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The Flow in a Profile Cascade with Separate Boundary Conditions for Vorticity and Bernoulli's Pressure on the Outflow

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Abstract: - We formulate strong and weak problems which are mathematical models of a flow of a viscous incompressible fluid through a profile cascade. We consider separate Dirichlet type boundary conditions for vorticity and pressure on the outflow. We prove the existence of a weak solution.

Key-Words: - Navier–Stokes equations, cascade of profiles, weak solution

1 Introduction

Modelling of a viscous incompressible flow in a 2D cascade represents a complicated theoretical problem especially due to the variety of boundary conditions on parts of the boundary of the flow field. Of first works, treating this subject, we can cite E. Martensen [11] and E. Meister [12]. While the boundary conditions on the inflow and on a profile are of the Dirichlet type, the reduction of the problem to one space period leads to a condition of a space–periodicity on another part of the boundary and finally, a different boundary condition is reasonable on the outflow. Concerning the situation on the outlet, the flow through a cascade has similar features as a flow through a channel. J. Heywood, R. Rannacher and S. Turek [4] explicitly did not involve any boundary condition on the outflow into the weak formulation and by means of a backward integration by parts have shown that this induces the so called "do nothing" boundary condition

$$-\nu \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}} + p\boldsymbol{n} = \boldsymbol{0}. \tag{1}$$

Here $\boldsymbol{u} = (u_1, u_2)$ is the velocity, p is the kinematic pressure and $\boldsymbol{n}$ denotes the outer normal to the boundary. However, this approach causes difficulties in attempts to prove the existence of a weak solution because condition (1) does not exclude a backward flow on the assumed outlet and the backward flow can eventually bring a non–controllable amount of kinetic energy back to the channel. Thus, the energy estimate breaks down. This problem can be avoided by appropriate tricks: S. Kračmar and J. Neustupa [5], [6] prescribed an additional boundary condition which restricted the kinetic energy brought back on the outflow and they therefore described and solved the problem by means of variational inequalities of the Navier–Stokes type. P. Kučera and Z. Skalák [7] solved the problem for "small" data. In our paper [2], we have subtracted the term $\frac{1}{2}\,(\boldsymbol{u}\cdot\boldsymbol{n})^-\boldsymbol{u}$ (where the superscript $-$ denotes the negative part) from the left–hand side of (1) and we obtained the boundary condition

$$-\nu \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}} + p\boldsymbol{n} - \frac{1}{2}\,(\boldsymbol{u}\cdot\boldsymbol{n})^-\boldsymbol{u} = \boldsymbol{h}. \tag{2}$$

This condition also enables us to restrict the kinetic energy brought back by the backward flow on the outlet and consequently, to derive the energy estimate and to prove the existence of a weak solution.

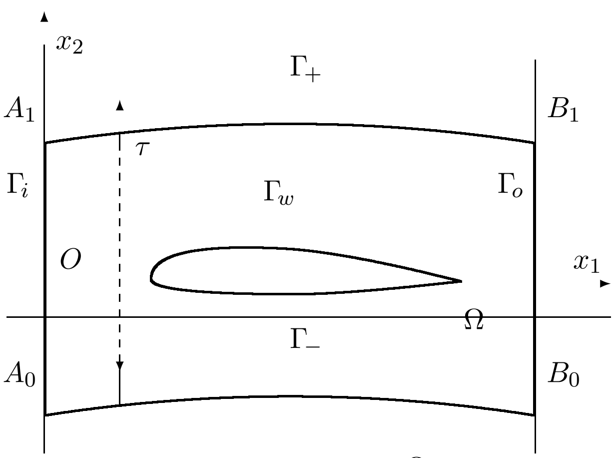

Figure 1: Domain Ω

The basic domain, which represents one spatial period of the flow field in the profile cascade, is denoted by Ω. The boundary condition used on the inflow Γ_i is

$$\boldsymbol{u}|_{\Gamma_i} = \boldsymbol{g}, \tag{3}$$

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where $\boldsymbol{g}$ represents the known distribution of the velocity. The boundary condition used on the profile Γ_w is the no–slip boundary condition

$$\boldsymbol{u}\,|_{\Gamma_w} \;=\; \boldsymbol{0}. \tag{4}$$

Furthermore, we consider the conditions of periodicity

$$\boldsymbol{u}(x_1, x_2+\tau) \;=\; \boldsymbol{u}(x_1, x_2), \tag{5}$$

$$\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}}(x_1, x_2+\tau) \;=\; -\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}}(x_1, x_2) \tag{6}$$

$$p(x_1, x_2+\tau) \;=\; p(x_1, x_2) \tag{7}$$

for $\boldsymbol{x} \equiv (x_1, x_2)$ on the artificial boundary Γ_-.

