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The Role of Modes in Asymptotic Dynamics of Solutions to the Homogeneous Navier-Stokes Equations

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Abstract: The main goal of the paper is the presentation of several results on the asymptotic dynamics of modes in strong global solutions to the homogeneous Navier-Stokes equations.

Key–Words: Navier-Stokes equations, Strong global solution, Asymptotic dynamics

1 Introduction

Let $T > 0$ and let $\Omega \subset \mathbf{R}^3$ be a smooth bounded domain. We deal with the homogeneous Navier-Stokes initial–boundary value problem which is defined by the equations

$$\frac{\partial w}{\partial t} + (w \cdot \nabla)w = -\nabla p + \nu \Delta w, \qquad (1)$$
$$\nabla \cdot w = 0 \qquad (2)$$

in $Q_T \equiv \Omega \times (0, T)$, by the initial condition

$$w(x, 0) = w_0(x), \quad \text{for every } x \in \Omega \qquad (3)$$

and by the homogeneous Dirichlet boundary conditions

$$w = 0 \quad \text{on } \partial\Omega \times (0, T). \qquad (4)$$

The unknown $w = (w_1, w_2, w_3)$ stands for velocity, p denotes pressure and $\nu > 0$ is the kinematic viscosity. The equations (1) and (2) describe the flow of a Newtonian viscous incompressible fluid (water) and its evolution in time. The Navier-Stokes equations (1) express the conservation of momentum and the equation of continuity (2) expresses the conservation of mass. Although the mathematical theory of the Navier-Stokes equations is deeply elaborated, many important questions still remain open. Especially, the question of the global in time existence of a smooth solution for arbitrarily large smooth initial data has not yet been solved and it belongs to the most challenging open problems od today's theory of partial differential equations. The survey of main results and open problems can be found e.g. in [3].

In this paper we deal with a special problem: we study the decay of a strong global solution of (1) - (4) for t approaching infinity. It is known that in this case every such solution decreases exponentially to zero for $t \mapsto \infty$ (in some norms). So, the nonlinear term $(w \cdot \nabla)w$ is "weak" for large t and does not have the strength to influence the flow in any substantial way. As a consequence, more precise information on the decay of the solution can be obtained.

To describe our problem more precisely, we will write the equations (1) - (4) in the following form (for the explanation see e.g. [2], [3] or [11]):

$$\frac{dw}{dt} + Aw + B(w, w) = 0, \qquad (5)$$
$$w(0) = w_0 \qquad (6)$$

and remind several basic concepts (see also [2], [3], [10] or [11]) concerning the equations (5) and (6):

- $L^2 = L^2(\Omega)$ is the Lebegue space with the norm $\| \cdot \|$.
- $W^{s,q} = W^{s,q}(\Omega), s \geq 0, q \geq 2$, are the Sobolev spaces endowed with the norm $\| \cdot \|_{s,q}$.
- $L^2_\sigma = L^2_\sigma(\Omega)$ is a subspace of $L^2(\Omega)^3$ which contains functions $\mathbf{u}$ whose divergence equals zero in Ω in the sense of distributions and $(\mathbf{u} \cdot \mathbf{n})|_{\partial\Omega} = \mathbf{0}$ in the sense of traces.
- P_σ is the orthogonal projection of $L^2(\Omega)^3$ onto L^2_σ.
- $B(w, w) = P_\sigma(w \cdot \nabla w)$.
- A is the Stokes operator on L^2_σ, $\mathcal{D}(A) = \{u \in (W^{2,2} \cap W^{1,2}_0)^3; \nabla \cdot u = 0\}$, $Aw = -P_\sigma \Delta w$ for every $w \in \mathcal{D}(A)$.
- $A^\alpha, \alpha \geq 0$ are the fractional powers of the Stokes operator.

- $e^{-At}, t \geq 0$, is the Stokes semigroup generated by the Stokes operator A.
- $\lambda_j, j \in N$, are the eigenvalues of A. It is known that $\{\lambda_j\}_{j=1}^{\infty}$ is a non-decreasing sequence of positive numbers and $\lim_{j\to\infty} \lambda_j = \infty$. Let for every $j \in N$, w_j be the eigenfunction of A associated with λ_j. If $w = \sum_j \alpha_j w_j$, let $P_n w = \sum_{j=1}^{n} \alpha_j w_j, \forall n \in N$.

In this paper we deal with strong global solutions of (5) and (6). Let $w_0 \in L^2_\sigma(\Omega)$. A function w from the space $C([0,\infty); L^2_\sigma(\Omega)) \cap C^1((0,\infty); D(A))$ is a strong global solution of (5) and (6), if the equation (5) is fulfilled for every $t > 0$ and $w(0) = w_0$. As was mentioned above it is not known whether or not there exists a strong global solution of (5) and (6) for every initial condition w_0 even if w_0 is smooth. On the other hand, substantial classes of initial conditions yielding strong global solutions of (5) and (6) were described in several papers, see e.g. [1] or [5]. So, the set of strong global solutions of (5) and (6) is sufficiently abundant. Let us also remark that every weak solution w of (5) and (6) (for its definition see e.g. [11]) is strong and global on a time interval $[t_0, \infty)$ if $t_0 = t_0(w)$ is sufficiently large, that is $w \in C([t_0,\infty); L^2_\sigma(\Omega)) \cap C^1(t_0,\infty); D(A))$. So, the results presented in this paper can be easily formulated also for weak solutions of (5) and (6).

Let us now explain the main problem to be solved in this paper. To this purpose, let us consider for a while the Stokes equations, that is the equation (5) is replaced by the equation

$$\frac{dw}{dt} + Aw = 0 \tag{7}$$

and solved together with (6). Suppose now for simplicity that the sequence of the eigenvalues of A is increasing. If $w_0 = \sum_j \alpha_j w_j$ is an initial condition then the solution of (7) and (6) can be written explicitly as

$$w(t) = \sum_j \alpha_j e^{-\lambda_j t} w_j,\ \forall t \geq 0. \tag{8}$$

If $\alpha_1 \neq 0$ then it is the first mode that prevails asymptotically in w for $t \to \infty$, by which we mean that

$$\lim_{t\to\infty} \frac{\|(I-P_1)w(t)\|}{\|P_1 w(t)\|} = 0.$$

Generally, if $\alpha_1 = \alpha_2 = \cdots = \alpha_{k-1} = 0$ for some $k \in N$ and $\alpha_k \neq 0$, then it is the k^{th} mode that prevails asymptotically in w for $t \to \infty$:

$$\lim_{t\to\infty} \frac{\|(I-P_k)w(t)\| + \|P_{k-1}w(t)\|}{\|(P_k - P_{k-1})w(t)\|} = 0.$$

We now ask, if similar results hold also for the strong global solutions of (5) and (6). We will show in Theorem 1, as our basic result, that if w is such a solution, then there exists a unique mode associated with w which prevails asymptotically in w for $t \to \infty$.

