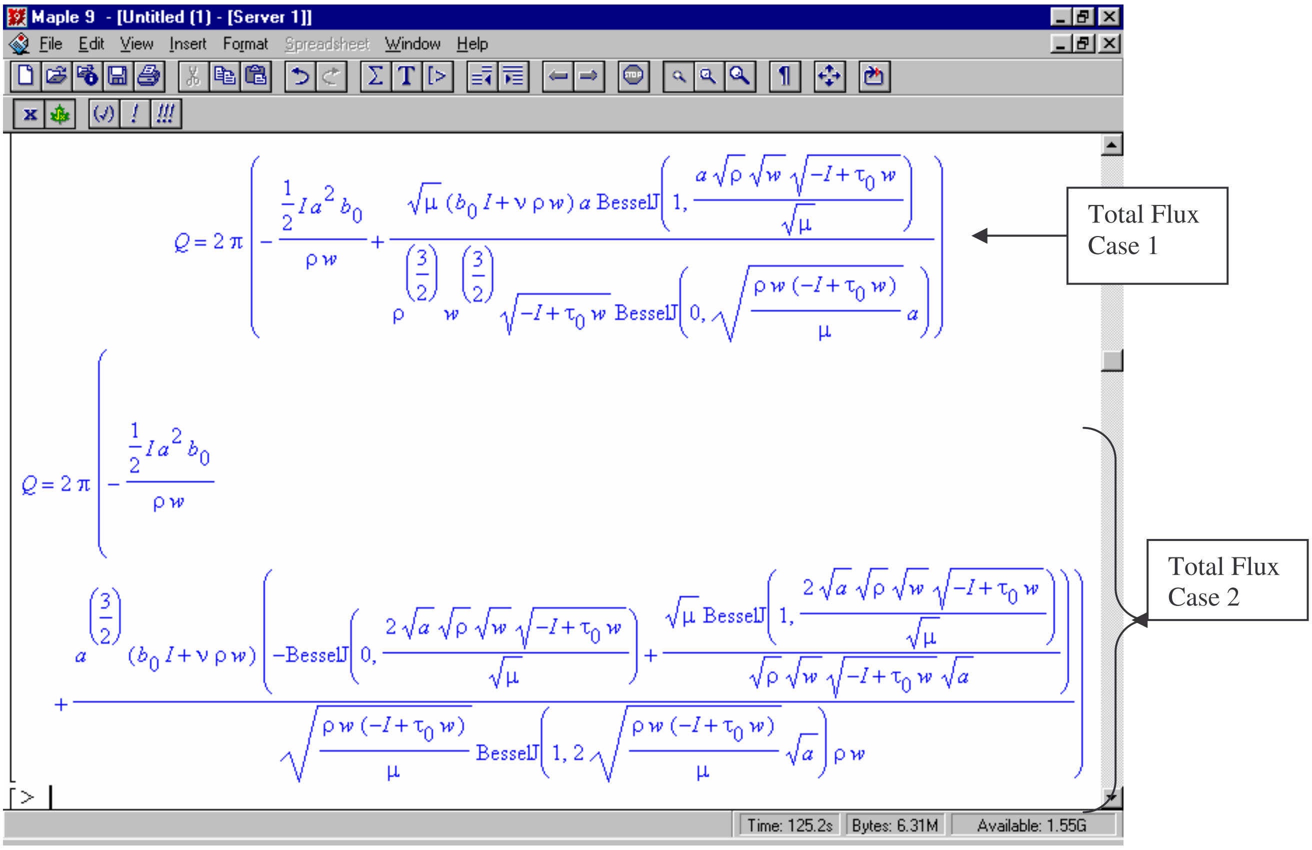

Figure 3. Total fluxes for cases first and second

Table 1. Pressure Gradient for periodic flow and Stability analysis for transient flow

Pressure Gradient for the periodic flow without memory and with constant viscosity $-\left(\frac{d}{dz}\mathrm{P}(z)\right) = -\frac{1}{3}\frac{(-w\,\rho\,\nu\,\pi\,a^4 + 24\,w\,\pi\,a^2\,\nu\,\tau_0\,u - 24\,w\,Q\,\tau_0\,u + 4\,w\,Q\,\rho\,a^2)\sin(w\,t)}{\pi\,a^4} - \frac{1}{3}\frac{(24\,u\,\pi\,a^2\,\nu - 24\,u\,Q)\cos(w\,t)}{\pi\,a^4}$
Characteristic Equation for the transient flow with memory and constant viscosity $\rho\,s^3\,a^2\,\tau_0 + (\rho\,a^2\,\tau_0\,\varepsilon + \rho\,a^2)\,s^2 + (\rho\,a^2\,\chi + \rho\,a^2\,\varepsilon + \alpha_n^2\,\mu)\,s + \alpha_n^2\,\mu\,\varepsilon = 0$ Hurwitz Inequalities for Stability: $0 < \alpha_n^2\,\mu\,\varepsilon^2\,\rho^2\,a^4\,\tau_0\,\chi + \alpha_n^2\,\mu\,\varepsilon^3\,\rho^2\,a^4\,\tau_0 + \alpha_n^2\,\mu\,\varepsilon\,\rho^2\,a^4\,\chi + \alpha_n^2\,\mu\,\varepsilon^2\,\rho^2\,a^4 + \alpha_n^4\,\mu^2\,\varepsilon\,\rho\,a^2$ $0 < \rho^2\,a^4\,\tau_0\,\varepsilon\,\chi + \rho^2\,a^4\,\tau_0\,\varepsilon^2 + \rho^2\,a^4\,\chi + \rho^2\,a^4\,\varepsilon + \rho\,a^2\,\alpha_n^2\,\mu$
Characteristic Equation for the transient flow with memory and variable viscosity $4\,\rho\,s^3\,a\,\tau_0 + (4\,\rho\,a\,\tau_0\,\varepsilon + 4\,a\,\rho)\,s^2 + (4\,\rho\,a\,\chi + 4\,\rho\,a\,\varepsilon + \alpha_n^2\,\mu)\,s + \alpha_n^2\,\mu\,\varepsilon = 0$ Hurwitz Inequalities for Stability $0 < 16\,\rho^2\,a^2\,\tau_0\,\varepsilon\,\chi + 16\,\rho^2\,a^2\,\tau_0\,\varepsilon^2 + 16\,\rho^2\,a^2\,\chi + 16\,\rho^2\,a^2\,\varepsilon + 4\,\rho\,a\,\alpha_n^2\,\mu$ $0 < 16\,\alpha_n^2\,\mu\,\varepsilon^2\,\rho^2\,a^2\,\tau_0\,\chi + 16\,\alpha_n^2\,\mu\,\varepsilon^3\,\rho^2\,a^2\,\tau_0 + 16\,\alpha_n^2\,\mu\,\varepsilon\,\rho^2\,a^2\,\chi + 16\,\alpha_n^2\,\mu\,\varepsilon^2\,\rho^2\,a^2 + 4\,\alpha_n^4\,\mu^2\,\varepsilon\,\rho\,a$