In [3], we solved the same problem with the boundary condition

$$-\nu\,\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}} + q\boldsymbol{n} \;=\; \boldsymbol{h} \tag{8}$$

(where $q = p + \frac{1}{2}\,|\boldsymbol{u}|^2$ is the so called Bernoulli's pressure) on the output Γ_o.

In the present paper, we study the 2D steady Navier–Stokes equation in the form

$$\omega(\boldsymbol{u})\,\boldsymbol{u}^{\perp} \;=\; -\nabla q + \nu\,(-\partial_2, \partial_1)\,\omega(\boldsymbol{u}) + \boldsymbol{f} \tag{9}$$

where $\omega(\boldsymbol{u}) = \partial_1 u_2 - \partial_2 u_1$ and $\boldsymbol{u}^{\perp} = (-u_2, u_1)$. $\omega(\boldsymbol{u})$ denotes the vorticity of the flow. The condition of incompressibility says that

$$\operatorname{div} \boldsymbol{u} \;=\; 0. \tag{10}$$

The condition used on Γ_o arises similarly as the "do nothing" condition (1) from the weak formulation of the problem which will be given in the next section. However, we can note that if the weak solution $\boldsymbol{u}$ is "smooth enough" then $\boldsymbol{u}$ and q satisfy

$$q = h_1, \qquad -\omega(\boldsymbol{u}) = h_2 \tag{11}$$

where $\boldsymbol{h} = (h_1, h_2)$ is a given function on Γ_o.

2 Weak formulation of the boundary–value problem

We denote by $(\,.\,,\,.\,)_0$ the scalar product of scalar–valued (respectively vector–valued, respectively tensor–valued) functions in $L^2(\Omega)$ or in $L^2(\Omega)^2$ or in $L^2(\Omega)^4$. $H^1(\Omega)$ is the usual Sobolev space with the scalar product $(\,.\,,\,.\,)_1$. We put $H^1(\Omega)^2 := H^1(\Omega) \times H^1(\Omega)$, the space of vector functions whose components belong to $H^1(\Omega)$, with the scalar product which is again denoted by $(\,.\,,\,.\,)_1$. The corresponding norm will be denoted by $\|\,.\,\|_1$. The symbol $\|\,.\,\|_{s;\,\partial\Omega}$ denotes the norm in the the Sobolev–Slobodetski space $H^s(\partial\Omega)$ or in $H^s(\partial\Omega)^2$. Furthermore, we use the following spaces and notation.

- $\mathcal{X} = \{\boldsymbol{v} \in C^{\infty}(\overline{\Omega})^2;\ \boldsymbol{v} = \boldsymbol{0}$ on $\Gamma_i \cup \Gamma_w,\ \boldsymbol{v}(x_1, x_2+\tau) = \boldsymbol{v}(x_1, x_2)\ \forall\,(x_1, x_2) \in \Gamma_-\}$
- X is the closure of $\mathcal{X}$ in $H^1(\Omega)^2$.
- $\mathcal{V} = \{\boldsymbol{v} \in \mathcal{X};\ \operatorname{div} \boldsymbol{v} = 0 \text{ in } \Omega\}$
- V is the closure of $\mathcal{V}$ in $H^1(\Omega)^2$.

It can be shown that

$$X \;=\; \{\boldsymbol{v} \in H^1(\Omega)^2;\ \boldsymbol{v} = \boldsymbol{0} \text{ in } \Gamma_i \cup \Gamma_w,\ \boldsymbol{v}(x_1, x_2+\tau) \;=\; \boldsymbol{v}(x_1, x_2) \text{ for } (x_1, x_2) \in \Gamma_-\}.$$

The identities on Γ_i, Γ_w and Γ_- are valid in the sense of traces. Using a standard procedure, we can prove that

$$V \;=\; \{\boldsymbol{v} \in X;\ \operatorname{div} \boldsymbol{v} = 0 \text{ a.e. in } \Omega\}.$$

Space V will be equipped by the norm $|||\,.\,|||$, induced by the scalar product

$$(\boldsymbol{u}, \boldsymbol{v})_V = (\nabla \boldsymbol{u}, \nabla \boldsymbol{v})_0\,. \tag{12}$$

It can be shown that the norm $|||\,.\,|||$ is equivalent with the norm $\|\,.\,\|_1$ in V.