The following theorem was proved in [6]: If w is a strong global solution of (5) and (6) then there exist constants $C_0, C_1, C, \delta_0 > 0$ such that $\|w(t)\|\|w(t+\delta)\|^{-1} \leq C_0$, $\|A^{1/2}w(t)\|\|w(t+\delta)\|^{-1} \leq C_1$ and $\|A^{1/2}w(t)\|\|w(t)\|^{-1} \leq C$ for all $\delta \in [0,\delta_0]$ and $t \geq 0$. We will see that the our results also lead to the improvement of this theorem.

Let us now formulate precisely the main result of this paper.

Theorem 1 *Let $w_0 \neq 0$ and w be a strong global solution of the Navier-Stokes equations (5) and (6). Then there exists a unique $n = n(w) \in N$ such that $\lambda_n < \lambda_{n+1}$ and if $\beta \in [0, 5/4)$ then*

$$\lim_{t\to\infty} \frac{\|A^\beta (I-P_n)w(t)\|}{\|A^\beta P_n w(t)\|} = 0. \tag{9}$$

If $l \in N$, $l \geq n$, $\lambda_l < \lambda_{l+1}$ and if $\omega \in (0, \min \lambda_{l+1} - \lambda_n, \lambda_n)$ then even

$$\lim_{t\to\infty} \frac{\|A^\beta (I-P_l)w(t)\|}{\|A^\beta P_n w(t)\|} e^{\omega t} = 0. \tag{10}$$

Let $\lambda_n > \lambda_1$. We denote by $k = k(w)$ the largest natural number such that $\lambda_k < \lambda_n$. ($n-k$ is the dimension of the space of all eigenfunctions associated to the eigenvalue λ_n). If $\alpha \in [0, \lambda_1)$ then

$$\lim_{t\to\infty} \frac{\|A^\beta (P_n - P_k)w(t)\|}{\|A^\beta P_k w(t)\|} e^{-\alpha t} = \infty. \tag{11}$$

Further,

$$\lim_{t\to\infty} \frac{\|A^\beta w(t)\|}{\|w(t)\|} = \lambda_n^\beta \tag{12}$$

and if $\gamma \in [0,\beta]$ then

$$\lim_{t\to\infty} \frac{\|A^\beta w(t)\|}{\|A^\gamma w(t+\delta)\|} = \lambda_n^{\beta-\gamma} e^{\lambda_n \delta} \tag{13}$$

uniformly on the sets $\{\delta; \delta \in [0, L]\}$, for every $L > 0$ and

$$\lim_{\delta\to 0_+} \frac{\|A^\beta w(t)\|}{\|A^\beta w(t+\delta)\|} = 1 \tag{14}$$

uniformly on the set $\{t; t \geq 0\}$. Finally, if $C_0 > 1$, then there exists $\delta_0 > 0$ such that

$$\frac{\|A^\beta w(t)\|}{\|A^\beta w(t+\delta)\|} < C_0, \tag{15}$$

for every $t \geq 0$ and $\delta \in [0, \delta_0]$.

Proceedings of the 2006 IASME/WSEAS International Conference on Continuum Mechanics, Chalkida, Greece, May 11-13, 2006 (pp142-150)

In the paper we will use the following theorem which was presented in [9]:

Theorem 2 *Let* $\varepsilon \in [0, 1/4)$*,* $w_0 \in \mathcal{D}(A^{1+\varepsilon})$ *and* $w_0 \neq 0$*. Let* w *be a strong global solution of the Navier-Stokes equations (5) and (6). Then there exist* $C_0 > 1$ *and* $\delta_0 \in (0, 1)$ *such that*

$$\frac{\|A^{1+\varepsilon}w(t)\|}{\|w(t+\delta)\|} \leq C_0, \ \forall t \geq 0, \ \forall \delta \in [0, \delta_0].$$

2 Proof of Theorem 1

We suppose in this section that w is a fixed strong global solution of (5) and (6) with $w_0 \neq 0$. We want to use Theorem 2 straightforwardly in our proofs and so we also suppose without lack of generality that $w_0 \in \mathcal{D}(A^{1+\varepsilon})$ for every $\varepsilon \in (0, 1/4)$.

Let us present at first a few propositions.

• If $\alpha \in [0, 1)$ then there exists $c_1 > 0$ such that (see [10])

$$\|A^\alpha e^{-At}u\| \leq \frac{c_1}{t^\alpha}\|u\|, \ \forall t > 0, \ \forall u \in L^2_\sigma. \tag{16}$$

• If $\gamma \in [3/4, 1)$ then there exists $c_2 > 0$ such that (see [6])

$$\|B(u, u)\| \leq c_2\|A^{1/2}u\| \ \|A^\gamma u\|, \ \forall u \in \mathcal{D}(A^\gamma). \tag{17}$$

• If $s \in (0, 1)$, then there exist $c_7, c_8 > 0$ such that (see [8])

$$c_7\|u\|_{2s,2} \leq \|A^s u\| \leq c_8\|u\|_{2s,2}, \ \forall u \in \mathcal{D}(A^s). \tag{18}$$

• See [4] for the following equality:

$$\mathcal{D}(A^s) = (W^{2s,2})^3 \cap L^2_\sigma, \ \forall s \in [0, 1/4). \tag{19}$$

• If $s \geq 0$ then there exists $c_6 > 0$ such that (see [4])

$$\|P_\sigma u\|_{s,2} \leq c_6\|u\|_{s,2}, \ \forall u \in (W^{s,2})^3. \tag{20}$$

• Let $s \in [0, 1]$ and $\eta > 3/2$. Then there exists $c_9 > 0$ such that

$$\|u \cdot \nabla u\|_{s,2} \leq c_9\|u\|_{\eta,2} \cdot \|u\|_{1+s,2} \tag{21}$$

for every $u \in (W^{\eta,2})^3 \cap (W^{1+s,2})^3$ (see [4]).