In order to derive formally the weak formulation of the problem (3)–(7), (9)–(11), we multiply equation (9) by an arbitrary test function $\boldsymbol{v} = (v_1, v_2) \in V$, integrate over Ω and apply Green's theorem and use the boundary conditions and the conditions of periodicity (3)–(7). We finally arrive at the equation

$$a(\boldsymbol{u}, \boldsymbol{v}) \;=\; (\boldsymbol{f}, \boldsymbol{v})_0 + b(\boldsymbol{h}, \boldsymbol{v}), \tag{13}$$

where

$$\begin{aligned}
a_1(\boldsymbol{u}, \boldsymbol{v}) \;&=\; (\omega(\boldsymbol{u}), \omega(\boldsymbol{v}))_0, \\
a_2(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) \;&=\; \int_{\Omega} \omega(\boldsymbol{u})\,\boldsymbol{v}^{\perp} \cdot \boldsymbol{w}\;\mathrm{d}\boldsymbol{x}, \\
a(\boldsymbol{u}, \boldsymbol{v}) \;&=\; a_1(\boldsymbol{u}, \boldsymbol{v}) + a_2(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}), \\
b(\boldsymbol{h}, \boldsymbol{v}) \;&=\; -\int_{\Gamma_o} \boldsymbol{h} \cdot \boldsymbol{v}\;\mathrm{d}S.
\end{aligned}$$

All these forms are defined for $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w} \in H^1(\Omega)^2$, $\boldsymbol{f} \in L^2(\Omega)^2$ and $\boldsymbol{h} \in L^2(\Gamma_o)^2$. Now the weak problem reads as follows:

Definition 1 *Let the function $\boldsymbol{g} \in H^s(\Gamma_i)^2$ (for some $s \in (\frac{1}{2}, 1]$) satisfy the condition $\boldsymbol{g}(A_1) = \boldsymbol{g}(A_0)$. Let $\boldsymbol{f} \in L^2(\Omega)^2$ and $\boldsymbol{h} \in L^2(\Gamma_o)^2$. The <u>weak solution</u> of the problem (3)–(7), (9)–(11) is a vector function $\boldsymbol{u} \in H^1(\Omega)^2$ which satisfies the identity (13) for all test functions $\boldsymbol{v} \in V$, the condition of incompressibility (10) a.e. in Ω, the boundary conditions (3), (4) in the sense of traces on Γ_i and Γ_w and the condition of periodicity (6) in the sense of traces on Γ_- and Γ_+.*

The pressure term q does not explicitly appear in the definition of the weak solution, however, as it is usual in the theory of the Navier–Stokes equations, it can be defined on the level of distributions or it can be recovered as a function from $H^1(\Omega)$, if the weak solution is sufficiently regular.

We further need an extension of the given function $\boldsymbol{g}$ from Γ_i onto the whole domain Ω. The existence of the appropriate extension is guaranteed by the next lemma. (See paper [2] for more details.)

Lemma 2 *There exists an extension of function $\boldsymbol{g}$ from Γ_i onto $\partial\Omega$ (we shall denote the extension again by $\boldsymbol{g}$) such that it belongs to $H^{1/2}(\partial\Omega)^2$, it equals zero on Γ_w, it satisfies the condition of periodicity (5) on Γ_- and Γ_+ and*

$$\int_{\partial\Omega} \boldsymbol{g}\cdot\mathbf{n}\,\mathrm{d}S \;=\; 0. \tag{14}$$

Moreover, there exists a constant $c_1 > 0$ independent of $\boldsymbol{g}$ such that

$$\|\boldsymbol{g}\|_{1/2;\,\partial\Omega} \;\leq\; c_1\,\|\boldsymbol{g}\|_{s;\,\Gamma_i}\,. \tag{15}$$

The proof can be found in [2]. The norms in (15) are the norms in the Sobolev–Slobodetski spaces $H^s(\Gamma_i)^2$ and $H^{1/2}(\partial\Omega)^2$. The next lemma, which is also taken from [2], shows that $\boldsymbol{g}$ can be extended from $\partial\Omega$ to Ω.

Lemma 3 *A function $\boldsymbol{g}\in H^{1/2}(\partial\Omega)^2$ which satisfies (14) can be extended to a function $\boldsymbol{g}^*\in H^1(\Omega)^2$ such that*

$$\boldsymbol{g}^*|_{\partial\Omega} \;=\; \boldsymbol{g} \quad \text{(in the sense of traces)}, \tag{16}$$

$$\operatorname{div}\boldsymbol{g}^* \;=\; 0 \quad \text{in } \Omega, \tag{17}$$

$$\|\boldsymbol{g}^*\|_1 \;\leq\; c_2\,\|\mathbf{g}\|_{1/2;\,\partial\Omega} \tag{18}$$

where the constant $c_2 > 0$ is independent of $\boldsymbol{g}$.

Now we can construct the weak solution $\boldsymbol{u}$ in the form $\boldsymbol{u} = \boldsymbol{g}^* + \boldsymbol{z}$ where $\boldsymbol{z}\in V$ is a new unknown function. Substituting $\boldsymbol{u} = \boldsymbol{g}^* + \boldsymbol{z}$ into the equation (13), we get the following problem: Find a function $\boldsymbol{z}\in V$ such that it satisfies the equation

$$a(\boldsymbol{g}^* + \boldsymbol{z}, \boldsymbol{v}) \;=\; (\boldsymbol{f}, \boldsymbol{v})_0 + b(\boldsymbol{h}, \boldsymbol{v}) \tag{19}$$

for all $\boldsymbol{v}\in V$. The final theorem on the existence reads as follows.

The next two lemmas will give a sufficient condition for coercivity of the form a.

Lemma 4 *There exist positive constants c_3 and c_4 such that*

$$a(\boldsymbol{g}^* + \boldsymbol{z}, \boldsymbol{z}) \;\geq\; |||\boldsymbol{z}|||\,\Big(\nu\,|||\boldsymbol{z}||| - \nu\,c_2\,c_1\,\|\boldsymbol{g}\|_{s;\,\Gamma_i} - c_3\,\|\boldsymbol{g}\|^2_{s;\,\Gamma_i} - c_4\,\|\boldsymbol{g}\|_{s;\,\Gamma_i}\,|||\boldsymbol{z}|||\Big) \tag{20}$$

for all $\boldsymbol{z}\in V$.

Proof. Using the definitions of the forms a, a_1 and a_2, we find that

$$a(\boldsymbol{g}^* + \boldsymbol{z}, \boldsymbol{z}) = a_1(\boldsymbol{z}, \boldsymbol{z}) + a_1(\boldsymbol{g}^*, \boldsymbol{z}) + a_2(\boldsymbol{g}^*, \boldsymbol{g}^*, \boldsymbol{z}) + a_2(\boldsymbol{g}^*, \boldsymbol{z}, \boldsymbol{z}) + a_2(\boldsymbol{z}, \boldsymbol{g}^*, \boldsymbol{z}) + a_2(\boldsymbol{z}, \boldsymbol{z}, \boldsymbol{z}).$$

Since $\boldsymbol{z}^\perp\cdot\boldsymbol{z} = 0$ in Ω, the terms $a_2(\boldsymbol{g}^*, \boldsymbol{z}, \boldsymbol{z})$ and $a_2(\boldsymbol{z}, \boldsymbol{z}, \boldsymbol{z})$ vanish. Hence,

$$a(\boldsymbol{g}^* + \boldsymbol{z}, \boldsymbol{z}) \;\geq\; a_1(\boldsymbol{z}, \boldsymbol{z}) - |a_1(\boldsymbol{g}^*, \boldsymbol{z})| - |a_2(\boldsymbol{g}^*, \boldsymbol{g}^*, \boldsymbol{z})| - |a_2(\boldsymbol{z}, \boldsymbol{g}^*, \boldsymbol{z})|. \tag{21}$$

We obviously have

$$a_1(\boldsymbol{z}, \boldsymbol{z}) \;=\; \nu(\nabla\boldsymbol{z}, \nabla\boldsymbol{z})_0 \;\geq\; \nu\,|||\boldsymbol{z}|||^2. \tag{22}$$