Theorem 1 will be proved at the end of this section as an immediate consequence of the following several lemmas. In the proofs c denotes a generic constant that can change from line to line.

Lemma 3 *Let* $\beta \in [0, 5/4)$*,* $\alpha \in [0, \lambda_1)$ *and* $B_0 > 0$*. Let further* $n \in N$*,* $\lambda_n < \lambda_{n+1}$*, and*

$$\liminf_{t\to\infty} \frac{\|A^\beta P_n w(t)\|}{\|A^\beta w(t)\|} e^{\alpha t} \geq B_0. \tag{22}$$

Then

$$\lim_{t\to\infty} \frac{\|A^\beta P_n w(t)\|}{\|A^\beta w(t)\|} = 1.$$

Proof: Let us put

$$g(t) = \frac{\|A^\beta(I - P_n)w(t)\|}{\|A^\beta P_n w(t)\|}. \tag{23}$$

Let $t \geq 0$ and $\delta \in (0, \delta_0]$. Firstly, it is clear that

$$\frac{\|A^\beta(I - P_n)e^{-A\delta}w(t)\|}{\|A^\beta P_n e^{-A\delta}w(t)\|} \leq e^{(\lambda_{n+1}-\lambda_n)\delta}\frac{\|A^\beta(I - P_n)w(t)\|}{\|A^\beta P_n w(t)\|}.$$

We will use the integral representation of w:

$$A^\beta w(t+\delta) = e^{-A\delta}A^\beta w(t) - \int_0^\delta A^\beta e^{-A(\delta-s)}B(w(t+s), w(t+s))\, ds. \tag{24}$$

If $J = \int_0^\delta \|A^\beta e^{-A(\delta-s)}B(w(t+s), w(t+s))\|ds$, we have

$$g(t+\delta) \leq \frac{\|A^\beta(I - P_n)e^{-A\delta}w(t)\| + J}{\|A^\beta P_n e^{-A\delta}w(t)\| - J} \tag{25}$$

Let $\beta \in (0, 1)$. Applying (16), (17) and Theorem 2, we obtain

$$J \leq c\frac{\delta^{1-\beta}}{1-\beta}\|A^\beta w(t)\| \ \|w(t)\|,$$

where we also used the fact that $t \mapsto \|w(t)\|$ is a decreasing function on $[0, \infty)$. Let $\beta \in [1, 5/4)$. We choose $\xi > 0$ such that $\beta + \xi < 5/4$ and $\gamma \in (3/4, 1)$ and denote $c(s) = c(\delta - s)^{\xi-1}$ and $\varrho = 2(\beta + \xi - 1)$. Using (16), (17), (19), (20), (21), (18) and Theorem 2, we obtain

$$\begin{aligned}
J &\leq \int_0^\delta c(s) \ \|A^{\beta+\xi-1}B(w(t+s), w(t+s)\| \ ds \\
&\leq \int_0^\delta c(s) \ \|B(w(t+s), w(t+s))\|_{\varrho,2} \ ds \leq \\
&\int_0^\delta c(s) \ \|w(t+s) \cdot \nabla w(t+s)\|_{\varrho,2} \ ds \leq \\
&\int_0^\delta c(s) \ \|w(t+s)\|_{2\gamma,2}\|w(t+s)\|_{1+\varrho,2} \ ds \leq \\
&\int_0^\delta c(s) \ \|A^\gamma w(t+s)\| \ \|A^{3/4}w(t+s)\| \ ds \leq \\
&\int_0^\delta c(s) \ \|w(t+s)\| \ \|w(t+s)\| \ ds \leq
\end{aligned}$$

Proceedings of the 2006 IASME/WSEAS International Conference on Continuum Mechanics, Chalkida, Greece, May 11-13, 2006 (pp142-150)

$$\int_0^\delta c(s)\,\|w(t)\|\,\|w(t)\|\,ds \le$$
$$\int_0^\delta c(s)\,\|A^\beta w(t)\|\,\|w(t)\|\,ds \le$$
$$c\frac{\delta^\xi}{\xi}\|A^\beta w(t)\|\,\|w(t)\|.$$

It follows from the assumption (22) that if t is sufficiently large then

$$\|A^\beta w(t)\| \le \frac{2e^{\alpha t}}{B_0} e^{\lambda_n \delta}\|A^\beta P_n e^{-A\delta} w(t)\|.$$

If we denote

$$M = c\delta^\theta \frac{2e^{\alpha t}}{B_0} e^{\lambda_n \delta}\|A^\beta P_n e^{-A\delta} w(t)\|\,\|w(t)\|, \quad (26)$$

we can continue in (25) and get

$$g(t+\delta) \le \frac{\|A^\beta (I-P_n) e^{-A\delta} w(t)\| + M}{\|A^\beta P_n e^{-A\delta} w(t)\| - M} \le$$
$$\frac{g(t) e^{-(\lambda_{n+1}-\lambda_n)\delta}}{1 - c\delta^\theta \|w(t)\| e^{\alpha t}} + \frac{c\delta^\theta \|w(t)\| e^{\alpha t}}{1 - c\delta^\theta \|w(t)\| e^{\alpha t}}, \quad (27)$$

where $\theta = \min(1-\beta, \xi)$. Let us fix $t_0 \ge 0$ such that

$$\alpha_0 = \frac{e^{-(\lambda_{n+1}-\lambda_n)\delta}}{1 - c\delta^\theta \|w(0)\| e^{-(\lambda_1-\alpha)t_0}} < 1$$

Since $\|w(t)\| e^{\alpha t} \le \|w(0)\| e^{-(\lambda_1-\alpha)t}$, we get for every $t \ge t_0$

$$g(t+\delta) \le \alpha_0 g(t) + \frac{c\delta^\theta \|w(0)\| e^{-(\lambda_1-\alpha)t}}{1 - c\delta^\theta \|w(0)\| e^{-(\lambda_1-\alpha)t}}.$$

Now it is clear, that $\limsup_{t\to\infty} g(t) < \infty$ and

$$0 \le \limsup_{t\to\infty} g(t+\delta) \le$$
$$\alpha_0 \limsup_{t\to\infty} g(t) = \alpha_0 \limsup_{t\to\infty} g(t+\delta).$$

Therefore,

$$\lim_{t\to\infty} g(t) = 0.$$

and the proof of Lemma 3 is complete. □

Definition 4 *Let $\beta \in (0, 5/4)$. By the use of Theorem 2 we can define a positive number $C = C(\beta)$ in the following way:*

$$C = C(\beta) = \limsup_{t\to\infty} \frac{\|A^\beta w(t)\|}{\|w(t)\|}.$$

Having defined C it is clear that there exists a unique $n = n(\beta) \in N$ such that

$$\lambda_n^\beta \le C(\beta) < \lambda_{n+1}^\beta. \quad (28)$$

Lemma 5 *Let $\beta \in (0, 5/4)$ and $C = C(\beta)$ and $n = n(\beta)$ be the numbers from Definition 4. Then*

$$\liminf_{t\to\infty} \frac{\|A^\beta P_n w(t)\|}{\|A^\beta w(t)\|} \ge \frac{\lambda_1^{2\beta}(\lambda_{n+1}^{2\beta} - C^2)}{C^2(\lambda_{n+1}^{2\beta} - \lambda_1^{2\beta})} > 0.$$

Proof: Let us define for every $t \ge 0$

$$\delta(t) = \frac{\|A^\beta P_n w(t)\|^2}{\|A^\beta w(t)\|^2}.$$

Then

$$\|w(t)\|^2 = \|P_n w(t)\|^2 + \|(I-P_n)w(t)\|^2 \le$$
$$\frac{\|A^\beta P_n w(t)\|^2}{\lambda_1^{2\beta}} + \frac{\|A^\beta (I-P_n) w(t)\|^2}{\lambda_{n+1}^{2\beta}} =$$
$$\frac{\delta(t)}{\lambda_1^{2\beta}}\|A^\beta w(t)\|^2 + \frac{(1-\delta(t))}{\lambda_{n+1}^{2\beta}}\|A^\beta w(t)\|^2 =$$
$$\|A^\beta w(t)\|^2 \left(\frac{\delta(t)\lambda_{n+1}^{2\beta} + (1-\delta(t))\lambda_1^{2\beta}}{\lambda_1^{2\beta}\lambda_{n+1}^{2\beta}} \right).$$

So, it holds for every nonnegative t that

$$\frac{\|A^\beta w(t)\|^2}{w(t)\|^2} \ge \left(\frac{\lambda_1^{2\beta}\lambda_{n+1}^{2\beta}}{\delta(t)\lambda_{n+1}^{2\beta} + (1-\delta(t))\lambda_1^{2\beta}} \right).$$

By the application of $\limsup_{t\to\infty}$ to the last inequality and using the definition of C we obtain

$$C^2 \ge \limsup_{t\to\infty} \frac{\lambda_1^{2\beta}\lambda_{n+1}^{2\beta}}{\delta(t)\lambda_{n+1}^{2\beta} + (1-\delta(t))\lambda_1^{2\beta}} =$$
$$\frac{\lambda_1^{2\beta}\lambda_{n+1}^{2\beta}}{\alpha\lambda_{n+1}^{2\beta} + (1-\alpha)\lambda_1^{2\beta}}, \quad (29)$$

where we put $\alpha = \liminf_{t\to\infty} \delta(t)$. The assertion of Lemma 5 now follows by elementary computation from (29). □

Corollary 6 *Let $\beta \in (0, 5/4)$ and $n = n(\beta)$ be the number from Definition 4. Then*

$$\lim_{t\to\infty} \frac{\|A^\beta P_n w(t)\|}{\|A^\beta w(t)\|} = 1.$$

Lemma 7 *Let $\beta \in (0, 5/4)$ and $n = n(\beta)$ be the number from Definition 4. Then*

$$\limsup_{t\to\infty} \frac{\|A^\beta w(t)\|}{\|w(t)\|} = \lambda_n^\beta.$$

Proof:

$$\frac{\|A^\beta w(t)\|^2}{\|w(t)\|^2} \leq \frac{\|A^\beta P_n w(t)\|^2}{\|P_n w(t)\|^2} + \frac{\|A^\beta (I-P_n) w(t)\|^2}{\|A^\beta w(t)\|^2} \frac{\|A^\beta w(t)\|^2}{\|w(t)\|^2}.$$

Since $\|A^\beta P_n w(t)\|^2/\|P_n w(t)\|^2 \leq \lambda_n^{2\beta}$, it follows from Lemma 3 and Theorem 2 that

$$\lambda_n^{2\beta} \leq C^2 = \limsup_{t\to\infty} \frac{\|A^\beta w(t)\|^2}{\|w(t)\|^2} \leq \lambda_n^{2\beta}$$

and the proof is complete. □

Lemma 8 *Let $0 < \gamma < \beta < 5/4$ and $n(\beta), n(\gamma)$ be the numbers from Definition 4. Then $n(\beta) = n(\gamma)$.*

Proof: We have

$$\|A^\gamma (I - P_{n(\beta)}) w(t)\| \leq c \|A^\beta (I - P_{n(\beta)}) w(t)\|$$

and

$$\|A^\gamma P_{n(\beta)} w(t)\| \geq \frac{1}{\lambda_{n(\beta)}^{\beta-\gamma}} \|A^\beta P_{n(\beta)} w(t)\|.$$

So,

$$\frac{\|A^\gamma (I - P_{n(\beta)}) w(t)\|}{\|A^\gamma P_{n(\beta)} w(t)\|} \leq \lambda_{n(\beta)}^{\beta-\gamma} \frac{\|A^\beta (I - P_{n(\beta)}) w(t)\|}{\|A^\beta P_{n(\beta)} w(t)\|}$$

and it follows from Corollary 6 that

$$\lim_{t\to\infty} \frac{\|A^\gamma (I - P_{n(\beta)}) w(t)\|}{\|A^\gamma P_{n(\beta)} w(t)\|} = 0. \qquad (30)$$