Let us further estimate the terms on the right-hand side of (21). If we use the Cauchy inequality, the continuous imbedding of $H^1(\Omega)$ into $L^4(\Omega)$, Green's theorem and the theorem on traces, we successively obtain

$$\begin{aligned}
|a_1(\boldsymbol{g}^*, \boldsymbol{z})| \;&=\; \nu\,(\nabla\boldsymbol{g}^*, \nabla\boldsymbol{z})_0 \\
&\leq\; \nu\,\|\boldsymbol{g}^*\|_1\,|||\boldsymbol{z}|||, \qquad (23)\\
|a_2(\boldsymbol{g}^*, \boldsymbol{g}^*, \boldsymbol{z})| \;&=\; \left|\int_\Omega \omega(\boldsymbol{g}^*)\,\boldsymbol{g}^{*\perp}\cdot\boldsymbol{z}\,\mathrm{d}\boldsymbol{x}\right| \\
&\leq\; \|\omega(\boldsymbol{g}^*)\|_0\,\|\boldsymbol{g}^*\|_{L^4}\,\|\boldsymbol{z}\|_{L^4} \\
&\leq\; c_5\,\|\boldsymbol{g}^*\|_1^2\,|||\boldsymbol{z}|||, \qquad (24)\\
|a_2(\boldsymbol{z}, \boldsymbol{g}^*, \boldsymbol{z})| \;&=\; \left|\int_\Omega \omega(\boldsymbol{z})\,\boldsymbol{g}^{*\perp}\cdot\boldsymbol{z}\,\mathrm{d}\boldsymbol{x}\right| \\
&\leq\; \|\omega(\boldsymbol{z})\|_0\,\|\boldsymbol{g}^*\|_{L^4}\,\|\boldsymbol{z}\|_{L^4} \\
&\leq\; c_6\,|||\boldsymbol{z}|||^2\,\|\boldsymbol{g}^*\|_1, \qquad (25)
\end{aligned}$$

Substituting (22)–(25) into (21) and using (15), (18), we get

$$\begin{aligned}
a(\boldsymbol{g}^* + \boldsymbol{z}, \boldsymbol{z}) \;&\geq\; \nu\,|||\boldsymbol{z}|||^2 - \nu\,\|\boldsymbol{g}^*\|_1\,|||\boldsymbol{z}||| \\
&\quad - c_5\,\|\boldsymbol{g}^*\|_1^2\,|||\boldsymbol{z}||| - c_6\,\|\boldsymbol{g}^*\|_1\,|||\boldsymbol{z}|||^2 \\
&\geq\; |||\boldsymbol{z}|||\,\Big(\nu\,|||\boldsymbol{z}||| - \nu\,c_1\,c_2\,\|\boldsymbol{g}\|_{s;\,\Gamma_i} - c_5\,c_1^2\,c_2^2\,\|\boldsymbol{g}\|^2_{s;\,\Gamma_i} \\
&\quad - c_6\,c_1\,c_2\,\|\boldsymbol{g}\|_{s;\,\Gamma_i}\,|||\boldsymbol{z}|||\Big). \qquad (26)
\end{aligned}$$

This completes the proof. □

Lemma 5 *There exists $\epsilon > 0$ such that if*

$$\|\boldsymbol{g}\|_{s;\,\Gamma_i} \;<\; \epsilon, \tag{27}$$

then the form $a(\boldsymbol{g}^* + \boldsymbol{z}, \boldsymbol{z})$ is coercive on the space V. It means that

$$\lim_{|||\boldsymbol{z}|||\to+\infty} a(\boldsymbol{g}^* + \boldsymbol{z}, \boldsymbol{z}) \;=\; +\infty. \tag{28}$$

Proof. Lemma 4 implies that it is sufficient to choose $\epsilon = \nu/c_4$. □

Theorem 6 (on the existence of a weak solution) *There exists* $\epsilon > 0$ *such that if* $\|\boldsymbol{g}\|_{s;\Gamma_i} < \epsilon$ *then there exists a solution* $\boldsymbol{u}$ *of the boundary–value problem defined in Definition 1.*

The proof is based on the Galerkin method. We construct a sequence of approximations $\{\boldsymbol{z}_n\}$ of function $\boldsymbol{z}$ from (19). Functions $\boldsymbol{z}_n$ are elements of finite–dimensional subspaces V_n of space V and they satisfy (19), however for $\boldsymbol{v}$ from V_n only. In order to prove the existence and boundedness of $\boldsymbol{z}_n$ in V_n, we need the bilinear form a to be coercive. This condition leads to the requirement of a sufficient smallness of the norm of function $\boldsymbol{g}$ in the space $L^s(\Gamma_i)^2$. Due to the reflexivity of space V, the sequence $\{\boldsymbol{z}_n\}$ contains a sub–sequence which converges weakly in V to a limit function $\boldsymbol{z}$. The sub–sequence converges strongly to $\boldsymbol{z}$ in $L^2(\Omega)^2$. We can prove that the function $\boldsymbol{u} = \boldsymbol{g}^* + \boldsymbol{z}$ is a weak solution of the problem (3)–(7), (9)–(11).