Since also

$$\frac{\|A^\gamma w(t)\|^2}{\|w(t)\|^2} \leq \frac{\|A^\gamma P_{n(\beta)} w(t)\|^2}{\|P_{n(\beta)} w(t)\|^2} + \frac{\|A^\gamma (I - P_{n(\beta)}) w(t)\|^2}{\|A^\gamma w(t)\|^2} \frac{\|A^\gamma w(t)\|^2}{\|w(t)\|^2},$$

it follows by the application of $\limsup_{t\to\infty}$ to the last inequality and by the use of Lemma 7, (30) and Theorem 2 that

$$\lambda_{n(\gamma)}^\gamma = \limsup_{t\to\infty} \frac{\|A^\gamma w(t)\|}{\|w(t)\|} \leq \lambda_{n(\beta)}^\gamma.$$

Therefore, $n(\gamma) \leq n(\beta)$. Suppose now that $n(\gamma) < n(\beta)$. Then

$$\frac{\|A^\beta (I - P_{n(\gamma)}) w(t)\|^2}{\|A^\beta w(t)\|^2} = \frac{\|A^\beta (P_{n(\beta)} - P_{n(\gamma)}) w(t)\|^2}{\|A^\beta w(t)\|^2} + \frac{\|A^\beta (I - P_{n(\beta)}) w(t)\|^2}{\|A^\beta w(t)\|^2} \leq \lambda_{n(\beta)}^{2(\beta-\gamma)} \times \frac{\|A^\gamma (P_{n(\beta)} - P_{n(\gamma)}) w(t)\|^2}{\|A^\gamma w(t)\|^2} \frac{\|A^\gamma w(t)\|^2}{\|A^\beta w(t)\|^2} + \frac{\|A^\beta (I - P_{n(\beta)}) w(t)\|^2}{\|A^\beta w(t)\|^2}$$

and using Corollary 6 we get

$$\lim_{t\to\infty} \frac{\|A^\beta (I - P_{n(\gamma)}) w(t)\|^2}{\|A^\beta w(t)\|^2} = 0. \qquad (31)$$

Since also

$$\frac{\|A^\beta w(t)\|^2}{\|w(t)\|^2} \leq \frac{\|A^\beta P_{n(\gamma)} w(t)\|^2}{\|P_{n(\gamma)} w(t)\|^2} + \frac{\|A^\beta (I - P_{n(\gamma)}) w(t)\|^2}{\|A^\beta w(t)\|^2} \frac{\|A^\beta w(t)\|^2}{\|w(t)\|^2},$$

using (31) and Theorem 2 we arrive at the inequality

$$\lambda_{n(\beta)}^\beta = \limsup_{t\to\infty} \frac{\|A^\beta w(t)\|}{\|w(t)\|} \leq \lambda_{n(\gamma)}^\beta.$$

Consequently, $n(\beta) \leq n(\gamma)$ which is the contradiction with the assumption that $n(\beta) > n(\gamma)$. Thus, $n(\beta) = n(\gamma)$ and Lemma 8 is proved. □

Definition 9 *Lemma 8 shows that the number n defined uniquely by the inequalities (28) for every $\beta \in (0, 5/4)$ does not depend on β. In fact, it depends only on the solution w. So, to express this dependance, we will write $n(w)$ instead of $n(\beta)$.*

Let $\lambda_{n(w)} > \lambda_1$. We will denote by $k(w)$ the largest natural number such that $\lambda_{k(w)} < \lambda_{n(w)}$. It is clear that $n(w) - k(w)$ is the (finite) dimension of the space of all eigenfunctions associated to the eigenvalue $\lambda_{n(w)}$.

Lemma 10 *Let $\beta \in [0, 5/4)$ and $n = n(w)$ be the number from Definition 9. Then*

$$\lim_{t\to\infty} \frac{\|A^\beta P_n w(t)\|}{\|A^\beta w(t)\|} = 1.$$

Proof: If $\beta > 0$ the proof follows immediately from Corollary 6 and Lemma 8. If $\beta = 0$ the proof is a consequence of the inequality

$$\frac{\|(I-P_n)w(t)\|}{\|P_n w(t)\|} \le c\lambda_n^{1/2}\frac{\|A^{1/2}(I-P_n)w(t)\|}{\|A^{1/2}P_n w(t)\|}.$$

Lemma 11 *Let $n = n(w)$ and $k = k(w)$ be the numbers from Definition 9 with $\lambda_n > \lambda_1$. Let $\alpha \in [0, \lambda_1)$. Then*

$$\liminf_{t\to\infty} \frac{\|A^\beta P_k w(t)\|}{\|A^\beta w(t)\|}e^{\alpha t} = 0, \tag{32}$$

for every $\beta \in [0, 5/4)$.

Proof: Let us fix $\beta \in (0, 5/4)$ and suppose by contradiction that

$$\liminf_{t\to\infty} \frac{\|A^\beta P_k w(t)\|}{\|A^\beta w(t)\|}e^{\alpha t} > 0.$$

We get by Lemma 3 that

$$\lim_{t\to\infty} \frac{\|A^\beta P_k w(t)\|}{\|A^\beta w(t)\|} = 1$$

and equivalently

$$\lim_{t\to\infty} \frac{\|A^\beta (I-P_k) w(t)\|}{\|A^\beta w(t)\|} = 0. \tag{33}$$

It is possible to write

$$\frac{\|A^\beta w(t)\|^2}{\|w(t)\|^2} \le \frac{\|A^\beta P_k w(t)\|^2}{\|P_k w(t)\|^2} + \frac{\|A^\beta (I-P_k) w(t)\|^2}{\|A^\beta w(t)\|^2}\frac{\|A^\beta w(t)\|^2}{\|w(t)\|^2}.$$

By the application of $\limsup_{t\to\infty}$ to the last inequality and the use of Lemma 7, (33) and Theorem 2 we obtain

$$\lambda_n^\beta = \limsup_{t\to\infty}\frac{\|A^\beta w(t)\|}{\|w(t)\|} \le \lambda_k^\beta$$

and it is the contradiction with the fact that $\lambda_k < \lambda_n$. So, (32) holds for every $\beta \in (0, 5/4)$. Finally, let $\beta = 0$. Then

$$\frac{\|P_k w(t)\|e^{\alpha t}}{\|w(t)\|} \le c\frac{\|A^{1/2}P_k w(t)\|e^{\alpha t}}{\|A^{1/2}w(t)\|}\frac{\|A^{1/2}w(t)\|}{\|w(t)\|}$$

and if we now apply $\liminf_{t\to\infty}$ to the last inequality and use (32) for $\beta = 1/2$ and Theorem 2, we get

$$\liminf_{t\to\infty}\frac{\|P_k w(t)\|}{\|w(t)\|}e^{\alpha t} = 0$$

and Lemma 11 is proved also for $\beta = 0$. □

Lemma 12 *Let $n = n(w)$ and $k = k(w)$ be the numbers from Definition 9 with $\lambda_n > \lambda_1$. Let $\alpha \in [0, \lambda_1)$. Then*

$$\lim_{t\to\infty}\frac{\|A^\beta (P_n - P_k)w(t)\|}{\|A^\beta P_k w(t)\|}e^{-\alpha t} = \infty,$$

for every $\beta \in [0, 5/4)$.

Proof: Let us denote

$$g(t) = \frac{\|(P_n - P_k)w(t)\|}{\|P_k w(t)\|}.$$

Firstly, due to Lemmas 11 and 10

$$\liminf_{t\to\infty}\frac{\|P_k w(t)\|}{\|P_n w(t)\|}e^{\alpha t} = \liminf_{t\to\infty}\frac{\|P_k w(t)\|e^{\alpha t}}{\|w(t)\|}\frac{\|w(t)\|}{\|P_n w(t)\|} = 0.$$

It implies that

$$\liminf_{t\to\infty}\frac{\|P_k w(t)\|}{\|(P_n - P_k)w(t)\|}e^{\alpha t} = \liminf_{t\to\infty}\left(\frac{\|P_k w(t)\|^2 e^{2\alpha t}}{\|P_n w(t)\|^2 - \|P_k w(t)\|^2}\right)^{1/2} = 0$$

and

$$\limsup_{t\to\infty} g(t)e^{-\alpha t} = \infty. \tag{34}$$

We will prove now that

$$\lim_{t\to\infty} g(t)e^{-\alpha t} = \infty. \tag{35}$$

Let us suppose by contradiction that (35) does not hold. Due to (34) then there exists an increasing sequence $\{t_j\}_{j=1}^\infty$ of positive numbers and $K > 1$ such that $t_j \to \infty$ for $j \to \infty$ and

$$g(t_j)e^{-\alpha t_j} = K,\ \forall j \in N.$$

Suppose now that for some sufficiently large t (which will be specified later) we have

$$g(t)e^{-\alpha t} = K. \tag{36}$$

Choose some $\delta \in (0, \delta_0]$ (δ_0 is from Theorem 2) and use the integral identity (24) for $\beta = 0$. We get

$$g(t+\delta) \le \frac{\|(P_n - P_k)e^{-A\delta}w(t)\| + J}{\|P_k e^{-A\delta}w(t)\| - J},$$

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where $J = \int_0^\delta \|B(w(t+s), w(t+s))\|ds$. Since

$$J \leq c\int_0^\delta \|A^{3/4}w(t+s))\|^2 ds \leq$$
$$c\int_0^\delta \|w(t+s)\|^2 ds \leq c\delta\|w(t)\|^2$$

and by Lemma 10 and (36)

$$\|w(t)\| \leq \sqrt{2}\|P_n w(t)\| =$$
$$\sqrt{2}\sqrt{e^{2\alpha t}K^2+1}\|P_k w(t)\| \leq$$
$$\sqrt{2}\sqrt{e^{2\alpha t}K^2+1}\, e^{\lambda_k\delta}\|P_k e^{-A\delta}w(t)\| \leq$$
$$cKe^{\alpha t}\|P_k e^{-A\delta}w(t)\|,$$

we have, using again (36) and denoting

$$M = cKe^{\alpha t}\delta\|P_k e^{-A\delta}w(t)\|\|w(t)\| \qquad (37)$$

that

$$g(t+\delta) \leq \frac{\|(P_n-P_k)e^{-A\delta}w(t)\| + M}{\|P_k e^{-A\delta}w(t)\| - M} \leq$$
$$\frac{\|(P_n-P_k)e^{-A\delta}w(t)\|}{\|P_k e^{-A\delta}w(t)\|}\frac{1}{1-cK\delta e^{\alpha t}\|w(t)\|} +$$
$$\frac{cK\delta e^{\alpha t}\|w(t)\|}{1-cK\delta e^{\alpha t}\|w(t)\|} \leq \frac{g(t)e^{-(\lambda_n-\lambda_k)\delta}}{1-cK\delta e^{\alpha t}\|w(t)\|} +$$
$$\frac{cK\delta e^{\alpha t}\|w(t)\|}{1-cK\delta e^{\alpha t}\|w(t)\|} =$$
$$K\frac{e^{-(\lambda_n-\lambda_k)\delta}e^{\alpha t} + c\delta e^{\alpha t}\|w(t)\|}{1-cK\delta e^{\alpha t}\|w(t)\|}.$$

Multiplying the last inequality by $e^{-\alpha(t+\delta)}$, we get

$$g(t+\delta)e^{-\alpha(t+\delta)} \leq$$
$$K\frac{e^{-(\alpha+\lambda_n-\lambda_k)\delta} + c\delta\|w(t)\|}{1-cK\delta e^{\alpha t}\|w(t)\|}.$$

If we now denote

$$f(\delta) = \frac{e^{-(\alpha+\lambda_n-\lambda_k)\delta} + c\delta\|w(t)\|}{1-cK\delta e^{\alpha t}\|w(t)\|},$$

then $f(0) = 1$ and $f'(0) = cKe^{\alpha t}\|w(t)\| + c\|w(t)\| - (\alpha+\lambda_n-\lambda_k)$. So, there exists $t^* > 0$ such that $f'(0) < 0$ for $t \geq t^*$, which yields the existence of $\delta_1 = \delta_1(t) > 0$ such that

$$g(t+\delta)e^{-\alpha(t+\delta)} < K, \qquad (38)$$

for every $\delta \in (0, \delta_1)$. Let $j_0 \in N$ be such a number that $t_{j_0} \geq t^*$. Then

$$g(t_{j_0}+\delta)e^{-\alpha(t_{j_0}+\delta)} < K, \qquad (39)$$

for every $\delta \in (0, \delta_1(t_{j_0}))$. Let us put

$$T = \sup\left\{t \geq t_{j_0}; g(\xi)e^{-\alpha\xi} \leq K,\ \forall\xi \in [t_{j_0}, t]\right\}.$$