3 Conclusion

The presented approach to the mathematical modelling of a flow of a viscous incompressible fluid through a 2D profile cascade differs from the previous models in the form in which the 2D Navier–Stokes equation is treated and which further reflects in the weak formulation. This concerns the viscous term and the nonlinear term. Writing the nonlinear term in the form $\omega(\boldsymbol{u})\,\boldsymbol{u}^\perp$ (and considering consequently Bernoulli's pressure $q \equiv p + \frac{1}{2}|\boldsymbol{u}|^2$ instead of the usual pressure p) has the advantage that the scalar product of $\omega(\boldsymbol{u})\,\boldsymbol{u}^\perp$ and $\boldsymbol{u}$ equals zero point-wise in Ω. Thus, in order to derive an apriori estimate of a solution, we do not need to integrate it by parts, we do not obtain any additional term on the outflow and we are therefore able to control a kinetic energy brought back by a possible backward flow. Considering the viscous term in the form $\nu(-\partial_2, \partial_1)\omega(\boldsymbol{u})$ on the other hand enables us to separate the boundary conditions prescribed for the vorticity $\omega(\boldsymbol{u})$ and q on the outflow, see (11).

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References:

[1] M. Feistauer: *Mathematical Methods in Fluid Dynamics.* Pitman Monographs and Surveys in Pure and Applied Mathematics 67, Longman Scientific & Technical, Harlow, 1993.

[2] M. Feistauer and T. Neustupa: On some aspects of analysis of incompressible flow through cascades of profiles. *Operator Theory, Advances and Applications*, Vol. 147, Birkhauser, Basel, 2004, 257–276.

[3] M. Feistauer and T. Neustupa: A Flow through a Cascade of Profiles with Natural Boundary Conditions Involving Bernoulli's Pressure on the Outflow. Preprint 2005.

[4] J. G. Heywood, R. Rannacher and S. Turek: Artificial boundaries and flux and pressure conditions for the incompressible Navier-Stokes equations. *Int. J. for Numerical Methods in Fluids* **22**, 1996, 325–352.

[5] S. Kračmar and J. Neustupa: Global existence of weak solutions of a nonsteady variational inequality of the Navier–Stokes type with mixed boundary conditions. *Proc. of the "International Symposium on Numerical Analysis"*, Charles University Prague, 1993, 156–177.

[6] S. Kračmar and J. Neustupa: A weak solvability of steady variational inequality of the Navier–Stokes type with mixed boundary conditions. *Nonlinear Analysis* **47**, 2001, 4169–4180.

[7] P. Kučera and Z. Skalák: Local solutions to the Navier-Stokes equations with mixed boundary conditions. *Acta Applicandae Mathematicae* **54**, No. 3, 1998, 275–288.

[8] P. Kučera: Solutions to the stationary Navier–Stokes equations with mixed boundary conditions in a bounded domain. *Proc. of the conference "Analysius, Numerics and Applications of Differential and Integral Equations"*, editors M.Bach, C.Constanda, G.C.Hsiao, A.M.Sandig & P.Werner, Pitman Research Notes in Mathematical Series 379, 1998, 127–132.

[9] A. Kufner, S. Fučík and O. John: *Function Spaces.* Academia, Praha, 1977.

[10] O. A. Ladyzhenskaya: *The Mathematical Theory of Viscous Incompressible Flow.* Gordon and Breach, New York 1969.

[11] E. Martensen: Berechnung der Druckverteilung an Gitterprofillen in ebener Potentialströmung mit einer Fredholmschen Integralgleichung. *Arch. Rat. Mech. Anal.* **3**, 1959, 253–270.

[12] E. Meister: *Randwertaufgaben der Funktionentheorie.* Teubner, Stuttgart, 1983.

[13] R. Temam: *Navier–Stokes Equations.* North–Holland, Amsterdam–New York–Oxford, 1977.