We have, due to (34) and (39), that $T \in (t_{j_0}, \infty)$ and

$$g(T)e^{-\alpha T} = K.$$

$T \geq t^*$, so using now (38) for $t = T$, we get

$$g(T+\delta)e^{-\alpha(T+\delta)} < K,\ \forall\delta \in (0, \delta_1(T))$$

and it is the contradiction with the definition of T. Therefore, (35) is proved. Finally, for every $\beta \in [0, 5/4)$

$$\lim_{t\to\infty}\frac{\|A^\beta(P_n-P_k)w(t)\|}{\|A^\beta P_k w(t)\|}e^{-\alpha t} \geq$$
$$\lim_{t\to\infty}\frac{\lambda_n^\beta\|(P_n-P_k)w(t)\|}{\lambda_k^\beta\|P_k w(t)\|}e^{-\alpha t} = \infty$$

and Lemma 12 is proved. □

Lemma 13 *Let $n = n(w)$ be the number from Definition 9. Then*

$$\lim_{t\to\infty}\frac{\|A^\beta w(t)\|}{\|w(t)\|} = \lambda_n^\beta,\ \forall\beta \in [0, 5/4).$$

Proof: Let $\lambda_{n(w)} = \lambda_1$. It follows then from Lemma 7 that

$$\lambda_1^\beta \leq \limsup_{t\to\infty}\frac{\|A^\beta w(t)\|}{\|w(t)\|} = \lambda_1^\beta$$

and Lemma 13 is proved immediately. Suppose that $\lambda_{n(w)} > \lambda_1$. By the use of Lemma 10 we get

$$\lim_{t\to\infty}\frac{\|A^\beta w(t)\|^2}{\|w(t)\|^2} = \lim_{t\to\infty}\frac{\|A^\beta P_k w(t)\|^2}{\|P_n w(t)\|^2} +$$
$$\lim_{t\to\infty}\frac{\|A^\beta(P_n-P_k)w(t)\|^2}{\|P_n w(t)\|^2} +$$
$$\lim_{t\to\infty}\frac{\|A^\beta(I-P_n)w(t)\|^2}{\|P_n w(t)\|^2}.$$

It follows from Lemma 12 and Lemma 10 that the first and third term from the right hand side of the last equality equal to zero and by the application of Lemma 12 the second term is equal to $\lambda_n^{2\beta}$. So, Lemma 13 is proved. □

If $\varepsilon \in (0, \lambda_n)$, then according to Lemma 13 there exists $t_0 = t_0(\varepsilon) > 0$ such that

$$\lambda_n - \varepsilon \leq \frac{\|A^{1/2}w(t)\|^2}{\|w(t)\|^2} \leq \lambda_n + \varepsilon,$$

for every $t \geq t_0$. It follows from (5) that

$$\frac{d}{dt}\|w(t)\|^2 + 2\|A^{1/2}w(t)\|^2 = 0$$

and therefore

$$\frac{d}{dt}\|w(t)\|^2 + 2(\lambda_n - \varepsilon)\|w(t)\|^2 \leq 0,$$
$$\frac{d}{dt}\|w(t)\|^2 + 2(\lambda_n + \varepsilon)\|w(t)\|^2 \geq 0,$$

for every $t \geq t_0$. It leads to the inequalities

$$\|w(t+\delta)\| \leq \|w(t)\|e^{-(\lambda_n-\varepsilon)\delta},$$
$$\|w(t+\delta)\| \geq \|w(t)\|e^{-(\lambda_n+\varepsilon)\delta}$$

and

$$e^{(\lambda_n-\varepsilon)\delta} \leq \frac{\|w(t)\|}{\|w(t+\delta)\|} \leq e^{(\lambda_n+\varepsilon)\delta}, \quad (40)$$

which hold for every $t \geq t_0$ and every $\delta > 0$. We will now use (40) to derive the following lemmas.

Lemma 14 *Let $n = n(w)$ be the number from Definition 9. Then for every $\beta \in [0, 5/4)$*

$$\lim_{t\to\infty} \frac{\|A^\beta w(t)\|}{\|w(t+\delta)\|} = \lambda_n^\beta e^{\lambda_n\delta} \quad (41)$$

and

$$\lim_{t\to\infty} \frac{\|A^\beta w(t)\|}{\|A^\beta w(t+\delta)\|} = e^{\lambda_n\delta}, \quad (42)$$

uniformly on the sets $\{\delta; \delta \in [0, L]\}$, for every $L > 0$.

Proof: By the application of $\lim_{t\to\infty}$ in (40) we obtain that

$$\lim_{t\to\infty} \frac{\|w(t)\|}{\|w(t+\delta)\|} = e^{\lambda_n\delta}, \quad (43)$$

uniformly on the sets $\{\delta; \delta \in [0, L]\}$, for every $L > 0$. Now we can write

$$\frac{\|A^\beta w(t)\|}{\|w(t+\delta)\|} = \frac{\|A^\beta w(t)\|}{\|w(t)\|}\,\frac{\|w(t)\|}{\|w(t+\delta)\|}$$

and

$$\frac{\|A^\beta w(t)\|}{\|A^\beta w(t+\delta)\|} = \frac{\|A^\beta w(t)\|}{\|w(t)\|} \times$$
$$\frac{\|w(t)\|}{\|w(t+\delta)\|}\,\frac{\|w(t+\delta)\|}{\|A^\beta w(t+\delta)\|} \quad (44)$$

and get (41) and (42) by the application of Lemma 13 and (43). □

Lemma 15 *Let $\beta \in [0, 5/4)$. Then*

$$\lim_{\delta\to 0_+} \frac{\|A^\beta w(t)\|}{\|A^\beta w(t+\delta)\|} = 1,$$

uniformly on the set $\{t; t \geq 0\}$.

Proof: By the application of $\lim_{\delta\to 0_+}$ in (40) we can obtain that

$$\lim_{\delta\to 0_+} \frac{\|w(t)\|}{\|w(t+\delta)\|} = 1, \quad (45)$$

uniformly on the set $\{t; t \geq 0\}$. Let ε be an arbitrary small positive number. Due to (44), (45) and Lemma 13 there exist $t_1 > 0$ and $\delta_1 > 0$ such that

$$\left|\frac{\|A^\beta w(t)\|}{\|A^\beta w(t+\delta)\|} - 1\right| < \varepsilon,$$

for every $t \geq t_1$ and $\delta \in [0, \delta_1]$. Because of the continuity of the function $\|A^\beta w(\cdot)\|$ on $[0, \infty)$, we also have the existence of δ_2 such that

$$\left|\frac{\|A^\beta w(t)\|}{\|A^\beta w(t+\delta)\|} - 1\right| < \varepsilon,$$

for every $t \in [0, t_1]$ and $\delta \in [0, \delta_2]$. If we put $\delta_3 = \min(\delta_1, \delta_2)$ then for every $t \geq 0$ and $\delta \in [0, \delta_3]$

$$\left|\frac{\|A^\beta w(t)\|}{\|A^\beta w(t+\delta)\|} - 1\right| < \varepsilon.$$

Since ε was arbitrary, Lemma 15 is proved. □

The following corollary of Lemma 15 is an improvement of Theorem 5 from [6].

Corollary 16 *Let $C_0 > 1$ and $\beta \in [0, 5/4)$. Then there exists $\delta_0 > 0$ such that*

$$\frac{\|A^\beta w(t)\|}{\|A^\beta w(t+\delta)\|} < C_0, \ \forall t \geq 0, \ \forall \delta \in [0, \delta_0].$$

Lemma 17 *Let $n = n(w)$ be the number from Definition 9, $l \in N$, $l \geq n$, $\lambda_l < \lambda_{l+1}$ and $\beta \in [0, 5/4)$. If $\omega \in (0, \min(\lambda_{l+1} - \lambda_n, \lambda_n))$ then*

$$\lim_{t\to\infty} \frac{\|A^\beta (I - P_l)w(t)\|}{\|A^\beta P_n w(t)\|} e^{\omega t} = 0.$$

Proof: We proceed similarly as in the proof of Lemma 3 up to the inequality (27). Instead of (27) we get

$$\frac{\|A^\beta (I - P_l)w(t+\delta)\|}{\|A^\beta P_n w(t+\delta)\|} \leq \frac{\|A^\beta (I - P_l)w(t)\|}{\|A^\beta P_n w(t)\|} \times$$
$$\frac{e^{-(\lambda_{l+1}-\lambda_n)\delta}}{1 - c\delta^\theta\|w(t)\|} + \frac{c\delta^\theta\|w(t)\|}{1 - c\delta^\theta\|w(t)\|}, \quad (46)$$

It follows now from Lemma 13 used for $\beta = 1/2$ and by some elementary computation that

$$\lim_{t\to\infty} \|w(t)\| e^{\omega t} = 0, \qquad (47)$$

where we used the assumption that $\omega < \lambda_n$. Let us put $g(t) = \|A^\beta(I-P_l)w(t)\|/\|A^\beta P_n w(t)\|$. Multiplying now (46) by $e^{\omega(t+\delta)}$ we get the inequality

$$g(t+\delta)e^{\omega(t+\delta)} \le \frac{e^{(\omega-(\lambda_{l+1}-\lambda_n))\delta}}{1-c\delta^\theta\|w(t)\|} g(t)e^{\omega t} + \frac{c\delta^\theta\|w(t)\|e^{\omega t}}{1-c\delta^\theta\|w(t)\|},$$

which holds for t sufficiently big. Let us fix $t_0 \ge 0$ such that $\alpha_0 = e^{[\omega-(\lambda_{l+1}-\lambda_n)]\delta}/(1-c\delta^\theta\|w(t_0)\|) < 1$ and put $\kappa = c\delta^\theta/(1-c\delta^\theta\|w(t_0)\|)$. So, we have

$$g(t+\delta)e^{\omega(t+\delta)} \le \alpha_0 g(t)e^{\omega(t)} + \kappa\|w(t)\|e^{\omega t}$$

for every $t \ge t_0$. Applying now $\limsup_{t\to\infty}$ to the last inequality and using (47) we can conclude immediately, that $\lim_{t\to\infty} g(t)e^{\omega t} = 0$. Lemma 17 is proved.

Proof of Theorem 1 Let the assumptions of Theorem 1 be satisfied, that is w is a strong global solution of the Navier-Stokes equations (5) and (6) with $w_0 \neq 0$. Since $w(t) \in D(A)^{1+\varepsilon}$ for every $\varepsilon \in [0, 1/4)$ and every $t \in (0,\infty)$, we can suppose without lack of generality that $w_0 \in D(A)^{1+\varepsilon}$. Let $n = n(w)$ and $k = k(w)$ be the numbers from Definition 9. Then (9) is a consequence of Lemma 10, (10) is a consequence of Lemma 17, (11) is a consequence of Lemma 12, (12) is a consequence of Lemma 13, (13) is a consequence of Lemma 14, (14) is a consequence of Lemma 15 and (15) is a consequence of Corollary 16. Theorem 1 is completely proved. □

3 Conclusion

Let us present, without proof, several additional results and one open problem. Let $n \in N$ and $\lambda_n < \lambda_{n+1}$. We define $G_n \subset D(A^\gamma)$ in the following way: $w_0 \in D(A^\gamma)$ belongs to G_n if and only if there exists a strong global solution w of (5) and (6) such that $w(0) = w_0$ and $n(w) = n$.

It is possible to show that G_n is not empty for any $n \in N$. In fact, every G_n is infinite, since if $w_0 \in G_n$ then evidently $w(t) \in G_n$ for every $t \ge 0$, where w is the strong global solution w of (5) and (6) with the initial condition w_0. Moreover, a certain subset of G_n is a part of a Lipschitz manifold in $D(A^\gamma)$.

Is it possible to prove some results concerning the "size" of the sets G_n? Is it true, for example, that $\overline{D(A^\gamma) \setminus \overline{G}_n} = D(A^\gamma)$ for $n \ge 2$?

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